NOTE ON THE COHOMOLOGY GROUPS OF ASSOCIATIVE ALGEBRAS

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The cohomology theory of associative algebras has been developed by G. Hochschild [1], [2], [3], and the 1-, 2-, and 3-dimensional cohomology groups have been interpreted with reference to classical notions of structure in his papers. Recently M. Ikeda has obtained, by a detailed analysis of Hochschild's modules, an interesting structural characterization of the class of algebras whose 2-dimensional cohomology groups are all zero [5].

In sections 1 and 2, we consider an algebra whose residue class algebra modulo its radical is separable, and offer a criterion for such algebra to have trivial $n(\geqq 2)$-dimensional cohomology group in terms of certain module, which is similar to Hochschild's module but is rather simpler.

In section 3, we consider the cases of dimensions 2 and 3. We offer another proof of Ikeda's theorem, and, under the assumption that A/N (N is the radical of A) is separable, a structural characterization of the class of algebras whose 3-dimensional cohomology groups are all zero.

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1. Let A be an associative algebra over a field F which possesses a unit element 1, and N be its radical. We assume, throughout this and the next section, that A/N is separable. Since 2-dimensional cohomology groups of A/N are all zero, A contains a subalgebra $\bar{A}$ such that A is decomposed into the direct (module) sum of $\bar{A}$ and N : $A = \bar{A} + N$. Evidently $\bar{A}$ is an algebra isomorphic to A/N, and hence separable. We denote elements of $\bar{A}$ by $\bar{a}$, $\bar{b}$, . . . and those of N by m_1, m_2, ...

With an A-A-module $\mathfrak{n}$ and a natural number n we denote, after Hochschild, the modules of all n-cochains, n-cocycles, n-coboundaries of A in $\mathfrak{n}$ by $C^n(A, \mathfrak{n})$, $Z^n(A, \mathfrak{n})$, $B^n(A, \mathfrak{n})$ respectively, and n-dimensional cohomology group of A in $\mathfrak{n}$ by $H^n(A, \mathfrak{n})$.

Let $P_n = A \times \ldots \times A$ be the n-fold direct product of the underlying vector space of A. We define the operations on P_n by setting

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$$(1)\qquad \begin{cases} a_0 * (a_1 \times \ldots \times a_n) = \sum_{i=0}^{n-1}(-1)^i a_0 \times \ldots \times a_i a_{i+1} \times \ldots \times a_n, \\ (a_1 \times a_2 \times \ldots \times a_n) * a_{n+1} = a_1 \times a_2 \times \ldots \times a_n a_{n+1}. \end{cases}$$

This makes P_n an A-A-module.[1] We call this the *n-dimensional Hochschild module* of A.

Lemma 1.1. *Let* $\mathfrak{n}$ *be an* A-A-*module. If* f *is an element of* $C^n(A, \mathfrak{n})$ *and* $\delta f\ (\bar{a}_1, a_2, \ldots, a_{n+1}) = 0$ *for any element* $\bar{a}_1$ *of* $\bar{A}$, *then there exists an element* g *of* $C^{n-1}(A, \mathfrak{n})$ *such that* $(f - \delta g)\ (\bar{a}_1, a_2, \ldots, a_n) = 0$ *for any element* $\bar{a}_1$ *of* $\bar{A}$.

Proof. Let $R(P_n, \mathfrak{n})$ be the module of all right operator homomorphisms from P_n into $\mathfrak{n}$. We define the operations of the elements of A for $F \in R(P_n, \mathfrak{n})$ by setting

$$(a \circ F)(a_1 \times a_2 \times \ldots \times a_n) = aF(a_1 \times a_2 \times \ldots \times a_n),$$
$$(F \circ a)(a_1 \times a_2 \times \ldots \times a_n) = F(a * (a_1 \times a_2 \times \ldots \times a_n)).$$

Under these operations, $R(P_n, \mathfrak{n})$ is an A-A-module.

For an $f \in C^n(A, \mathfrak{n})$ having the property in the lemma we define an element $F(f)$ of $C^1(\bar{A}, R(P_n, \mathfrak{n}))$ by the relation $F(f)(\bar{a}_1)(a_2 \times \ldots \times a_{n+1}) = f(\bar{a}_1, a_2, \ldots, a_n)a_{n+1}$. Then we can verify, from the property of f, that $\delta F(f) = 0$. Since $\bar{A}$ is separable, there exists an element G of $R(P_n, \mathfrak{n})$ such that $F(f)(\bar{a}) = \delta G(\bar{a}) = \bar{a} \circ G - G \circ \bar{a}$. We define $g \in C^{n-1}(A, \mathfrak{n})$ by setting

$$g(a_1, a_2, \ldots, a_{n-1}) = G(a_1 \times a_2 \times \ldots \times a_{n-1} \times 1),$$

then we see, from the property of G, that g satisfies the requierment of the lemma.

Now let $Q_{n-1} = N \times A \times \ldots \times A$ be the direct product of the vector spaces of N and $(n-2)$-fold direct product of A. We define the operations of the element of A, $\bar{A}$ on Q_{n-1}, on the right and left sides, respectively, by setting

$$(2)\qquad \begin{cases} (m_1 \times a_2 \times \ldots \times a_{n-1}) * a_n = \sum_{i=1}^{n-1}(-1)^{n-i-1} m_1 \times \ldots \times a_i a_{i+1} \times \ldots \times a_n, \\ \bar{a}_0 * (m_1 \times a_2 \times \ldots \times a_{n-1}) = \bar{a}_0 m_1 \times a_2 \times \ldots \times a_{n-1}. \end{cases}$$

This makes Q_{n-1} an $\bar{A}$-A-module.

We denote by $\bar{L}(Q_{n-1}, \mathfrak{n})$ the module of all $\bar{A}$-(left) operator homomorphisms from Q_{n-1} into $\mathfrak{n}$, and define the operations of the elements of A for $F \in \bar{L}(Q_{n-1}, \mathfrak{n})$ by setting

$$(3)\qquad \begin{cases} (a \circ F)(m_1 \times a_2 \times \ldots \times a_{n-1}) = F((m_1 \times a_2 \times \ldots \times a_{n-1}) * a) \\ (F \circ a)(m_1 \times a_2 \times \ldots \times a_{n-1}) = F(m_1 \times a_2 \times \ldots \times a_{n-1})a. \end{cases}$$

[1] A module $\mathfrak{m}$ is called an A-A-module if $\mathfrak{m}$ is A-left and right module and satisfies $a(mb) = (am)b$ $(a, b \in A,\ m \in \mathfrak{m})$.

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Under these operations $\bar{L}(Q_{n-1}, \mathfrak{n})$ is an A-A-module.

THEOREM 1.1. *Let* $\mathfrak{n}$ *be a module such that* $N\mathfrak{n} = \mathfrak{n}N = 0$. *Then (under the assumption that* A/N *is separable)*

$$H^n(A, \mathfrak{n}) \simeq H^1(A, \bar{L}(Q_{n-1}, \mathfrak{n})) \qquad (n \geqq 2).$$

Proof. Denote by $\bar{C}^n(A, \mathfrak{n})$ the module of all n-cochains f such that $f(\bar{a}_1, a_2, \ldots, a_n) = 0$ for any element $\bar{a}_1$ of $\bar{A}$, and set $\bar{Z}^n(A, \mathfrak{n}) = Z^n(A, \mathfrak{n}) \frown \bar{C}^n(A, \mathfrak{n})$, $\bar{B}^n(A, \mathfrak{n}) = B^n(A, \mathfrak{n}) \frown \bar{C}^n(A, \mathfrak{n})$. From Lemma 1.1 every cohomology class contains an element of $\bar{Z}^n(A, \mathfrak{n})$, and hence $H^n(A, \mathfrak{n})$ is isomorphic to $\bar{Z}^n(A, \mathfrak{n})/\bar{B}^n(A, \mathfrak{n})$. With an element f of $\bar{Z}^n(A, \mathfrak{n})$ and an element a_n of A, we define a linear mapping $F(f)(a_n)$ from Q_{n-1} into $\mathfrak{n}$ by the relation $F(f)(a_n)(m_1 \times a_2 \times \ldots \times a_{n-1}) = f(m_1, a_2, \ldots, a_n)$. Since $\delta f(\bar{a}, m_1, a_2, \ldots, a_n) = \bar{a}f(m_1, a_2, \ldots, a_n) - f(\bar{a}m_1, a_2, \ldots, a_n) = 0$, $F(f)(a_n)$ is an element of $\bar{L}(Q_{n-1}, \mathfrak{n})$ and $F(f)$ is an element of $C^1(A, \bar{L}(Q_{n-1}, \mathfrak{n}))$. Taking account of the assumed property of $\mathfrak{n}$ we see by direct computations that $(\delta F(f)(a_n, a_{n+1}))(m_1 \times \ldots \times a_{n-1}) = \delta f(m_1, a_2, \ldots, a_{n+1}) = 0$, and hence $F(f) \in Z^1(A, \bar{L}(Q_{n-1}, \mathfrak{n}))$.

Now let f be an element of $\bar{B}^n(A, \mathfrak{n})$. Then there exists an element g' of $C^{n-1}(A, \mathfrak{n})$ such that $f = \delta g'$. Since $\delta g'(\bar{a}_1, a_2, \ldots, a_n) = 0$ for $\bar{a}_1 \in \bar{A}$, from Lemma 1.1 there exists an element h of $C^{n-2}(A, \mathfrak{n})$ such that $(g' - \delta h)(\bar{a}_1, a_2, \ldots, a_{n-1}) = 0$ for $\bar{a}_1 \in \bar{A}$. Set $g = g' - \delta h$, then $f = \delta g$ and $g \in \bar{C}^{n-1}(A, \mathfrak{n})$. Since $f(\bar{a}_0, m_1, a_2, \ldots, a_{n-1}) = \delta g(\bar{a}_0, m_1, a_2, \ldots, a_{n-1}) = \bar{a}_0 g(m_1, a_2, \ldots, a_{n-1}) - g(\bar{a}_0 m, a_2, \ldots, a_{n-1}) = 0$, if we set $G(m_1 \times a_2 \times \ldots \times a_{n-1}) = g(m_1, a_2, \ldots, a_{n-1})$ then $G \in \bar{L}(Q_{n-1}, \mathfrak{n})$. By direct computations we can verify that $F(f)(a) = \pm \delta G$, and hence the mapping $f \to F(f)$ induces a homomorphism from $H^n(A, \mathfrak{n})$ into $H^1(A, \bar{L}(Q_{n-1}, \mathfrak{n}))$.

Conversely, if F is an element of $Z^1(A, \bar{L}(Q_{n-1}, \mathfrak{n}))$ we define an element f of $\bar{C}^n(A, \mathfrak{n})$ by setting

$$f(\bar{a}_1, a_2, \ldots, a_n) = 0 \quad \text{for} \quad \bar{a}_1 \in \bar{A},$$
$$f(m_1, a_2, \ldots, a_n) = F(a_n)(m_1 \times \ldots \times a_{n-1}) \quad \text{for} \quad m_1 \in N.$$

Then it is easily seen that f is an element of $\bar{Z}^n(A, \mathfrak{n})$ and $F = F(f)$. This shows that $H^n(A, \mathfrak{n})$ is mapped onto $H^1(A, \bar{L}(Q_{n-1}, \mathfrak{n}))$ by the above mapping. Further if $F(f)$ is a coboundary, that is, $F(f) = \delta G$, then we see that $f = \delta g$, where g is an element of $\bar{C}^{n-1}(A, \mathfrak{n})$ defined by the relations $g(m_1, a_2, \ldots, a_{n-1}) = G(m_1 \times a_2 \times \ldots \times a_{n-1})$, for $m_1 \in N$, and $g(\bar{a}_1, a_2, \ldots, a_{n-1}) = 0$, for $\bar{a}_1 \in \bar{A}$. This shows that the above homomorphism is an isomorphism.

2. In this section, we recall some definitions and properties about the module extensions and offer a criterion for A to have trivial n-dimensional cohomology groups in terms of Q_{n-1}.

Let $\mathfrak{m}$ and $\mathfrak{n}$ be two modules with the same operator domain Ω. We call

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a third Ω-module $\mathfrak{M}$ an (Ω-)*extension of* $\mathfrak{n}$ by $\mathfrak{m}$ if $\mathfrak{M}$ contains $\mathfrak{n}$ and $\mathfrak{M}/\mathfrak{n}\cong\mathfrak{m}$. If an extension $\mathfrak{M}$ of $\mathfrak{n}$ by $\mathfrak{m}$ contains an (Ω-)submodule $\mathfrak{m}'$ such that $\mathfrak{m}$ is the direct sum $\mathfrak{M}=\mathfrak{n}+\mathfrak{m}'$, then we say that $\mathfrak{m}$ *splits*. If for any Ω-module $\mathfrak{n}$ every extension of $\mathfrak{n}$ by $\mathfrak{m}$ splits, we call $\mathfrak{m}$ an (M_0)-module.

Now let $\mathfrak{m}$ and $\mathfrak{n}$ be two $\bar{A}$-A-modules and $\mathfrak{M}$ be an ($\bar{A}$-A-)extensions of $\mathfrak{n}$ by $\mathfrak{m}$. For $u\in\mathfrak{m}$, take a system of linear representatives $\{B_u\}$. Then

$$(4)\qquad \begin{cases} \bar{a}B_u = B_{\bar{a}u}+\beta(\bar{a},\ u) & (\bar{a}\in\bar{A},\ \beta(\bar{a},\ u)\in\mathfrak{n}),\\ B_u a = B_{ua}+\gamma(u,\ a) & (a\in A,\ \gamma(u,\ a)\in\mathfrak{n}). \end{cases}$$

$\beta(\bar{a},\ u)$ and $\gamma(u,\ a)$ are linear in $\bar{a},\ a,\ u$. From the associative relations $\bar{a}(\bar{b}B_u)=(\bar{a}\bar{b})B_u$, $(\bar{a}B_u)b=\bar{a}(B_u b)$, $(B_u a)b=B_u(ab)$, we have

$$(5)\qquad \begin{cases} \bar{a}\beta(\bar{b},\ u)+\beta(\bar{a},\ \bar{b}u)-\beta(\bar{a}\bar{b},\ u)=0,\\ \beta(\bar{a},\ ub)-\beta(\bar{a},\ u)b=\gamma(\bar{a}u,\ b)-\bar{a}\gamma(u,\ b),\\ \gamma(u,\ a)b+\gamma(ua,\ b)-\gamma(u,\ ab)=0. \end{cases}$$

The structure of $\mathfrak{M}$ is completely determined by $\{\beta,\ \gamma\}$, and conversely if $\{\beta,\gamma\}$ sutisfies the relations (5) we have an extension of $\mathfrak{n}$ by $\mathfrak{m}$, by (4). We call $\{\beta,\ \gamma\}$ satisfying (5) *a factor system.* Two factor systems $\{\beta_1,\ \gamma_1\}$ and $\{\beta_2,\ \gamma_2\}$ are called *associated* if there exists a linear mapping λ from $\mathfrak{m}$ into $\mathfrak{n}$ satisfying the relations

$$(6)\qquad \begin{cases} \beta_2(\bar{a},\ u)=\beta_1(\bar{a},\ u)+\{\bar{a}\lambda(u)-\lambda(\bar{a}u)\},\\ \gamma_2(u,\ a)=\gamma_1(u,\ a)+\{\lambda(u)a-\lambda(ua)\}. \end{cases}$$

As is well known, $\{\beta_1,\ \gamma_1\}$ and $\{\beta_2,\ \gamma_2\}$ are associated if and only if they define equivalent extensions.[2)]

We denote by $\bar{L}(\mathfrak{m},\ \mathfrak{n})$ the module of all $\bar{A}$-(left) operator homomorphisms from $\mathfrak{m}$ into $\mathfrak{n}$, and, defining the operations as (3), we make this an A-A-module. Since every ($\bar{A}$-A-)extension of $\mathfrak{n}$ by $\mathfrak{m}$ is ($\bar{A}$-)left inessential,[3)] by an argument similar to those in [3] or [6], we can verify the following lemma.

LEMMA 2.1. *Let* $\mathfrak{m}$ *and* $\mathfrak{n}$ *be two* $\bar{A}$-A-*modules. Then all extensions of* $\mathfrak{n}$ *by* $\mathfrak{m}$ *split if and only if* $H^1(A,\ \bar{L}(\mathfrak{m},\ \mathfrak{n}))=0$.

Let next

$$\bar{A}=\sum_{\kappa=1}^{k}\bar{A}e_\kappa=\sum_{\kappa=1}^{k}e_\kappa\bar{A}$$

be direct decompositions of $\bar{A}$ into indecomposable left and right ideals, and

2) Two extensions $\mathfrak{M}_1$, $\mathfrak{M}_2$ of $\mathfrak{n}$ by $\mathfrak{m}$ are called equivalent if there exists an isomorphism between $\mathfrak{M}_1$ and $\mathfrak{M}_2$ which leaves invariant each element of $\mathfrak{n}$ as well as the isomorphism from $\mathfrak{M}_i/\mathfrak{n}$ to $\mathfrak{m}$.

3) An $\bar{A}$-A-extension $\mathfrak{M}$ of $\mathfrak{n}$ by $\mathfrak{m}$ is called ($\bar{A}$-) left inessential if M splits as an $\bar{A}$-(left) extension.

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$\{e_\kappa\}$ be mutually othogonal primitive idempotents. Then

$$A=\sum_{\kappa=1}^{k}Ae_\kappa=\sum_{\kappa=1}^{k}e_\kappa A$$

are direct decompositions of A into indecomposable left and right ideals.

The structure theorem of (M_0)-modules states (see [7]):

LEMMA 2.2. *An A-right module* $\mathfrak{m}$ *is an* (M_0)*-module if and only if* $\mathfrak{m}1$ *is a direct sum of submodules isomorphic to* $e_\kappa A$.

Now we have

LEMMA 2.3. *Let* $\mathfrak{m}$ *be an* $\bar{A}$*-A-module, and suppose that* $1u=u$ *for* $u\in\mathfrak{m}$. $\mathfrak{m}$ *is an* (M_0)*-module as an* $\bar{A}$*-A-module if and only if it is so as an A-(right) module.*

Proof. i) Let $\mathfrak{m}$ be an (M_0)-module as an $\bar{A}$-A-module. Then $1\mathfrak{m}1=\mathfrak{m}1$ is a direct sum of submodules isomorphic to $\bar{A}e_\kappa\times e_\lambda A$, and hence as A-right module directly decomposed into a direct sum of submodules isomorphic to $e_\lambda A$. This shows that $\mathfrak{m}$ is an (M_0)-module as A-right module.

ii) Let $\mathfrak{m}$ be an (M_0)-module as A-right module. It is sufficient to prove that for any $\bar{A}$-A-module $\mathfrak{n}$ such that $\mathfrak{n}N=0$, every extension of $\mathfrak{n}$ by $\mathfrak{m}$ splits. Let $\mathfrak{n}$ be such a module, and $\{\beta,\gamma\}$ a factor system. Since $\bar{A}$ is separable, we can assume that $\beta(\bar{a},u)=\gamma(u,\bar{a})=0$. Then $\{\beta,\gamma\}$ satisfies the relations

$$(7)\qquad\begin{cases}\text{i)}\ \beta(\bar{a},u)=\gamma(u,\bar{a})=0,\\ \text{ii)}\ \gamma(\bar{a}u,m)-\bar{a}\gamma(u,m)=0,\\ \text{iii)}\ \gamma(u,m)\bar{b}-\gamma(u,m\bar{b})=0,\\ \text{iv)}\ \gamma(u\bar{a},m)-\gamma(u,\bar{a}m)=0.\end{cases}$$

And the extension determined by $\{\beta,\gamma\}$ splits if and only if there exists a linear mapping λ from $\mathfrak{m}$ into $\mathfrak{n}$ satisfying the relations

$$(8)\qquad\begin{cases}\beta(\bar{a},u)=\bar{a}\lambda(u)-\lambda(\bar{a}u)=0,\\ \gamma(u,\bar{a})=\lambda(u)\bar{a}-\lambda(u\bar{a})=0,\\ \gamma(u,m)=-\lambda(um).\end{cases}$$

Since $\mathfrak{m}$ is an (M_0)-module as an A-right module, there exists a linear mapping λ' satisfying the relations

$$(9)\qquad\begin{cases}\gamma(u,\bar{a})=\lambda'(u)\bar{a}-\lambda'(u\bar{a})=0,\\ \gamma(u,m)=-\lambda'(um).\end{cases}$$

Now, since $\mathfrak{m}$ is completely reducible as $\bar{A}$-$\bar{A}$-module, $\mathfrak{m}$ is decomposed into a direct sum of $\mathfrak{m}N$ and an another $\bar{A}$-$\bar{A}$-submodule $\mathfrak{m}_0$; $\mathfrak{m}=\mathfrak{m}N+\mathfrak{m}_0$. From (7) ii) and iii), λ' induces an $\bar{A}$-$\bar{A}$-operater homomorphism from $\mathfrak{m}N$ into $\mathfrak{n}$. Hence if we define a mapping λ from $\mathfrak{m}$ into $\mathfrak{n}$ by setting

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$$\lambda(um) = \lambda'(um),$$
$$\lambda(u_0) = 0 \quad \text{for} \quad u_0 \in \mathfrak{m}_0,$$

then λ satisfies the relations (8), and the extension determined by $\{\beta, \gamma\}$ splits.

LEMMA 2.4. *$H^n(A, \mathfrak{n}) = 0$ for every A-A-module $\mathfrak{n}$ if (and only if) it holds for every A-A-module $\mathfrak{n}$ such that $N\mathfrak{n} = \mathfrak{n}N = 0$.*

Proof. Suppose that $H^n(A, \mathfrak{n}) = 0$ for all $\mathfrak{n}$ such that $N\mathfrak{n} = \mathfrak{n}N = 0$. Let $\mathfrak{m}$ be an A-A-module and $\mathfrak{m} = \mathfrak{m}_0 \supset \mathfrak{m}_1 \supset \mathfrak{m}_1 \supset \ldots \supset \mathfrak{m}_t = 0$ be a composition series of $\mathfrak{m}$. In case $t = 1$, $N\mathfrak{m} = \mathfrak{m}N = 0$ and hence $H^n(A, \mathfrak{m}) = 0$. Now suppose that $H^n(A, \mathfrak{n}) = 0$ for all $\mathfrak{n}$ with a length of composition series less than t, and consider an $f \in Z^n(A, \mathfrak{m})$. Set $\bar{f}(a_1, \ldots, a_n) \equiv f(a_1, \ldots, a_n) \bmod \mathfrak{m}_{t-1}$, then $\bar{f} \in Z^n(A, \mathfrak{m}/\mathfrak{m}_{t-1})$. Since the length of conposition series is equal to $t-1$, $\bar{f} \in B^n(A, \mathfrak{m}/\mathfrak{m}_{t-1})$. Hence, there exists an element g_1 of $C^{n-1}(A, \mathfrak{m})$ such that $\bar{f}(a_1, \ldots, a_n) \equiv \delta g_1(a_1, \ldots, a_n) \bmod m_{t-1}$. Since $f - \delta g_1 \in Z^n(A, \mathfrak{m}_{t-1})$ and $N\mathfrak{m}_{t-1} = \mathfrak{m}_{t-1}N = 0$, there exists a $g_2 \in C^{n-1}(A, \mathfrak{m}_{t-1})$ such that $f - \delta g_1 = \delta g_2$. This shows that $f \in B^n(A, \mathfrak{m})$, and hence $H^n(A, \mathfrak{m}) = 0$.

By an argument similar to those in the above proof, we have

LEMMA 2.5. *An A-right module $\mathfrak{m}$ is an (M_0)-module if (and only if), for any A-right module $\mathfrak{n}$ such that $\mathfrak{n}N = 0$, all extensions of $\mathfrak{n}$ by $\mathfrak{m}$ split.*

Now, from Theorem 1.1, Lemmas 2.1, 2.3, 2.4, and 2.5, we have immediately the following theorem.

THEOREM 2.1. *(Under the assumption that A/N is separable[1]) all n-dimensional cohomology groups of A are zero if and only if Q_{n-1} is an (M_0)-module as an A-right module.*

3. In this section, we shall consider the cases of dimension 2 and 3.

It was shown in [1] that the class of algebras whose 2-dimensional cohomology groups are all zero coinsides with the class of absolutely segregated algebras.

Since Q_1 is isomorphic to N as an A-right module, we have immediately the following theorem, which is a special case of Ikeda's theorem.

THEOREM 3.1. *Let A be an algebra such that A/N is seperable. Then A is absolutely segregated if and only if N is an (M_0)-module as an A-right module.*

In order to prove the seperability of A/N for an absolutely segregated algebra A, we mention the following lemma.

LEMMA 3.1. *If an algebra A over an algebraically closed field F is absolutely segregated then the rank of $e_\kappa A e_\kappa$ over F, denoted by $[e_\kappa A e_\kappa]$, is equal to 1.*

Proof. Since F is algebraically closed, A/N is seperable. From theorem

[1] Cf. a note at the end.

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3.1, N is an (M_0)-module as an A-right module.

Let $t_{\kappa\lambda}$ be the number of factors isomorphic to $e_\lambda A$ in a direct decomposition of $e_\kappa N$ into directly indecomposable submodules: $e_\kappa N \cong \sum_\lambda t_{\kappa\lambda} e_\lambda A$. We assume that the indices are so arranged as $[e_1 A] \leqq [e_2 A] \leqq \ldots \leqq [e_k A]$. Then $\kappa < \lambda$ implies $t_{\kappa\lambda} = 0$. Set $c_{\kappa\lambda} = [e_\kappa A e_\lambda]$, $C = (c_{\kappa\lambda})$, and $T = (t_{\kappa\lambda})$. From $e_\kappa N e_\lambda \cong \sum_\mu t_{\kappa\mu} e_\mu A e_\lambda$, we have

$$C(E-T) = E \qquad (E\colon \text{ unit matrix}).$$

Since the matrix $E-T$ is

$$\begin{pmatrix} 1 & \cdot & \cdot & -t_{\kappa\lambda} \\ \cdot & & \cdot & \\ \cdot & & & \cdot \\ 0 & \cdot & \cdot & 1 \end{pmatrix},$$

its inverse matrix C is of from

$$\begin{pmatrix} 1 & \cdot & \cdot & c_{\kappa\lambda} \\ \cdot & & \cdot & \\ \cdot & & & \cdot \\ 0 & \cdot & \cdot & 1 \end{pmatrix}.$$

This shows that $c_{\kappa\kappa} = [e_\kappa A e_\kappa] = 1$.

As was shown in the proof of "only if" part of Theorem in §5 of [5], it is concluded rather easily from lemma 3.1 that A/N is separable if A is an absolutely segregated algebra. Combining this fact with Theorem 3.1 we have immediately

THEOREM 3.2. (Ikeda's Theorem). *An algebra with unit element is absolutely segregated if and only if*

i) *A/N is separable,*

ii) *N is an (M_0)-module as A-right module.*

Next, supposing that A/N is separable, we consider the case of dimension 3. Let $N \otimes A$ be a direct product of underlying vector spaces of N and A, and define the operation for $m \otimes b \in A$, as usual, by setting

$$(m \otimes b)a = m \otimes ba.$$

Then $N \otimes A$ is an A-right module. The mapping $m \otimes b \rightarrow mb$ induces an A-(right) operator homomorphism from $N \otimes A$ on N. We denote its kernel by N_0. Then we have

LEMMA 3.1. $Q_2 * 1 \cong N_0$ *(as A-right modules).*

Proof. Since $(m \times a) * 1 = m \times a - ma \times 1$, $m \times a$ is contained in $Q_2 * 1$ if and only if $ma = 0$. If $m \times b \in Q_2 * 1$, then $(m \times b) * a = m \times ba - mb \times a = m \times ba$. Hence

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the mapping $m \otimes b \to m \times b$ induces an isomorphisms from N_0 onto $Q_2 * 1$.

From this lemma and theorem 2.1, we have immediately

THEOREM 3.3. *Let A/N be separable. Then 3-dimensional cohomology groups of A are all zero if and only if N_0 is an (M_0)-module as an A-right module.*

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Added in proof: Recently T. Nakayama and M. Ikeda have proved jointly that if n-dimensional cohomology groups of A are all zero then A/N is separable. Using this theorem, Theorem 2.1 and 3.3 are improved as follows:

THEOREM 2.1′: *Let A be an algebra with unit element. Then n-dimensional cohomology groups of A are all zero if and only if*

i) *A/N is separable,*

ii) *Q_{n-1} is an (M_0)-module as an A-right module.*

THEOREM 3.3′: *Let A be an algebra with unit element. Then 3-dimensional cohomology groups are all zero if and only if*

i) *A/N is separable,*

ii) *N_0 is an (M_0)-module as an A-right module.*

As is easily seen, Theorem 2.1′ is an actual generalization of Ikeda's theorem.

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