WEAKLY PRIME SUBMODULES AND PRIME SUBMODULES

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Abstract. A proper submodule \(N\) of an \(R\)-module \(M\) is called a weakly prime submodule, if for each submodule \(K\) of \(M\) and elements \(a, b\) of \(R\), \(abK \subseteq N\), implies that \(aK \subseteq N\) or \(bK \subseteq N\). In this paper we will study weakly prime submodules and we shall compare weakly prime submodules with prime submodules.

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1. Introduction. Throughout this paper all rings are commutative with identity and all modules are unitary. Also we consider \(R\) to be a ring and \(M\) a unitary \(R\)-module.

Let \(N\) be a proper submodule of \(M\). It is said that \(N\) is a prime submodule of \(M\), if the condition \(ra \in N, r \in R\) and \(a \in M\) implies that \(a \in N\) or \(rM \subseteq N\). In this case, if \(P = (N : M) = \{t \in R \mid tM \subseteq N\}\), we say that \(N\) is a \(P\)-prime submodule of \(M\), and it is easy to see that \(P\) is a prime ideal of \(R\). Prime submodules have been studied in several papers such as [1–4], [6–8], [10].

A proper submodule \(N\) of \(M\) is called a weakly prime submodule, if for each submodule \(K\) of \(M\) and elements \(a, b\) of \(R\), \(abK \subseteq N\), implies that \(aK \subseteq N\), or \(bK \subseteq N\).

Weakly prime submodules have been introduced and studied in [5]. If we consider \(R\) as an \(R\)-module, then prime submodules and weakly prime submodules are exactly prime ideals of \(R\). More generally for every multiplication module any submodule is a prime submodule if and only if it is a weakly prime submodule. For every \(R\)-module, it is easy to see that any prime submodule is a weakly prime submodule, but the converse is not always correct. For example let \(R\) be a ring with \(\text{dim } R \neq 0\), and \(P \subset Q\) a chain of prime ideals of \(R\). Then it is easy to see that for the free \(R\)-module \(R \oplus R\), the submodule \(P \oplus Q\) is a weakly prime submodule which is not a prime submodule.

Recall that a proper submodule \(N\) of a module \(M\) is said to be a primary submodule if the condition \(ra \in N, r \in R\) and \(a \in M\), implies that \(a \in N\) or \(r^nM \subseteq N\), for some positive number \(n\).

In this note, we will find some relations between prime submodules and weakly prime submodules. It is proved that any weakly prime submodule is a prime submodule if and only it is a primary submodule. Also any irreducible and weakly prime submodule is a prime submodule.

It is proved that:

1. If \(F\) is a flat \(R\)-module and \(N\) a weakly prime submodule of \(M\) such that \(F \otimes N \neq F \otimes M\), then \(F \otimes N\) is a weakly prime submodule of \(F \otimes M\).

2. If \(F\) is a faithfully flat \(R\)-module and \(N\) a submodule of \(M\), then \(N\) is a weakly prime submodule of \(M\), if and only if \(F \otimes N\) is a weakly prime submodule of \(F \otimes M\).
2. Some comparisons. In the following, we compare some properties of weakly prime submodules with properties of prime submodules.

**Lemma 2.1.** Let $M$ be an $R$-module and $N$ a proper submodule of $M$.

(i) $N$ is a weakly prime submodule if and only if for every submodule $K$ of $M$ not contained in $N$, $(N : K)$ is a prime ideal of $R$. In particular $(N : M)$ is a prime ideal of $R$.

(ii) Let $N$ be a weakly prime submodule of $M$. Then for all submodules $K$ and $L$ of $M$ not contained in $N$, $(N : K) \subseteq (N : L)$ or $(N : L) \subseteq (N : K)$.

**Proof.** The proof is obvious.

**Corollary 2.2.** Let $M$ be an $R$-module and $N$ a proper submodule of $M$. Then $N$ is a prime submodule if and only if $N$ is primary and weakly prime.

**Proof.** Let $N$ be primary and weakly prime, and $rx \in N$, where $x \notin N$. Then there exists a positive number $n$ such that for each $y \in M \setminus N$, $r^n y \in N$, i.e., $r^n \in (N : y)$. By Lemma 2.1, (i), $(N : y)$ is a prime ideal, then $r \in (N : y)$. Hence for each $y \in M$, we have, $ry \in N$, that is, $rM \subseteq N$. The converse is clear.

**Theorem 2.3.** Let $M$ be an $R$-module and $N$ a proper submodule of $M$. The following are equivalent.

(i) $N$ is a weakly prime submodule.

(ii) For any $x, y \in M$, if $(N : x) \neq (N : y)$, then $N = (N + Rx) \cap (N + Ry)$.

**Proof.** (i) $\implies$ (ii) Let $r \in (N : x) \setminus (N : y)$, where $r \in R$, i.e., $rx \in N$ and $ry \notin N$. Since by Lemma 2.1, (i), $(N : y)$ is a prime ideal, it is easy to see that $(N : ry) = (N : r)$. If $t \in (N + Rx) \cap (N + Ry)$, then $t = n_1 + r_1 x = n_2 + r_2 y$, where $n_1, n_2 \in N$ and $r_1, r_2 \in R$. Note that $rt = r n_1 + r_1 rx = rn_2 + r_2 ry$ and $r_1 rx, n_1, n_2 \in N$, so $r_2 y \in N$, that is, $r_2 \in (N : ry) = (N : y)$. Since $r_2 y \in N$, we have $t = n_2 + r_2 y \in N$.

(ii) $\implies$ (i) It is enough to show that if $r_1 r_2 a \in N$, where $r_1, r_2 \in R$, $a \in M$ and $r_1 a \notin N$, then $r_2 a \in N$. We have, $r_1 \in (N : r_2 a) \setminus (N : a)$, so $(N : r_2 a) \neq (N : a)$. Put $x = r_2 a$, $y = a$, then by our assumption we have, $N = (N + Rr_2 a) \cap (N + Ra)$. Evidently, $r_2 a \in (N + Ra) \cap (N + Rr_2 a) = N$.

From the definition of prime submodule, it is easy to see that if $N$ is a prime submodule of an $R$-module $M$ and $x, y \in M$ such that $rx \in N$, where $r \in R$, then $N = N + Rx$, or $N = N + Rry$. Compare this note with the following corollary, part (i).

**Corollary 2.4.** Let $M$ be an $R$-module, $N$ a weakly prime submodule of $M$ and $x, y \in M$.

(i) If $rx \in N$ where $r \in R$, then $N = (N + Rx) \cap (N + Rry)$.

(ii) If $N$ is an irreducible submodule, then $N$ is a prime submodule.

**Proof.** (i) If $ry \notin N$, then obviously, $N = (N + Rx) \cap (N + Rry)$. Now let $ry \notin N$. So $(N : x) \neq (N : y)$, and by Theorem 2.3, we have $N \subseteq (N + Rx) \cap (N + Rry) \subseteq (N + Rx) \cap (N + Rry) = N$.

(ii) Let $rx \in N$ where $r \in R$. By part (i), for each $y \in M$, we have, $N = (N + Rx) \cap (N + Rry)$, and since $N$ is irreducible, $x \in N$ or $ry \in N$.

**Proposition 2.5.** Let $A_i$, $1 \leq i \leq n$ be a finite collection of ideals of a ring $R$ and let $M$ be the free $R$-module $\bigoplus_{i=1}^{n} R$. Then $\bigoplus_{i=1}^{n} A_i$ is a weakly prime submodule of $M$ if and only if $\{ A_i | A_i \neq R \}$ is a non-empty chain of prime ideals of $R$. 

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3. Weakly prime submodules and flat modules. Let $M$ be an $R$-module and $N$ a submodule of $M$. In this section for every $a \in R$, we consider $(N : a)$ to be:

$$(N : a) = \{m \in M \mid am \in N\}.$$ 

It is easy to see that $(N : a)$ is a submodule of $M$ containing $N$. The following lemma will give us a characterization of weakly prime submodules.

**Lemma 3.1.** Let $M$ be an $R$-module and $N$ a proper submodule of $M$. Then $N$ is a weakly prime submodule of $M$ if and only if for every $a, b \in R$, $(N : ab) = (N : a)$ or $(N : ab) = (N : b)$.

**Proof.** Let $N$ be a weakly prime submodule of $M$. It is easy to see that $(N : ab) = (N : a) \cup (N : b)$. Now since $(N : a) \cup (N : b) = (N : ab)$ is a submodule of $M$, we have $(N : a) \subseteq (N : b)$ or $(N : b) \subseteq (N : a)$. Hence $(N : ab) = (N : a)$, or $(N : ab) = (N : b)$.

For the converse let $abm \in N$, where $a, b \in R$ and $m \in M$. By our assumption we may suppose that $(N : ab) = (N : a)$. Thus $m \in (N : ab) = (N : a)$, that is, $am \in N$. So $N$ is a weakly prime submodule of $M$. □

**Lemma 3.2.** Let $M$ be an $R$-module, $N$ a submodule of $M$ and $a \in R$. Then for every flat $R$-module $F$, we have $F \otimes (N : a) = (F \otimes N : a)$.

**Proof.** Clearly $F \otimes (N : a) \subseteq (F \otimes N : a)$. Consider the exact sequence $0 \longrightarrow (N : a) \longrightarrow M \xrightarrow{g_a} M \otimes \frac{M}{N}$, where $g_a(m) = am + N$, $\forall m \in M$. Since $F$ is a flat module and $\theta : F \otimes M \otimes \frac{M}{N} \longrightarrow F \otimes M$ induced by $\theta(f \otimes (m + N)) = (f \otimes m) + F \otimes N$, $\forall m \in M, \forall f \in F$ is an isomorphism, we have the following exact sequence

$$0 \longrightarrow F \otimes (N : a) \xrightarrow{(1 \otimes g'_a)} F \otimes M \otimes \frac{M}{N},$$

where $(1 \otimes g'_a)(f \otimes m) = a(f \otimes m) + F \otimes N$, $\forall m \in M, \forall f \in F$. Consequently $F \otimes (N : a) = \text{Ker}(1 \otimes g'_a) = (F \otimes N : a)$. □

**Theorem 3.3.** Let $M$ be an $R$-module.

(i) If $F$ is a flat $R$-module and $N$ a weakly prime submodule of $M$ such that $F \otimes N \neq F \otimes M$, then $F \otimes N$ is a weakly prime submodule of $F \otimes M$.

(ii) Let $F$ be a faithfully flat $R$-module. Then $N$ is a weakly prime submodule of $M$ if and only if $F \otimes N$ is a weakly prime submodule of $F \otimes M$.

**Proof.** (i) Let $a, b \in R$. By Lemma 3.1, we may suppose that $(N : ab) = (N : a)$. Now, by Lemma 3.2, we have $(F \otimes N : ab) = F \otimes (N : ab) = F \otimes (N : a) = (F \otimes N : a)$, that is, $(F \otimes N : ab) = (F \otimes N : a)$. Hence by Lemma 3.1, $F \otimes N$ is a weakly prime submodule of $F \otimes M$.

(ii) Let $N$ be a weakly prime submodule of $M$ and $F \otimes N = F \otimes M$. Therefore, $0 \longrightarrow F \otimes N \xrightarrow{(1 \otimes g'_a)} F \otimes M \longrightarrow 0$ is an exact sequence, and since $F$ is a faithfully flat module, then $0 \longrightarrow N \xrightarrow{(1 \otimes g'_a)} M \longrightarrow 0$ is an exact sequence. Hence $N = M$, which is a contradiction. So $F \otimes N \neq F \otimes M$. Now by part i), $F \otimes N$ is a weakly prime submodule of $F \otimes M$.

Conversely suppose that $F \otimes N$ is a weakly prime submodule of $F \otimes M$. We have, $F \otimes N \neq F \otimes M$ and obviously $N \neq M$. Let $a, b \in R$. We may assume that $(F \otimes N : a) = (F \otimes N : b)$. □
$ab = (F \otimes N : a)$, by Lemma 3.1. Then by Lemma 3.2, we have $F \otimes (N : a) = (F \otimes N : a) = (F \otimes N : ab) = F \otimes (N : ab)$. So $0 \rightarrow F \otimes (N : a) \rightarrow F \otimes (N : ab) \rightarrow 0$ is an exact sequence, and since $F$ is faithfully flat, $0 \rightarrow (N : a) \rightarrow (N : ab) \rightarrow 0$ is an exact sequence, which implies that $(N : a) = (N : ab)$. Hence by Lemma 3.1, $N$ is a weakly prime submodule of $M$. \hfill \Box

A theorem similar to Theorem 3.3 for prime submodules has been proved in [2]. It is easy to see that a proper submodule $N$ of an $R$-module $M$ is a prime submodule if and only if for every $a \in R$, $(N : a) = N$ or $(N : a) = M$. Now by a proof similar to that of Theorem 3.3, we can show this theorem for prime submodules, which is different from the mentioned proof in [2].

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References