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Embeddability of Some Three-Dimensional Weakly Pseudoconvex CR Structures

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Abstract. We prove that a class of perturbations of standard CR structure on the boundary of three-dimensional complex ellipsoid $E_{p,q}$ can be realized as hypersurfaces on $\mathbb{C}^2$, which generalizes the result of Burns and Epstein on the embeddability of some perturbations of standard CR structure on S^3.

1 Statement of Results

The examples given by Nirenberg (*cf.* [JT]) show that not all three-dimensional strongly pseudoconvex CR manifolds can be realized as hypersurfaces in $\mathbb{C}^2$. So, it is an interesting and fundamental problem to decide what three-dimensional CR manifolds can be realized as hypersurfaces in $\mathbb{C}^2$ or submanifolds in $\mathbb{C}^N$. This is an active topic in recent years [BD] [BE] [C2] [E1] [E2] [JT] [K] [L]. When the CR structure is strongly pseudoconvex, the problem is only interesting in three-dimensional case since a theorem of Boutet de Monvel states that any compact $(2n+1)$ dimensional CR manifold can be realized as a submanifold in $\mathbb{C}^N$ for some N, provided $n > 1$.

Burns and Epstein considered perturbations of the standard CR structure on a three-dimensional sphere [BE] (see also [BD]). They proved that sufficiently small perturbations of standard CR structure with "positive" Fourier coefficients are embeddable and the "generic" perturbations are nonembeddable. Such results are generalized to three-dimensional circle bundles in [E1] [L]. Epstein also obtained a deep relative index theorem on the space of embeddable CR structures [E2]. Compared with strongly pseudoconvex CR structure, the embeddability of weakly pseudoconvex CR structure is not well studied (*cf.* [C2] [K2]). In this paper, we prove that perturbations with "positive" Fourier coefficients of standard CR structure on the boundary of complex ellipsoid $E_{p,q}$ can be realized as hypersurfaces in $\mathbb{C}^2$, which generalizes Burns and Epstein's result on S^3. Further results about embeddability of pseudoconvex CR structures of finite type are in progress.

Let M be a real hypersurface in $\mathbb{C}^2$, and TM and $T\mathbb{C}^2$ be tangent spaces to M and $\mathbb{C}^2$, respectively. The complexified tangent space $\mathbb{C} \otimes T\mathbb{C}^2$ has the decomposition $T^{1,0}\mathbb{C}^2 \oplus T^{0,1}\mathbb{C}^2$ into holomorphic and antiholomorphic vectors. Using coordinates z, w on $\mathbb{C}^2$, we have that $T^{1,0}\mathbb{C}^2$ and $T^{0,1}\mathbb{C}^2$ are spanned by $\{\frac{\partial}{\partial z}, \frac{\partial}{\partial w}\}$ and $\{\frac{\partial}{\partial \bar{z}}, \frac{\partial}{\partial \bar{w}}\}$, respectively. Set

$$T^{1,0}(M) = \mathbb{C} \otimes TM \cap T^{1,0}\mathbb{C}^2|_M. \tag{1.1}$$

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Then $T^{0,1}(M) = \overline{T^{1,0}(M)}$ and $T^{0,1}(M) \cap T^{1,0}(M) = \{0\}$. $T^{1,0}(M)$ is always a one dimensional complex subbundle of $\mathbb{C} \otimes TM$, and is called *the complex tangential space of* M.

Now let M be an abstract real three-dimensional manifold and V be a one dimensional complex subbundle of $\mathbb{C} \otimes TM$. If $V \cap \bar{V} = \{0\}$, (M, V) is called a CR *manifold* and V is its complex tangential space, which is also denoted by $T^{1,0}(M)$. The CR structure is called *pseudoconvex* if we can choose a vector field T transverse to $T^{1,0}(M) \oplus T^{0,1}(M)$ such that

$$[Z, \bar{Z}] = -i\lambda T, \quad \mod Z, \bar{Z}. \tag{1.2}$$

where $\bar{Z}$ is a local section of $T^{0,1}(M)$, $\lambda \geq 0$ is the *Levi form* of M.

Let $\Omega^{0,1}(M)$ be the dual of $T^{0,1}(M)$. Now, we define the $\bar{\partial}_b$ *operator* on CR manifold M by

$$\bar{\partial}_b f = \bar{Z} f \theta^{\bar{1}}, \tag{1.3}$$

where $\theta^{\bar{1}} \in \Omega^{0,1}(M)$ is normalized by $\theta^{\bar{1}}(\bar{Z}) = 1$, $\theta^{\bar{1}}(Z) = 0$ and $\theta^{\bar{1}}(T) = 0$.

A distribution f is called *a* CR *function* if $\bar{\partial}_b f = 0$. A CR manifold M is called *embeddable* in $\mathbb{C}^N$ for some positive integer N if there exist N smooth CR functions $\phi_1, \dots, \phi_N$ on M such that mapping $\Phi = (\phi_1, \dots, \phi_N) \colon M \to \mathbb{C}^N$ is an embedding. Then $\Phi_* Z$ is a complex tangential vector of submanifold $\Phi(M)$ with CR structure induced by standard complex structure of $\mathbb{C}^N$.

We assume that there is a Riemannian metric $\langle \cdot, \cdot \rangle$ on M. This, in turn, defines a Hermitian L^2 structure on sections of $\mathbb{C} \otimes T^*(M)$, $\Omega^{1,0}(M)$ and $\Omega^{0,1}(M)$. We require the Riemannian metric being *compatible with the* CR *structure*, *i.e.*, $T^{1,0}(M)$ and $T^{0,1}(M)$ are orthogonal under the Hermitian metric. Define a Hermitian inner product on the global sections of $\Omega^{1,0}(M)$,

$$(\phi, \psi) = \int_M \langle \phi, \psi \rangle \, dV, \tag{1.4}$$

where dV denotes the volume element. We denote by $L^2(M, \Omega^{0,1})$ the completion of the space $C^\infty(M, \Omega^{0,1})$ under the L^2 norm. We denote the L^2 closure of $\bar{\partial}_b$ also by $\bar{\partial}_b$. Recall the definition of L^2 closure of $\bar{\partial}_b$. First define $\mathrm{Dom}(\bar{\partial}_b) \subset L^2(M)$ to be the space of all $\phi \in L^2(M)$ such that there exists a sequence of $C^\infty(M)$ functions $\{\phi_\nu\}$, with $\phi = \lim \phi_\nu$ in $L^2(M)$ and $\{\bar{\partial}_b \phi_\nu\}$ is a Cauchy sequence in $L^2(M, \Omega^{0,1})$. We denote the L^2 adjoint of $\bar{\partial}_b$ by $\bar{\partial}_b^*$. Finally define the *Kohn Laplacian* $\Box_b$ by

$$\mathrm{Dom}(\Box_b) = \{\phi \in \mathrm{Dom}(\bar{\partial}_b) \,;\, \bar{\partial}_b \phi \in \mathrm{Dom}(\bar{\partial}_b^*)\}. \tag{1.5}$$

Then, for $\phi \in \mathrm{Dom}(\Box_b)$, we define

$$\Box_b \phi = \bar{\partial}_b^* \bar{\partial}_b \phi. \tag{1.6}$$

The embeddability of a CR structure connects with an analytic property of the associated $\bar{\partial}_b$ operator by the following theorem due to Boutet de Monvel and Kohn [E, p. 5] [K2].

Theorem A *A strongly pseudoconvex* CR *structure on a compact manifold is embeddable if and only if the range of the associated* $\bar{\partial}_b$ *operator is closed.*

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Such characterization for the embeddability of three-dimensional pseudoconvex CR structures of finite type also holds. The CR structure is called *of type m at point* $P \in M$ if

$$X_1 \cdots X_{m-2}\lambda(P) \neq 0 \tag{1.7}$$

for some $X_j \in \{Z, \bar{Z}\}$, $j = 1, \dots, m-2$, where λ is defined by (1.2) and

$$X_1 \cdots X_l \lambda(P) = 0 \tag{1.8}$$

for all $X_j \in \{Z, \bar{Z}\}$, $j = 1, \dots, l$ and $l < m-2$. The CR structure is called *of finite type* if the type of each point of M is less than a fixed positive integer.

Theorem B **([C], [K2])** *A pseudoconvex* CR *structure of finite type on a three-dimensional compact manifold is embeddable if and only if the range of the associated* $\bar{\partial}_b$ *operator is closed.*

It is easy to see that small revision of arguments in [JT] gives the proof of existence of nonembeddable CR structure of finite type. We omit the details. The main problem is to decide what three-dimensional pseudoconvex CR manifolds of finite type can be realized as real submanifolds in $\mathbb{C}^N$.

In the last two decades, the regularity theory of $\bar{\partial}_b$ operator on three-dimensional pseudoconvex CR manifolds of finite type has been established [C1] [FK] [S], which makes it possible to investigate the problem of embeddability of such CR manifolds.

Let Z be a global nowhere-vanishing section of $T^{1,0}(M)$, $\dim M = 3$. *A perturbation of* CR *structure on M* is defined as

$$Z^{\psi} = Z + \psi \bar{Z} \tag{1.9}$$

for some smooth function ψ on M. Then the complex tangential space $T^{1,0}_{\psi}(M)$ associated with this CR structure is spanned by Z^{ψ}, and

$$T^{1,0}_{\psi}(M) \oplus \overline{T^{1,0}_{\psi}(M)} = T^{1,0}(M) \oplus \overline{T^{1,0}(M)}.$$

Note that

$$[Z^{\psi}, \bar{Z}^{\psi}] = -i\lambda(1 - |\psi|^2)T, \quad \mod Z, \bar{Z}. \tag{1.10}$$

Z^{ψ} is still pseudoconvex with Levi form $\lambda^{\psi} = (1 - |\psi|^2)\lambda \geq 0$ if $|\psi| < 1$. $\lambda^{\psi}(P) = 0$ if and only if $\lambda(P) = 0$ for each $P \in M$.

The simplest pseudoconvex domain of finite type is the *complex ellipsoid* in $\mathbb{C}^2$,

$$E_{p,q} = \{(z, w)\ ;\ |z|^{2p} + |w|^{2q} \leq 1\}. \tag{1.11}$$

where p, q are positive integers. Let (p, q) be the greatest common divisor of p and q and $p' = \frac{p}{(p,q)}$, $q' = \frac{q}{(p,q)}$.

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The following S^1 action

$$U_\phi\colon (z,w) \to (e^{iq'\phi}z, e^{ip'\phi}w), \quad \phi \in [0,2\pi) \tag{1.12}$$

is free, since $e^{iq'\phi} = e^{ip'\phi} = 1$ if and only if $\phi = 0$ by p' and q' being relatively prime. It is obvious that U_ϕ is a free S^1 action on the boundary $bE_{p,q}$ of complex ellipsoid. Note that for a function $f \in C_0^\infty(\mathbb{C}^2)$,

$$(U_\phi^*) f(z,w,\bar z,\bar w) = f(e^{iq'\phi}z, e^{ip'\phi}w, e^{-iq'\phi}\bar z, e^{-ip'\phi}\bar w), \tag{1.13}$$

the real vector field corresponding to U_ϕ is

$$T = i\Big(q'z\frac{\partial}{\partial z} + p'w\frac{\partial}{\partial w} - q'\bar z\frac{\partial}{\partial \bar z} - p'\bar w\frac{\partial}{\partial \bar w}\Big). \tag{1.14}$$

We decompose $L^2(bE_{p,q})$ according to the action of U_ϕ. For integer m, let

$$F_m = \{f \in L^2(bE_{p,q})\ ;\ U_\phi^* f = e^{im\phi} f\} \tag{1.15}$$

and

$$\mathcal{F}_m = \bigoplus_{k\geq m} F_k. \tag{1.16}$$

Note that $z \in F_{q'}$, $w \in F_{p'}$, $\bar z \in F_{-q'}$, $\bar w \in F_{-p'}$. Each polynomial lies in $\mathcal{F}_m$ for some integer m. Hence, $L^2(M) = \bigoplus_{m=-\infty}^{\infty} F_m$. Our main theorem is:

Theorem 1.1 *If ψ has sufficiently small C^4 norm, the* CR *structure Z^ψ on $bE_{p,q}$ with $\psi \in \mathcal{F}_{2(p'+q')}$ can be realized as a compact hypersurface in $\mathbb{C}^2$ which is a deformation of $bE_{p,q}$.*

See [BE] and [E1] for the theorem in the case of $p = q = 1$.

Note z and w are CR functions on $E_{p,q}$. We only need to solve equations

$$\bar Z^\psi(z+\xi) = 0 \text{ and } \bar Z^\psi(w+\xi') = 0. \tag{1.17}$$

Then, $z+\xi$ and $w+\xi'$ are CR functions of CR structure on $bE_{p,q}$ determined by Z^ψ, and they obviously define a diffeomorphism from $bE_{p,q}$ to a hypersurface in $\mathbb{C}^2$ if the C^1 norms of ξ and ξ' are sufficiently small.

To solve (1.17), we should solve equations of type $\bar Z u = v$. Note $\bar Z$ is not locally solvable, *i.e.*, there exists smooth function v such that the above equation does not have a solution. So we need information of the range of $\bar Z$. By direct calculation, we find that the structure equations of the standard CR structure on $bE_{p,q}$ is quite simple. By using these equations, we find sufficient information about the range of $\bar\partial_b$ and the kernel of $\bar\partial_b^*$. This is done in Section 2. The main theorem is proved in Section 3.

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2 Properties of $\bar\partial_b$ on $bE_{p,q}$

The purpose of this section is to describe the range of $\bar\partial_b$ and the kernel of $\bar\partial_b^*$ on $bE_{p,q}$. It can be easily checked that

$$Z = qw^{q-1}\bar{w}^q \frac{\partial}{\partial z} - pz^{p-1}\bar{z}^p \frac{\partial}{\partial w} \tag{2.1}$$

is a complex tangential vector on the boundary $bE_{p,q}$ of the complex ellipsoid. By simple calculation,

$$[Z, \bar{Z}] = -i\lambda T \tag{2.2}$$

with T defined as in (1.14) and

$$\lambda = pq(p,q)|z|^{2p-2}|w|^{2q-2}. \tag{2.3}$$

It is easy to see that T is transverse to the span $\{Z, \bar{Z}\}$. The complex ellipsoid is weakly pseudoconvex, since the degenerate locus $\lambda = 0$ is the union of two circles $\{(z,0)\ ;\ |z| = 1\}$ and $\{(0,w) : |w| = 1\}$. Points not lying on these two circles are strongly pseudoconvex. It is easy to see that the type of points in $\{(z,0)\ ;\ |z| = 1\}$ is $2q$ and the type of points in $\{(0,w) : |w| = 1\}$ is $2p$. Therefore, the boundary $bE_{p,q}$ of complex ellipsoid is of type $\max\{2p, 2q\}$.

Let's calculate $U_{\phi*}Z$. For $P = (z,w)$ and $Q = (z',w') = (U_\phi z, U_\phi w) = (e^{iq'\phi}z, e^{ip'\phi}w)$,

$$\begin{aligned}(2.4)\quad (U_{\phi*}Z)(Q)f(Q,\bar{Q}) &= Z(P)f(U_\phi P, \overline{U_\phi P}) \\ &= \Big(qw^{q-1}\bar{w}^q \frac{\partial}{\partial z} - pz^{p-1}\bar{z}^p \frac{\partial}{\partial w}\Big) f(e^{iq'\phi}z, e^{ip'\phi}w, e^{-iq'\phi}\bar{z}, e^{-ip'\phi}\bar{w}) \\ &= \Big(\Big(qe^{iq'\phi}w^{q-1}\bar{w}^q \frac{\partial}{\partial z} - pe^{ip'\phi}z^{p-1}\bar{z}^p \frac{\partial}{\partial w}\Big) f\Big) \\ &\qquad (e^{iq'\phi}z, e^{ip'\phi}w, e^{-iq'\phi}\bar{z}, e^{-ip'\phi}\bar{w}) \\ &= e^{i(p'+q')\phi}\Big(\Big(qw'^{q-1}\overline{w'}^q \frac{\partial}{\partial z'} - pz'^{p-1}\overline{z'}^p \frac{\partial}{\partial w'}\Big) f\Big)(z', w', \bar{z}', \bar{w}').\end{aligned}$$

Hence,

$$U_{\phi*}Z = e^{i(p'+q')\phi}Z. \tag{2.5}$$

It is obvious that

$$U_{\phi*}T = T. \tag{2.6}$$

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It follows from (2.5) that

$$[T, Z] = -i(p' + q')Z, \tag{2.7}$$

by a theorem about Lie derivatives [BC, pp. 16–17]. (2.7) can also be checked by direct calculation of the Lie bracket.

Let θ, θ^1 and $\theta^{\bar{1}}$ be the dual of T, Z and $\bar{Z}$, which are globally nowhere vanishing 1-forms. The operator $\bar{\partial}_b$ is defined as (1.5). By the definition of exterior differentiation, for a C^∞ k-form ω and $(k+1)$-tuples of C^∞ vectors $V_1, \dots, V_{k+1}$,

$$\begin{aligned} &d\omega(V_1, \dots, V_{k+1}) \\ &\quad = \sum_j (-1)^{j-1} V_j \omega(V_1, \dots, V_{j-1}, V_{j+1}, \dots, V_{k+1}) \\ &\qquad + \sum_{j<j'} (-1)^{j+j'} \omega([V_j, V_{j'}], \dots, V_{j-1}, V_{j+1}, \dots, V_{j'-1}, V_{j'+1}, \dots, V_{k+1}), \end{aligned} \tag{2.8}$$

the duals of (2.2) and (2.7) are

$$\begin{cases} d\theta = i\lambda \theta^1 \wedge \theta^{\bar{1}}, \\ d\theta^1 = i(p' + q')\theta \wedge \theta^1, \\ d\theta^{\bar{1}} = -i(p' + q')\theta \wedge \theta^{\bar{1}}, \end{cases} \tag{2.9}$$

which are *the structure equations of* CR *manifold* $bE_{p,q}$. The dual of (2.5) is

$$U_\phi^* \theta^1 = e^{i(p'+q')\phi} \theta^1, \tag{2.10}$$

and obviously

$$U_\phi^* \theta = \theta. \tag{2.11}$$

Since T, Z and $\bar{Z}$ are globally nowhere vanishing sections of the complexified tangent space $T(M) \otimes \mathbb{C}$, we can define an inner product on $T(M) \otimes \mathbb{C}$ by requiring T, Z and $\bar{Z}$ to be an orthonomal basis. So, an inner product is defined on $\Omega^{0,1}(M)$. Define the volume element

$$dV = i\theta \wedge \theta^1 \wedge \theta^{\bar{1}}. \tag{2.12}$$

Proposition 2.1

(1) *$F_m \perp F_{m'}$ for $m \neq m'$;*
(2) *For each integer m,*

$$Z \colon F_m \cap \operatorname{Dom} Z \to F_{m-p'-q'}, \quad \bar{Z} \colon F_m \cap \operatorname{Dom} \bar{Z} \to F_{m+p'+q'} \tag{2.13}$$

and

$$\Box_b \colon F_m \cap \operatorname{Dom} \Box_b \to F_m, \tag{2.14}$$

where the Kohn Laplacian $\Box_b = \bar{\partial}_b^* \bar{\partial}_b = -Z\bar{Z}$.

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Proof (1) Since $U_\phi^*(\theta\wedge\theta^1\wedge\bar\theta^1)=\theta\wedge\theta^1\wedge\bar\theta^1$ by (2.10–2.11), the volume element dV is invariant under the action U_ϕ. For $f\in F_k$, $g\in F_{k'}$, $k\neq k'$,

$$\int_M f\bar g\,dV=\int_M U_\phi^*(f\bar g)U_\phi^*\,dV=e^{i(k-k')\phi}\int_M f\bar g\,dV. \tag{2.15}$$

where each number $\phi\in[0,2\pi)$. It follows that $\int_M f\bar g\,dV=0$.

(2) It follows from (2.10) that $U_\phi^*\theta^{\bar 1}=e^{-i(p'+q')\phi}\theta^{\bar 1}$. Noting that exterior forms are invariant under coordinate transformations, for $u\in F_m$,

$$\begin{aligned} e^{im\phi}\bar Zu\cdot\theta^{\bar 1}&=e^{im\phi}\bar\partial_b u=\bar\partial_b U_\phi^* u=U_\phi^*(\bar\partial_b u)=U_\phi^*(\bar Zu\cdot\theta^{\bar 1})\\ &=U_\phi^*(\bar Zu)U_\phi^*\theta^{\bar 1}=e^{-i(p'+q')\phi}U_\phi^*(\bar Zu)\theta^{\bar 1}, \end{aligned}\tag{2.16}$$

we find

$$U_\phi^*(\bar Zu)=e^{i(m+p'+q')\phi}\bar Zu. \tag{2.17}$$

This completes the proof of Proposition 2.1.

For example, $w\in F_{p'}$ and $Zw=-pz^{p-1}\bar z^p\in F_{-q'}$, which satisfies (2.13).

By the structure equations (2.9), the Lie derivative of dV is

$$\begin{aligned} L_{\bar Z}dV&=\sqrt{-1}\big(di(\bar Z)+i(\bar Z)d\big)(\theta\wedge\theta^1\wedge\theta^{\bar 1})\\ &=-\sqrt{-1}d(\theta\wedge\theta^1)=0, \end{aligned}\tag{2.18}$$

by the formula of Lie derivative $L=di+id$, where

$$\big(i(X)\omega\big)(X_1,\dots,X_k)=\omega(X,X_1,\dots,X_k)$$

for any $(k+1)$-form ω and vector fields $X,X_1,\dots,X_k$. This property of volume element is very important in finding $\ker\bar\partial_b^*$. The inner product on $\Omega^{0,1}(M)$ can be defined as follows. For $\omega,\omega'\in\Omega^{0,1}$, $\omega=f\theta^{\bar 1}$ and $\omega'=f'\bar\theta^1$ for some functions f,f' on M, define

$$\langle\omega,\omega'\rangle=\int_M f\bar f'\,dV. \tag{2.19}$$

Proposition 2.2 *The formal adjoint $\bar\partial_b^*$ of $\bar\partial_b$ is*

$$\bar\partial_b^*(f\theta^{\bar 1})=-Zf. \tag{2.20}$$

Proof For $u\in C^\infty(M)$, $\eta=v\theta^{\bar 1}\in\Omega^{0,1}(M)$,

$$\langle\bar\partial_b u,\eta\rangle=\int_M\bar Zu\bar v\,dV=-\int_M u\overline{Zv}\,dV-\int_M u\bar vL_{\bar Z}\,dV, \tag{2.21}$$

which implies (2.20) by (2.18). The second equality follows from $\int_M L_{\bar Z}(u\bar v\,dV)=\int_M\big(di(\bar Z)+i(\bar Z)\,d\big)(u\bar v\,dV)=\int_M d\big(i(\bar Z)(u\bar v\,dV)\big)=0$ by Stokes' formula and M having no boundary.

It follows that the formal adjoints of Z and $\bar Z$ are $-\bar Z$ and $-Z$, respectively.

Proposition 2.3 *The kernel of $\bar\partial_b^*$ is orthogonal to $\mathcal F_1$, i.e., the range of $\bar\partial_b$ contains $\{f\theta^{\bar 1}\,;\,f\in\mathcal F_1\}$.*

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Proof Let $\eta = v\theta^{\bar{1}} \in \ker \bar{\partial}_b^*$. There exists a sequence of

$$\eta^\nu = v^\nu \theta^{\bar{1}} \in C^\infty\big(M, T^{0,1}(M)\big)$$

such that $\eta^\nu \to \eta$ in $L^2\big(M, T^{0,1}(M)\big)$ and $\bar{\partial}_b^* \eta^\nu \to 0$ in $L^2(M)$. Let

$$\eta = \sum_m \eta_m = \sum_m v_m \theta^{\bar{1}} \quad \text{and} \quad \eta^\nu = \sum_m \eta_m^\nu = \sum_m v_m^\nu \theta^{\bar{1}}, \tag{2.22}$$

where $v_m \in F_m$, $v_m^\nu \in F_m \cap C^\infty(M)$ for each m, ν. Then, $v_m^\nu \to v_m$ and $\bar{\partial}_b^* \eta_m^\nu = -Z v_m^\nu \to 0$ in $L^2(M)$ by F_m being mutually orthogonal by Proposition 2.1 and the fact that (2.20) holds for $f \in C^\infty(M)$. Now, for each m,

$$\begin{aligned}
\|\bar{\partial}_b^* \eta_m^\nu\|^2 &= \langle \bar{\partial}_b \bar{\partial}_b^* \eta_m^\nu, \eta_m^\nu \rangle = -\int_M \bar{Z} Z v_m^\nu \cdot \bar{v}_m^\nu \, dV \\
&= -\frac{1}{2} \int_M (\bar{Z} Z + Z \bar{Z}) v_m^\nu \cdot \bar{v}_m^\nu \, dV + \frac{1}{2} \int_M [Z, \bar{Z}] v_m^\nu \cdot \bar{v}_m^\nu \, dV \\
&= \frac{1}{2} \int_M (|Z v_m^\nu|^2 + |\bar{Z} v_m^\nu|^2) \, dV - \frac{i}{2} \int_M \lambda T v_m^\nu \cdot \bar{v}_m^\nu \, dV
\end{aligned} \tag{2.23}$$

where λ is defined by (2.3). We have used Stokes' formula (2.21) to get the last equality. By the definition of F_m (1.15), $T v_m^\nu = i m v_m^\nu$. When $m > 0$, the last integral in (2.23) is $\frac{m}{2} \int_M \lambda v_m^\nu \cdot \bar{v}_m^\nu \, dV \geq 0$. So,

$$\frac{m}{2} \int_M \lambda |v_m^\nu|^2 \, dV \leq \|\bar{\partial}_b^* \eta_m^\nu\|^2 \to 0 \tag{2.24}$$

as $\nu \to \infty$. It follows from Fatou's Lemma that

$$\int_M \lambda |v_m|^2 \, dV \leq \lim_{\nu \to \infty} \int_M \lambda |v_m^\nu|^2 \, dV = 0.$$

Since λ vanishes only on two circles, $v_m = 0$ for all positive integers. The proposition is proved. ■

Proposition 2.3 for three-dimensional circle bundles (including S^3) was established in [E1] by using Kodaira's vanishing theorem for positive line bundles on a Riemann surface. Here we use direct calculation, which is a version of Bochner's technique to prove the vanishing theorem.

3 Proof of the Theorem

To prove Theorem 1.1, we only need to solve equation

$$\bar{Z}^\psi (h + \xi) = 0, \tag{3.1}$$

for $h = z$ or w, *i.e.*,

$$\bar{Z}\xi = -\psi Z(h+\xi) \tag{3.2}$$

To solve (3.2), we use iteration,

$$\xi_0 = 0, \quad \text{and} \quad \bar{Z}\xi_n = -\psi Z(h+\xi_{n-1}). \tag{3.3}$$

Proposition 3.1 *Suppose* $h \in \ker \bar{\partial}_b \cap \mathcal{F}_1 \cap C^\infty$, $\psi \in \mathcal{F}_{2(p'+q')}$ *and* $\psi \in C^k$ *for some positive integer* $k \geq 4$. *Then, (3.3) have solutions* $\xi_n \in \mathcal{F}_1$ *for all* $n = 1, 2, \ldots$.

Before the proof of this proposition, we summarize the regularity property of the $\bar{\partial}_b$ operator given by Smith in the following theorem (see Theorems 1.2, 1.3, 4.17, and 4.18 in [S]). Let $\Psi^m_\rho(M)$ be a class of operators defined in [S, p. 139], where $m \in \mathbb{R}^1$, ρ is a symbol.

Theorem 3.2 **([S])**

(1) *If* $T_1 \in \Psi^{m_1}_\rho(M)$ *and* $T_2 \in \Psi^{m_2}_\rho(M)$, *then the composition* $T_1 \circ T_2 \in \Psi^{m_1+m_2}_\rho(M)$.
(2) *If* $T_1 \in \Psi^{m_1}_\rho(M)$, *then* $T_1^* \in \Psi^{m_1}_\rho(M)$.
(3) *If* $T \in \Psi^m_\rho(M)$, $m \leq 0$, *then* T *is a bounded operator from* $L^2_s(M)$ *to* $L^2_s(M)$, *where* $L^2_s(M)$ *is the standard Sobolev space of degree* s *on* M.
(4) *Suppose* M *to be a three-dimensional* CR *manifold of finite type and the range of* $\bar{\partial}_b$ *to be closed. There exists a mapping* $P\colon L^2(M) \to L^2(M)$, *such that*

$$\begin{aligned} \bar{Z}P &= I - S_2 \\ P\bar{Z} &= I - S_1 \end{aligned} \tag{3.4}$$

where Z *is a globally nowhere-vanishing complex tangential vector,* $S_1, S_2 \in \Psi^0_\rho(M)$, $P \in \Psi^{-1}_\rho(M)$, S_1 *and* S_2 *are the Szegö projections* $S_1\colon L^2(M) \to \ker \bar{Z} \cap L^2(M)$, $S_2\colon L^2(M) \to \ker Z \cap L^2(M)$.
(5) $G = \Box_b^{-1} = P \circ P^* \in \Psi^{-2}_\rho(M)$ *and* $Z, \bar{Z} \in \Psi^1_\rho(M)$.

If $\Omega^{0,1}(M)$ has a globally nowhere-vanishing section $\theta^{\bar{1}}$, then we can identify $L^2(M, \Omega^{0,1})$ with $L^2(M)$ by mapping $f\theta^{\bar{1}} \to f$. Under this identification, the L^2 closure of $\bar{\partial}_b$ and $\bar{Z}$ are the same on $L^2(M)$, and the L^2 closure of $\bar{\partial}_b^*$ and $-Z$ are the same on $L^2(M)$ by Proposition 2.2.

Proof of Proposition 3.1 Suppose $\xi_{n-1} \in \mathcal{F}_1 \cap \mathrm{Dom}(Z)$. Since $h \in \mathcal{F}_1$ and $\psi \in \mathcal{F}_{2(p'+q')}$, it follows from Proposition 2.1 that

$$\psi Z(h+\xi_{n-1}) \in \mathcal{F}_{p'+q'+1}. \tag{3.5}$$

Hence, $\psi Z(h+\xi_{n-1}) \perp$ kernel Z and $\psi Z(h+\xi_{n-1}) \in$ the range of $\bar{Z}$ by Proposition 2.3. Since the complex ellipsoid $E_{p,q}$ is embedded in $\mathbb{C}^2$, the range of $\bar{\partial}_b$ is closed by Theorem B. We can apply Theorem 3.2. Note

$$(I - S_2)\big(\psi Z(h+\xi_{n-1})\big) = \psi Z(h+\xi_{n-1}) \tag{3.6}$$

by the definition of S_2. Now we can apply the first equation of (3.4) of Theorem 3.2 to find a solution ξ_n of (3.3) with

$$\xi_n = -P(\psi Z)(h + \xi_{n-1}). \tag{3.7}$$

We claim $\xi_n \perp \ker \bar{Z}$. It follows from the second equation of (3.4) that if $v \in$ the range of $\bar{Z}$, *i.e.*, $v = \bar{Z}v'$ for some v', then $Pv = P\bar{Z}v' = v' - S_1 v'$, *i.e.*, $Pv \perp \operatorname{kernel} \bar{Z}$. Therefore, $\xi_n \perp \ker \bar{Z}$ by $(\psi Z)(h + \xi_{n-1}) \in$ the range of $\bar{Z}$.

Let's check $\xi_n \in \mathcal{F}_1$. Let $\xi_n = \xi_n' + \xi_n''$ with $\xi_n' \in \mathcal{F}_1$ and $\xi_n'' \perp \mathcal{F}_1$. Then,

$$\bar{Z}\xi_n' + \bar{Z}\xi_n'' = \bar{Z}\xi_n = -\bar{Z}P(\psi Z)(h + \xi_{n-1}) = -\psi Z(h + \xi_{n-1}) \in \mathcal{F}_{p'+q'+1} \tag{3.8}$$

as above. Note $\bar{Z}\xi_n' \in \mathcal{F}_{p'+q'+1}$ and $\bar{Z}\xi_n'' \perp \mathcal{F}_{p'+q'+1}$ by Proposition 2.1. It follows that $\bar{Z}\xi_n'' = 0$, by F_m being mutually orthogonal. Then, $\xi_n'' = 0$ by $\xi_n \perp \ker \bar{Z}$. So, $\xi_n \in \mathcal{F}_1$.

ξ_n is smooth and hence in $\operatorname{Dom}(Z)$ by the arguments in the following proof of Proposition 3.3. We can iterate equations (3.3) now. The proposition is proved. ■

Since $z \in F_{q'}$, $w \in F_{p'}$, we can apply Proposition 3.1 to $h = z$ or w. Now what remains is to prove that the sequence ξ_n converges in appropriate topology for ψ small. Theorem 1.1 follows from the following proposition, *i.e.*, we find C^1 solutions of (1.17).

Proposition 3.3 *Suppose $h \in \ker \bar{\partial}_b \cap \mathcal{F}_1 \cap C^\infty$, $\psi \in \mathcal{F}_{2(p'+q')}$ and ψ has sufficiently small C^{k+1} norm for some positive integer $k \geq 3$. Then*

$$\bar{Z}^\psi(h + \xi) = 0 \tag{3.9}$$

has a unique C^α ($\alpha = k - \frac{3}{2}$) solution orthogonal to $\ker \bar{\partial}_b$ *with*

$$\xi \in \mathcal{F}_1 \quad \text{and } \|\xi\|_{C^\alpha} \leq C\|\psi\|_{C^{k+1}} \tag{3.10}$$

for some constant $C > 0$.

Proof Note $PZ \in \Psi^0_\rho(M)$ and $P \in \Psi^{-1}_\rho(M)$ are bounded on $L^2(M)$ by Theorem 3.2(1), (3), (5). As operators on $L^2(M)$,

$$\begin{aligned} \|P(\psi Z)\| &\leq \|PZ\psi\| + \|P[\psi, Z]\| \\ &\leq \|PZ\| \cdot \|\psi\|_{C^0(M)} + \|P\| \cdot \|[\psi, Z]\|_{C^0(M)} \leq C\|\psi\|_{C^1(M)} \end{aligned} \tag{3.11}$$

for some constant $C > 0$. Thus, if $\|\psi\|_{C^1(M)}$ is sufficiently small, $\|\xi_n - \xi_{n-1}\|_{L^2(M)} \leq \rho\|\xi_{n-1} - \xi_{n-2}\|_{L^2(M)}$ with constant $\rho < 1$ by (3.7) and (3.11). Hence, the sequence ξ_n obtained in Proposition 3.1 converges to a solution of (3.2) in $L^2(M)$.

Now fix a positive integer k. By the definition of Sobolev space $L^2_k(M)$,

$$\|\psi \cdot f\|_{L^2_k(M)} \leq C_1\|\psi\|_{C^k(M)} \cdot \|f\|_{L^2_k(M)} \tag{3.12}$$

for some constant $C_1 > 0$. Thus, as operators on $L^2_k(M)$,

$$\begin{aligned} \|P(\psi Z)\| &\leq \|PZ\psi\| + \|P[\psi, Z]\| \\ &\leq \|PZ\| \cdot \|\psi\|_{C^k(M)} + \|P\| \cdot \|[\psi, Z]\|_{C^k(M)} \leq C\|\psi\|_{C^{k+1}(M)} \end{aligned} \tag{3.13}$$

for some constant $C > 0$, by $PZ \in \Psi^0_\rho(M)$ and $P \in \Psi^{-1}_\rho(M)$ bounded on $L^2_k(M)$ by Theorem 3.2(1), (3), (5). Thus, if $\|\psi\|_{C^{k+1}(M)}$ sufficiently small, $\|\xi_n - \xi_{n-1}\|_{L^2_k(M)} \leq \rho\|\xi_{n-1} - \xi_{n-2}\|_{L^2_k(M)}$ with constant $\rho < 1$ by (3.7) and (3.13). Hence, the sequence ξ_n obtained in Proposition 3.1 converges to a solution of (3.2) in $L^2_k(M)$. Finally, we use the Sobolev imbedding $L^2_k(M) \hookrightarrow C^\alpha$ with $\alpha \leq k - \frac{3}{2}$. Proposition 3.3 is proved.

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