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Topological dynamics and algebras with an involution

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In this paper we extend such dynamical concepts as weak almost periodicity and simple equicontinuity of topological dynamics to the context of groups of $*$-automorphisms on a C^*-algebra with unit. If (A, G) is a C^* flow and if $S(A)$ is the state space of A, we show that (A, G) is weakly almost periodic if and only if $(S(A), G)$ is weakly almost periodic, and that, if A is a von Neumann algebra, then A is G-finite if and only if G is simple equicontinuous on the unit ball of A with respect to the weak$*$-topology.

1. Introduction

Some dynamical concepts such as almost periodicity and minimality of topological dynamics have been extended to the context of groups of $*$-automorphisms on a C^*-algebra with unit by Laison and Laison [4]. They showed, among others, that a C^* flow (A, G) is almost periodic if and only if the induced transformation group $(S(A), \hat{G})$ is almost periodic, where $S(A)$ is the state space of A and

$$\hat{G} = \{g : S(A) \to S(A), \hat{g}(\beta) = p \cdot g^{-1}\} .$$

We shall extend such concepts as weak almost periodicity and simple equicontinuity in somewhat similar fashion. We show that a C^* flow (A, G) is weakly almost periodic if and only if $(S(A), G)$ is weakly almost periodic, and that, if A is, in particular, a C^*-algebra, then A is G-finite if and only if G is simple equicontinuous on the unit ball B

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of A with respect to the weak*-topology.

2. Weak almost periodicity in C^* flow

A transformation group (X, T), where X is a uniform space, is weakly almost periodic if for each $f \in C^*(X)$, the set of all bounded real-valued continuous functions on X, the set $\{f^t : t \in T\}$, where $f^t(x) = f(x \cdot t)$, is a weakly relatively compact subset of $C^*(X)$. (X, T) is simple equicontinuous if its transition group is simple equicontinuous. Readers are referred to [1] and [8] for simple equicontinuity in function spaces. It is proved in [8] that weak almost periodicity and simple equicontinuity are identical concepts in (X, T) if each orbit closure in (X, T) is compact.

If A is a C^*-algebra with unit, and G a group of *-automorphisms of A, G endowed with the pointwise topology on A is a topological group, [3], and (A, G) is a transformation group under the action $(a, g) \to g(a)$. We shall call such a pair a C^* flow.

DEFINITION. Let (A, G) be a C^* flow. Then $a \in A$ is weakly almost periodic if each orbit $G(a) = \{g(a) : g \in G\}$ has weakly compact closure in A. The C^* flow (A, G) is weakly almost periodic if every $a \in A$ is weakly almost periodic.

REMARK. If (X, T) is a compact transformation group, then there is a natural C^* flow associated with (X, T). If $C(X)$ denotes the commutative C^*-algebra of continuous complex-valued functions on X with the usual supremum norm, then, for each $t \in T$, we may define a mapping $\tilde{t} : C(X) \to C(X)$ by $\tilde{t}(a)(x) = a(x \cdot t)$ for each $a \in C(X)$ and each $x \in X$. Then $\tilde{t}$ is a *-automorphism of the C^*-algebra $C(X)$, and if $\tilde{T} = \{\tilde{t} : t \in T\}$, then $(C(X), \tilde{T})$ is a C^* flow. It is easy to see that (X, T) is weakly almost periodic if and only if the C^* flow $(C(X), \tilde{T})$ is weakly almost periodic.

If A is a C^*-algebra, let $S(A)$ denote the state space, with the weak*-topology, of A, and let $S_E(A)$ denote the pure states of A. Then $S(A)$ is compact, and has as a subbase of the uniform structure all subsets of the form

$$\{(p, q) : |p(a)-q(a)| < \varepsilon\}$$

where $a \in A$ and $\varepsilon > 0$. If $K \subset A^*$, we may define, for each $a \in A$, a function $\hat{a} : K \to \mathbb{C}$ by $\hat{a}(p) = p(a)$ for each $p \in K$; then A may be weak topologically embedded into $C(S(A))$ by the mapping $a \to \hat{a}$. Here we say that A is weakly topologically embedded into $C(S(A))$ by the mapping $a \to \hat{a}$ if the mapping $a \to \hat{a}$ is a topological map when both A and $C(S(A))$ are equipped with weak topologies.

THEOREM 1. *Let A be a C^*-algebra, and F a norm bounded subset of A. Then the following are equivalent:*

(1) *F is weakly relatively compact in A;*

(2) *F is simple equicontinuous on $\overline{S_E(A)}$;*

(3) *F is simple equicontinuous on $S(A)$.*

Proof. (1) $\Rightarrow$ (2). Since F is weakly relatively compact in A, F is simple equicontinuous [1, Theorem 5.5]. Hence F is simple equicontinuous on $\overline{S_E(A)}$.

(2) $\Rightarrow$ (1). It is easy to see that if F is a bounded subset of A and $K \subset A^*$, then F is simple equicontinuous on K if and only if $\hat{F}$ is weakly relatively compact in $C(K)$, the space of bounded continuous complex-valued functions on K. Thus it is sufficient to show that $\hat{F}$ is weakly relatively compact in $C(S(A))$. It in turn suffices to show that $\hat{F}$ is weakly sequentially compact on $S(A)$ [2, Theorem 14, p. 269]. For this purpose let $(\hat{a}_n)$ be a sequence in $\hat{F}$. Then, since $\hat{F}$ is pointwise relatively compact, $\hat{F} \subset C(\overline{S_E(A)})$ is quasi-equicontinuous in the sense of [2] (see also [8, Theorem 2.4]). Thus $\hat{F}$ is weakly sequentially compact on $\overline{S_E(A)}$. Hence there is a subsequence $(\hat{a}_k)$ of $(\hat{a}_n)$ which converges weakly to a_0, and for each $p \in \overline{S_E(A)}$, $p(a_k) \to p(a_0)$. Since $S(A)$ is the closed convex hull of $S_E(A)$, for each $q \in S(A)$, there is a normalized measure u_q on $S_E(A)$ such that

$$\hat{a}(q) = \int_{S_E(A)} \hat{a}(p)du_q(p)$$

for each $a \in A$ by [6, Proposition 12]; that is,

$$q(a) = \int_{S_E(A)} p(a)du_q(p)$$

for each $a \in A$. Thus

$$\lim_{k\to\infty} q(a_k) = \lim_{k\to\infty} \int_{S_E(A)} p(a_k)du_q(p)$$
$$= \int_{S_E(A)} \lim_{k\to\infty} p(a_k)du_q(p) = \int_{S_E(A)} p(a_0)du_q(p)$$
$$= q(a_0) .$$

Hence $(\hat{a}_k)$ converges pointwise to $\hat{a}_0$ in $C(S(A))$, which completes the proof.

(3) ⇒ (1). Assume F is simple equicontinuous on S. If

$$F_1 = \{\mathrm{re}(a) : a \in F\} \cup \{\mathrm{im}(a) : a \in F\} ,$$

then F_1 is simple equicontinuous on S, and hence is weakly relatively compact when embedded in $C(S(A))$. Since the embedding is weakly topological, F_1 is weakly relatively compact in A. Since $F \subset F_1 + iF_1$, F is weakly relatively compact as desired.

For each $p \in S(A)$ and each $g \in G$, $p \circ g \in S(A)$, and if $p \in S_E(A)$, so is $p \circ g$. Furthermore, the mapping of $(p, g) \to p \circ g$ of $S(A) \times G \to S(A)$ is continuous. Hence $(S(A), G)$ and $(\overline{S_E(A)}, G)$ are compact transformation groups [3].

THEOREM 2. *Let* (A, G) *be a* C^* *flow. Then the following are equivalent:*

(1) (A, G) *is weakly almost periodic;*

(2) $(\overline{S_E(A)}, G)$ *is weakly almost periodic;*

(3) $(S(A), G)$ *is weakly almost periodic.*

Proof. This follows from Theorem 1 above and Theorem 2.5 of [8].

3. Simple equicontinuity in W*-algebras

Let M be a W*-algebra with predual M_*. For a group of *-automorphisms G of M, M is said to be G-finite if, for each non-zero positive element a in M, there is a G-invariant normal state p of M such that $p(x) \neq 0$. Størmer [7] proved that M is G-finite if and only if for each $\phi \in M_*$ the set $W = \{\phi \circ g : g \in G\}$ is weakly relatively compact in M_*. Since W is weakly relatively compact if and only if W is simple equicontinuous [1, Theorem 5.5], hence M is G-finite if and only if for each $\phi \in M_*$, $\{\phi \circ g : g \in G\}$ is simple equicontinuous. If G is simple equicontinuous, $\{\phi \circ g : g \in G\}$ is simple equicontinuous for each $\phi \in M_*$; hence M is G-finite if G is simple equicontinuous. A group of *-automorphisms G of M is said to be quasi-equicontinuous on the unit ball B of M with respect to the weak*-topology τ if the closure of G in the product space $(B, \tau)^B$ is a set of continuous functions from (B, τ) into (B, τ) [5]. Note that B is a G-invariant subset of M, and we have a compact transformation group (B, G). The following lemma is easy and the proof will be omitted.

LEMMA 4. *Let K be a G-invariant subset of M. Then G is simple equicontinuous (or quasi-equicontinuous) on K with respect to the weak*-topology if and only if (K, G) is simple equicontinuous (respectively quasi-equicontinuous).*

THEOREM 5. *If G is a group of *-automorphisms of a W*-algebra M, then the following are equivalent:*

(1) *M is G-finite;*

(2) *G is quasi-equicontinuous on B with respect to the weak*-topology;*

(3) *G is simple equicontinuous on B with respect to the weak*-topology.*

Proof. (1) ⇒ (2) is proved in [5].

(2) ⇒ (3). Let G be quasi-equicontinuous on B with respect to the weak*-topology. Then the enveloping semigroup of (B, G) is contained in $C(B, G)$, and hence (B, G) is weakly almost periodic by the remark

following Theorem 2.5 of [8]. Hence (B, G) is simple equicontinuous, and G is simple equicontinuous on B with respect to the weak*-topology.

(3) ⇒ (1) follows from the remark made earlier in this section.

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