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Operator theory on quaternionic Hilbert spaces

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This thesis is concerned with the spectral theory of continuous real-linear operators on quaternionic Hilbert space and with functional calculi for certain classes of such operators.

A brief review is given of some results concerning functions of a quaternion variable, including a characterisation of the class of conjugate-symmetric functions - that is, those functions which are symmetric with respect to the automorphisms of the quaternion division ring.

The elementary theory of quaternionic Hilbert space is reviewed, covering, inter alia, the Riesz representation theorem, orthogonality, and dimension theory. The real and complex Hilbert space structures which can be imposed on a quaternionic Hilbert space are considered in detail.

The pivotal result is the fact that any continuous real-linear operator A on a quaternionic Hilbert space has a unique decomposition $A = A_0 + i_1A_1 + i_2A_2 + i_3A_3$, where the A_ν are continuous linear operators and (i_1, i_2, i_3) is any right-handed orthonormal triad of vector quaternions. Specific examples are discussed and it is shown that the colinear and complex-linear operators fit naturally into this characterisation.

The spectrum, point spectrum, continuous spectrum, residual spectrum,

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and resolvent set of a real-linear operator A are defined in the traditional way. If A is complex-linear, the terms representative spectrum, and so forth, are used for the corresponding complex Hilbert space concepts; if A is linear, these definitions do not depend on the choice of unit vector quaternion u to act as $\sqrt{-1}$. The spectral mapping theorem is proved for linear operators and real polynomials; however, a counter-example shows that, for complex-linear operators, one must be content with the "representative spectral mapping theorem". The spectral mapping theorem is also established for real-linear operators and the inverse function, the quaternion conjugation, or any automorphism of the quaternions.

In the spectral theorem for normal linear operators discussed by Viswanath [3], a unitary skew-hermitian operator J is used to replace the multiplication operator iI on complex Hilbert space. The present author avoids this by using a colinear multiplication operator uI (where u is a unit vector quaternion) and the complex Hilbert space spectral theorem to define $f(A)$ for complex-linear A and suitable $f : \tilde{\sigma}(A) \to C$ where $\tilde{\sigma}$ denotes representative spectrum. If A is linear this functional calculus can be extended to conjugate-symmetric functions $f : \sigma(A) \to Q$. In this case $f(A)$ is linear and independent of u.

A continuous real-linear operator A is sesquihermitian if the components A_ν are hermitian; this condition is shown to be independent of the choice of quaternion basis. The joint spectral distribution $T(A_0, A_1, A_2, A_3)$, as defined by Anderson [1], provides a functional calculus for sesquihermitian operators and real-valued C^∞-functions on Q. This calculus is independent of the quaternion basis and extends naturally to quaternion-valued functions to give a continuous quaternion-linear mapping from the algebra of these functions to that of sesquihermitian operators. The mapping is not, in general, multiplicative unless the components A_ν commute, in which case it agrees with that for several commuting operators on complex Hilbert space.

The joint spectrum $\sigma_W(A)$ of A is the support of $T(A_0, A_1, A_2, A_3)$. The convex hull of $\sigma_W(A)$ coincides with the closure of the numerical range of A and contains the spectrum.

In an Appendix, an alternative definition of joint spectrum, suggested by Harte [3], is considered.

References

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