AUTOMORPHISM GROUPS OF ORTHOMODULAR LATTICES

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Every group is the automorphism group of an orthomodular lattice.

In this note we prove that every group is the automorphism group of
some orthomodular lattice. The proof makes use of the following results:
Birkhoff's construction of a lattice with a given group as its automorphism
group [4]; the embedding of a lattice L in an orthomodular lattice
generated by the chains in L [6]; the existence of Boolean algebras with
trivial automorphism groups [7] and the pasting of orthomodular lattices at
atoms [5]. The case of finite groups was solved in [11], it is an easy
application of results in [10]. For basic facts about orthomodular
lattices the reader is referred to [5].

Let G be a group provided with a ordinal-type well ordering $<$ such
that the unit $e \in G$ is the smallest element. The lattice L consists of
the elements of the following disjoint sets: $\{0\}$, $\{1\}$, G, $G \times G$ and the
set U of two-element subsets $\{x, y\}$ of G . The order structure $\leq$ on
L has 0 as the smallest, 1 as the largest element and G as its set
of atoms. Furthermore we define

  (i) $x, y < \{x, y\} \in U$ for $x, y \in G$,

  (ii) $(x, y) \leq (u, v)$ if $x = u$ and $y \leq v$ for
        $(x, y), (u, v) \in G \times G$,

  (iii) $x \leq (y, z)$ if $x^{-1}y \leq z$ for $x \in G$, $(y, z) \in G \times G$,

and

(iv) $\{x,\ y\} \le (x,\ z)$ for $\{x,\ y\} \in U$, $(x,\ z) \in G \times G$ if
       $y \le (x,\ z)$.

The maximal chains in L are of the form

$$0 < x < (x,\ e) < \ldots < (x,\ a) < \ldots < 1$$

or

$$0 < x < \{x,\ y\} < (x,\ b_0) < \ldots < (x,\ a) < \ldots < 1$$

or

$$0 < x < \{x,\ y\} < (y,\ c_0) < \ldots < (y,\ d) < \ldots < 1$$

where b_0 respectively c_0 are the smallest elements in G with

$y^{-1}x \le b_0$ respectively $x^{-1}y \le c_0$. Observe that the only non-trivial

joins and meets in L are $x \lor y = \{x,\ y\} = (x,\ b_0) \land (y,\ c_0)$.

The automorphisms g of L are given by

$$g(0) = 0 ,\quad g(1) = 1 ,\quad g(x) = bx ,$$

$$g(x,\ y) = (bx,\ y) ,\quad g(\{x,\ y\}) = \{bx,\ by\}$$

for a fixed element $b \in G$ ([4], pp. 44-47). Therefore G is isomorphic
to the automorphism group of L .

The lattice L generates an orthomodular lattice M as follows (for
details see [6]): let $(C_i)_{i \in I}$ by the set of finite chains in L
containing $0, 1$ and define $D_{ij} = C_i \cap C_j$. Every C_i generates a
Boolean algebra B_i . In the union $M = \bigcup\limits_{i \in I} B_i$ the elements of B_i and
B_j which are generated by the common subchain D_{ij} are identified. M is
an orthomodular lattice with the $'$ and $\le$ structure induced by the B_i .
The atoms of M are the elements of G , of $H = \{\{x,\ y\} \land x' \mid x,\ y \in G\}$,
of $P = \left\{ (x,\ b_0) \land \{x,\ y\}' \mid \{x,\ y\} \in U,\ b_0 \text{ minimal with } y^{-1}x \le b_0 \right\}$ and of
$Q = \{(x,\ c) \land (x,\ a)' \mid x \in G,\ c \text{ covers } a\}$.

An automorphism of M maps atoms onto atoms. Therefore we now extend
M to an orthomodular lattice N with the same set of atoms by pasting

suitable automorphism-free Boolean algebras at the atoms of $M - L$ in such a manner that N has the (uniquely) extended automorphisms of L as its automorphisms. Let B_τ be a Boolean algebra of power $2^{\aleph_\tau}$ (τ an ordinal) with $\text{Aut}(B_\tau) = 1$ [7]. Then B_τ has at most one atom. We can assume that it has exactly one atom since if it has no atom then $2 \times B_\tau$ has one atom and $\text{Aut}(2 \times B_\tau) = 1$.

LEMMA. *If $a \in C$ is an atom of the orthomodular lattice C and D is obtained from the disjoint union of C and B_τ by identifying the two zero elements, the two unit elements, the atom $a \in C$ with the atom $b \in B_\tau$ and the element $a' \in C$ with $b' \in B_\tau$ then D with the $'$ and $\leq$ structure induced by C and B_τ is an orthomodular lattice.*

This is a special case of Greechie's paste job ([5], pp. 42-44). We assume now that $|G|$ is infinite. Then all blocks (maximal Boolean subalgebras) of M have the same cardinality m . Choose the ordinal number η such that $2^{\aleph_\eta} > m$. To every atom in H respectively P we paste one copy of B_η respectively $B_{\eta+1}$. Enumerate the elements of the well-ordered chain G by

$$e = a_{\eta+2} < a_{\eta+3} < \ldots < a_\sigma < \ldots .$$

At every atom $(x, a_\sigma) \wedge (x, a_{\sigma-1})' \in Q$ (if it exists for σ) we paste one copy of B_σ . The lemma assures that the pasting N of M with all these Boolean algebras is an orthomodular lattice.

If an atom p of N is mapped by an automorphism of N to an atom $q \in N$ then the blocks containing p are mapped onto the blocks containing q . For different σ_1, σ_2 , B_{σ_1} is not isomorphic to B_{σ_2} and B_{σ_1} is, for $\sigma_1 \geq \eta$, also not isomorphic to some block in M by cardinality reasons. Hence an automorphism of N can only map atoms of L onto atoms of L , atoms of H respectively P onto atoms of H respectively P and in Q only the atoms of the form $(x, a_\sigma) \wedge (x, a_{\sigma-1})'$ and $(y, a_\sigma) \wedge (y, a_{\sigma-1})'$ can be mapped onto one another. Every automorphism

of L therefore extends uniquely to an automorphism of N and N has no other automorphisms which shows that $\text{Aut}(N)$ is isomorphic to G.

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