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Evolution and stability of Laplace-like resonances under tidal dissipation

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Abstract. The Laplace resonance is a configuration that involves the commensurability between the mean motions of three small bodies revolving around a massive central one. This resonance was first observed in the case of the three inner Galilean satellites, Io, Europa, and Ganymede. In this work the Laplace resonance is generalised by considering a system of three satellites orbiting a planet that are involved in mean motion resonances. These Laplace-like resonances are classified in three categories: first-order (2:1&2:1, 3:2&3:2, 2:1&3:2), second-order (3:1&3:1) and mixed-order resonances (2:1&3:1). In order to study the dynamics of the system we implement a model that includes the gravitational interaction with the central body, the mutual gravitational interactions of the satellites, the effects due to the oblateness of the central body and the secular interaction of a fourth satellite and a distant star. Along with these contributions we include the tidal interaction between the central body and the innermost satellite. We study the survival of the Laplace-like resonances and the evolution of the orbital elements of the satellites under the tidal effects. Moreover, we study the possibility of capture into resonance of the fourth satellite.

Keywords. Celestial mechanics; Planets and satellites; General methods; Numerical.

1. Introduction

The Laplace resonance is a notorious case of resonant configuration belonging to the class of three-body resonances, *i.e.*, resonances involving a commensurability between the mean motions of three small bodies revolving around a massive central body. A prominent example in our solar system is the configuration of the three Galilean satellites Io, Europa, and Ganymede, whose mean motions n_k satisfy the near-commensurability

$$n_I - 3n_E + 2n_G \approx 0\,. \tag{1.1}$$

The so-called Laplace angle is defined as

$$\phi_L = \lambda_1 - 3\lambda_2 + 2\lambda_3\,, \tag{1.2}$$

and it librates around π with a small amplitude.

In this work we generalize the relations that involve the mean longitudes of the three satellites, so that they describe a chain of resonances given by the form $j{:}k\&m{:}n$. We will consider first-order resonances, when $j-k=1$ and $m-n=1$, second-order resonances, when $j-k=2$ and $m-n=2$, and, finally, mixed-order resonances, when $j-k=1$ and

$m-n=2$ or $j-k=2$ and $m-n=1$. In particular, we will study the resonant chains: 2:1&3:2, 3:2&3:2, 2:1&3:1, 3:1&3:1.

In order to perform numerical integrations of the equations of motion we use the parameters that correspond to the Galilean satellites. However this study can be easily applied to many examples of resonant chains of systems of satellites in the solar system, as well as of extrasolar planetary systems. For example, a resonant chain is observed in the system of satellites of Pluto. Specifically, Styx, Nix, Kerberos, and Hydra are near 3:1, 4:1, 5:1 and 6:1 resonance with Charon, respectively. There are also many examples of extrasolar planetary systems. Kepler-31 is a system of three planets in a 2:1&2:1 resonance, YZ Ceti is in a 3:2&3:2 resonance and Kepler-305 in a 3:2&2:1 resonance (Pichierri *et al.* (2019)). Moreover, three of the four planets of V1298 Tauri are in a 2:1&3:2 resonant chain (David *et al.* (2019)).

Following Celletti *et al.* (2021), we study the dynamical evolution of a system of three satellites orbiting around a central planet and they are involved in a first-, second- or mixed-order resonant chain. We implement a model which consists of the gravitational interaction among the central planet and the satellites, the mutual gravitational interaction of the satellites, the effects due to the oblateness of the planet and the secular interaction of a distant star and a fourth satellite. Finally we include the tidal interaction between the planet and the innermost satellite.

2. Hamiltonian model

In this section we present the Hamiltonian model adopted, which consists of the following contributions: (a) the gravitational interaction of the satellites due to the central planet, (b) the mutual gravitational interactons of the satellites, (c) the effects due to the oblateness of the planet and (d) the secular gravitational interaction due to a distant star and a fourth satellite. In the following, m_0 is the mass of the central planet, and m_j is the mass of the j-th satellite. The orbital elements of the j-th satellite are: the semi-major axis a_j, the eccentricity e_j, the inclination I_j with respect to the equatorial reference frame, the mean longitude λ_j, the longitude of the pericentre ϖ_j, the longitude of the ascending node Ω_j and the auxiliary variable $s_j = \sin(I_j/2)$.

2.1. *Keplerian part*

The Keplerian part of the Hamiltonian can be expressed using the semi-major axes of the satellites as

$$\mathcal{H} = -\frac{\mathcal{G}M_1\mu_1}{2a_1} - \frac{\mathcal{G}M_2\mu_2}{2a_2} - \frac{\mathcal{G}M_3\mu_3}{2a_3} - \frac{\mathcal{G}M_4\mu_4}{2a_4}\,, \tag{2.1}$$

where the following auxilliary variables are used:

$$\begin{aligned} &\mu_1 = \frac{m_0 m_1}{M_1}, \qquad \mu_2 = \frac{M_1 m_2}{M_2}, \qquad \mu_3 = \frac{M_2 m_3}{M_3}, \qquad \mu_4 = \frac{M_3 m_4}{M_4} \\ &M_1 = m_0 + m_1, \quad M_2 = M_1 + m_2, \quad M_3 = M_2 + m_3, \quad M_4 = M_3 + m_4\,. \end{aligned} \tag{2.2}$$

2.2. *Satellite interaction*

In order to express the Hamiltonian with respect to the orbital elements, the direct and the indirect part of the satellite-satellite interactions are expanded and truncated up to second order in the eccentricities and the inclinations (see for example Murray & Dermott (1999)). The Hamiltonian that corresponds to the satellite interaction is given by:

$$\mathcal{H} = \mathcal{H}_{(1,2)} + \mathcal{H}_{(2,3)} + \mathcal{H}_{(1,3)}\,, \tag{2.3}$$

where

$$\begin{aligned}
\mathcal{H}_{(1,2)} = &-\frac{\mathcal{G}m_1m_2}{a_2}\big(B_0(\alpha_{1,2}) + f_1^{1,2}({e_1}^2+{e_2}^2)\\
&+ f_2^{1,2}e_1\cos(j\lambda_2-k\lambda_1-\varpi_1) + f_3^{1,2}e_2\cos(j\lambda_2-k\lambda_1-\varpi_2)\\
&+ f_4^{1,2}{e_1}^2\cos(2j\lambda_2-2k\lambda_1-2\varpi_1) + f_5^{1,2}{e_2}^2\cos(2j\lambda_2-2k\lambda_1-2\varpi_2)\\
&+ f_6^{1,2}e_1e_2\cos(2j\lambda_2-2k\lambda_1-\varpi_1-\varpi_2) + f_7^{1,2}e_1e_2\cos(\varpi_2-\varpi_1)\\
&+ f_8^{1,2}{s_1}^2\cos(2j\lambda_2-2k\lambda_1-2\Omega_1) + f_9^{1,2}{s_2}^2\cos(2j\lambda_2-2k\lambda_1-2\Omega_2)\\
&+ f_{10}^{1,2}s_1s_2\cos(2j\lambda_2-2k\lambda_1-\Omega_1-\Omega_2) + f_{11}^{1,2}s_1s_2\cos(\Omega_1-\Omega_2)\big)
\end{aligned} \tag{2.4}$$

$$\begin{aligned}
\mathcal{H}_{(2,3)} = &-\frac{\mathcal{G}m_2m_3}{a_4}\big(B_0(\alpha_{2,3}) + f_1^{2,3}({e_2}^2+{e_3}^2)\\
&+ f_2^{2,3}e_2\cos(m\lambda_3-n\lambda_2-\varpi_2) + f_3^{2,3}e_3\cos(m\lambda_3-n\lambda_2-\varpi_3)\\
&+ f_4^{2,3}{e_2}^2\cos(2m\lambda_3-2n\lambda_2-2\varpi_2) + f_5^{2,3}{e_3}^2\cos(2m\lambda_3-2n\lambda_2-2\varpi_3)\\
&+ f_6^{2,3}e_2e_3\cos(2m\lambda_3-2n\lambda_2-\varpi_2-\varpi_3) + f_7^{2,3}e_2e_3\cos(\varpi_3-\varpi_2)\\
&+ f_8^{2,3}{s_2}^2\cos(2m\lambda_3-2n\lambda_2-2\Omega_2) + f_9^{2,3}{s_3}^2\cos(2m\lambda_3-2n\lambda_2-2\Omega_3)\\
&+ f_{10}^{2,3}s_2s_3\cos(2m\lambda_3-2n\lambda_2-\Omega_2-\Omega_3) + f_{11}^{2,3}s_2s_3\cos(\Omega_2-\Omega_3)\big)
\end{aligned} \tag{2.5}$$

$$\begin{aligned}
\mathcal{H}_{1,3} = &-\frac{\mathcal{G}m_1m_3}{a_3}\big(B_0(\alpha_{1,3}) + f_1^{1,3}(e_1^2+e_3^2)\\
&+ f_7^{1,3}e_1e_3\cos(\varpi_3-\varpi_1) + f_{11}^{1,3}s_1s_3\cos(\Omega_1-\Omega_3)\big)\ .
\end{aligned} \tag{2.6}$$

In the above expressions $\alpha_{ij} = a_i/a_j$ is the ratio of the semi-major axes of the i-th and j-th satellite, $B_0(\alpha_{i,j}) = \frac{1}{2}{b_{1/2}}^{(0)}(\alpha_{i,j}) - 1$ and the functions $f^{i,j}$ are linear combinations of the Laplace coefficients ${b_s}^{(n)}$ and their derivatives (Murray & Dermott (1999), Ellis & Murray (2000)). Equations (2.4) and (2.5) are valid for a first-order resonant chain where $j-k = m-n = 1$, however the expansion to the second- and mixed-order cases is straightforward.

2.3. *Secular interaction of the fourth satellite*

The secular gravitational attraction of the distant star ($\sigma = S$) or of a fourth satellite ($\sigma = 4$) is given by:

$$\begin{aligned}
\mathcal{H}_\sigma = -\sum_{i=1}^{\sigma}\Bigg[&\frac{\mathcal{G}m_im_\sigma}{a_\sigma}\Bigg\{\frac{1}{2}b_{1/2}^{(0)}\left(\frac{a_i}{a_\sigma}\right) - 1 + \frac{1}{8}\frac{a_i}{a_\sigma}b_{3/2}^{(1)}\left(\frac{a_i}{a_\sigma}\right)(e_i^2+e_\sigma^2)\\
&-\frac{1}{2}\frac{a_i}{a_\sigma}b_{3/2}^{(1)}\left(\frac{a_i}{a_\sigma}\right)(s_i^2+s_\sigma^2)\Bigg\}\Bigg]\ .
\end{aligned} \tag{2.7}$$

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2.4. *Oblateness of the central planet*

The contribution due to the oblateness of the central planet, limited to the secular terms, is given by:

$$\begin{aligned}\mathcal{H}_{obl} = -\sum_{i=1}^{4} \frac{\mathcal{G} M_i \mu_i}{2a_i} \Bigg[& J_2 \left(\frac{R_P}{a_i}\right)^2 \left(1 + \frac{3}{2} e_i^2 - \frac{3}{2} s_i^2\right) \\ & - \frac{3}{4} J_4 \left(\frac{R_P}{a_i}\right)^4 \left(1 + \frac{5}{2} e_i^2 - \frac{5}{2} s_i^2\right)\Bigg] ,\end{aligned} \tag{2.8}$$

where R_P is the radius of the planet and J_2 is the spherical harmonic coefficient of degree two. To obtain the final formulation of the Hamiltonian, we express it in modified Delaunay variables, which are defined as $(j = 1, 2, 3)$:

$$\begin{aligned} L_j &= \mu\sqrt{\mathcal{G} M_j \alpha_j} \,, && \lambda_j \\ P_j &= L_j (1 - \sqrt{1 - e_j^2}) \,, && p_j = -\varpi_j \\ \Sigma_j &= L_j \sqrt{1 - e_j^2} (1 - \cos I_j) \,, && \sigma_j = \Omega_j \,. \end{aligned} \tag{2.9}$$

3. Tidal interaction

In order to take into account the tidal interaction between the central body (considered to be fast-rotating) and the closest satellite, we include in the model the following equations describing the evolution of semi-major axis, eccentricity, and inclination of the satellite (see Ferraz-Mello *et al.* (2008)):

$$\begin{aligned} \frac{\dot{a}}{a} &= \frac{2}{3} c \left(1 + \frac{51}{4} e^2 - D(7e^2 + {S_B}^2)\right) \\ \frac{\dot{e}}{e} &= -\frac{1}{3} c \left(7D - \frac{19}{4}\right) \\ \dot{I} &= -\frac{3}{4} S_B c (1 + 2D) \,. \end{aligned} \tag{3.1}$$

The parameters c and D are defined as:

$$c = \frac{9}{2} \frac{k_0}{Q_0} \frac{m_1}{m_0} \left(\frac{R_0}{a_1}\right)^5 n \,, \qquad D = \frac{Q_0}{Q_1} \frac{k_1}{k_0} \left(\frac{R_1}{R_0}\right)^5 \left(\frac{m_0}{m_1}\right)^2 \tag{3.2}$$

following the notation of Malhotra (1991), where m_α are the masses, k_α/Q_α the tidal ratios, R_α the radii of the first satellite and the central planet, n is the mean motion of the satellite, and $S_B = \sin(I_1)$. As the tidal interaction is quite weaker than the gravitational forces, longer integrations are needed in order to appreciate its effects. We then multiply them by an enhancing parameter α before including the equations above in the model. This is equivalent to a rescaling of time (see Showman & Malhotra (1997), Lari *et al.* (2020), Celletti *et al.* (2021)). Equations (3.1) are translated in Delaunay variables and then added to the equations for the variables L_j, P_j, Σ_j that can be derived from the Hamiltonian.

4. Dynamical evolution

Having defined the model, we study the dynamical evolution of the system. The equations of motion are integrated numerically applying an 8-th order Runge-Kutta scheme implemented in a Mathematica program. The parameters used in this study are listed in Table 1 and Table 2.

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Table 1. Values of the masses m_j, the eccentricities e_j and the inclinations I_j of the four satellites considered. These values correspond to those of the Galilean satellites.

Satellite	m_j	e_j	I_j
S_1	8.933×10^{22}	4.721×10^{-3}	3.758×10^{-2}
S_2	4.797×10^{22}	9.819×10^{-3}	4.622×10^{-1}
S_3	1.482×10^{23}	1.458×10^{-3}	2.069×10^{-1}
S_4	1.076×10^{23}	7.44×10^{-3}	1.996×10^{-1}

Table 2. Initial values of the semi-major axes of the four satellites, expressed in km.

	2:1&2:1	3:2&3:2	2:1&3:2	3:1&3:1	2:1&3:1
S_1	4.22×10^5	4.22×10^5	4.22×10^5	4.22×10^5	4.22×10^5
S_2	6.713×10^5	5.53×10^5	6.713×10^5	8.78×10^5	6.713×10^5
S_3	10.705×10^5	7.25×10^5	8.78×10^5	18.26×10^5	13.94×10^5
S_4	18.828×10^5	12.751×10^5	15.442×10^5	32.115×10^5	24.517×10^5

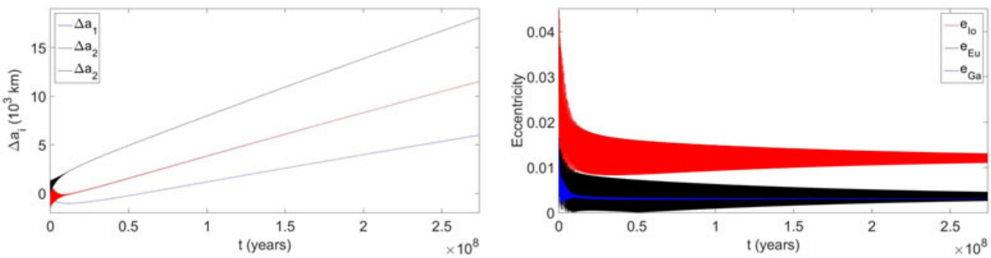

Figure 1. Evolution of the sem-major axes (left) and the eccentricities (right) of the three satellites in the case of the 2:1&3:2 first-order resonance. The tidal effects are multiplied by $\alpha = 10^4$.

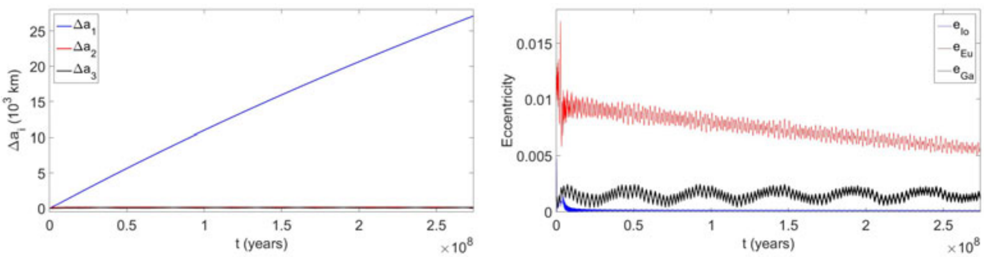

Figure 2. Evolution of the sem-major axes (left) and the eccentricities (right) of the three satellites in the case of the 2:1&3:1 mixed-order resonance. The tidal effects are multiplied by $\alpha = 10^5$.

In the case of the first-order resonances the three satellites move outwards due to the tidal interaction between the innermost satellite and the central planet (an example for the 2:1&3:2 resonance is given in the left panel of Fig. 1). The angular momentum is transferred through the tidal interaction from the central planet to the first satellite, from the first satellite to the second one, and so on. The resonant configuration is therefore maintained. However in the case of the 2:1&3:1 mixed-order resonance the two inner satellites move outwards, while the semi-major axis of the outer one remains constant, as shown in the left panel of Fig. 2. In this case the resonant configuration is destroyed.

The eccentricities of the satellites in the case of a first-order resonance converge to limiting values that are lower than the initial ones. An example is shown in the right panel of Fig. 1. In the mixed-order case the eccentricity of the innermost satellite converges to zero, the eccentricity of the second satellite is decreasing and the eccentricity of the third satellite oscillates around a value close to the initial one (see right panel of Fig. 2).

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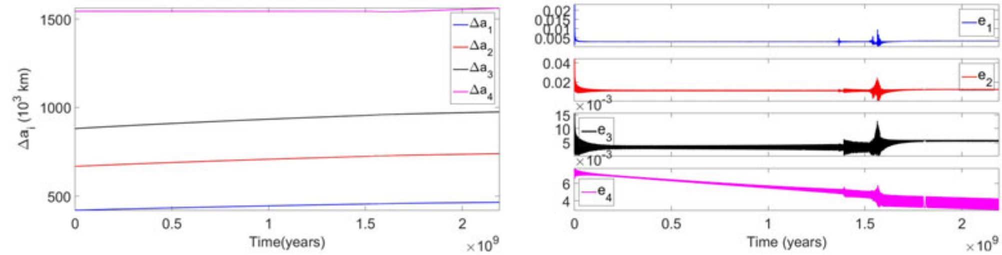

Figure 3. Evolution of the semi-major axes (left) and eccentricities (right) of the four satellites. The tidal effects are multiplied by $\alpha = 10^4$.

5. Capture into resonance of the fourth satellite

In this section we modify the Hamiltonian model in order to include the gravitational interaction of the fourth satellite (i.e., not limited to its secular part) in order to study the possibility of capture into resonance. We sample 100 different initial conditions for the mean longitude of the fourth satellite and integrate numerically the equations of motion. In the cases of the first-order resonances the fourth satellite is captured into resonance in all the different cases studied. In the case of the 2:1&3:2 resonance the fourth satellite is captured in a 2:1 resonance with the third satellite and the resonant argument $\phi_{outer} = \lambda_2 - 3\lambda_3 + 2\lambda_4$ librates around π. In the case of the 3:2&3:2 resonance the resonant angles $2\lambda_4 - \lambda_3 - \varpi_3$ and $2\lambda_4 - \lambda_3 - \varpi_4$ rotate, but the resonant argument ϕ_{outer} librates around π.

The semi-major axes of the satellites increase in a similar rate after the capture into resonance of the fourth satellite, which takes place after 1.5 Gyr (left panel of Fig. 3). The eccentricities of the first and second satellite remain the same after the capture of the fourth satellite, the eccentricity of the third satellite is larger after the capture than before and the eccentricity of the fourth satellite decreases (right panel of Fig. 3).

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