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Motivic Haar Measure on Reductive Groups

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Abstract. We define a motivic analogue of the Haar measure for groups of the form $G(k((t)))$, where k is an algebraically closed field of characteristic zero, and G is a reductive algebraic group defined over k. A classical Haar measure on such groups does not exist since they are not locally compact. We use the theory of motivic integration introduced by M. Kontsevich to define an additive function on a certain natural Boolean algebra of subsets of $G(k((t)))$. This function takes values in the so-called dimensional completion of the Grothendieck ring of the category of varieties over the base field. It is invariant under translations by all elements of $G(k((t)))$, and therefore we call it a motivic analogue of Haar measure. We give an explicit construction of the motivic Haar measure, and then prove that the result is independent of all the choices that are made in the process.

Introduction

In this paper we define a version of Haar measure on groups that arise when taking the set of points of an algebraic group over a "large" local field. For an algebraic group G defined over an algebraically closed field k of characteristic zero, we consider the set of its points $G(F)$ over the field $F = k((t))$ of Laurent series with coefficients in k. Since F is a local field, it can be expected that $G(F)$ would be in many ways analogous to a p-adic group. However, there is no hope for a Haar measure on $G(F)$ in the usual sense, since, unlike the p-adic situation, the set $G(F)$ is not locally compact. By means of the theory of motivic integration introduced by M. Kontsevich [11], we define a "variety-valued" invariant measure on $G(F)$ in the case when G is reductive, and give an explicit formula for such a measure.

The main purpose of this paper, however, is not just to show the existence of motivic Haar measure, but to provide an example of using what we know about p-adic integrals to make conclusions about motivic integrals. Namely, the main technical result is Proposition 22, and its proof essentially consists in extracting an equality of motivic integrals from a well-known equality of the analogous p-adic integrals. Statements of this kind can be hard to prove due to the algebraic geometric nature of motivic integrals. On the other hand, recent work by J. Denef and F. Loeser, and by J. Sebag provides some evidence that "motivic" statements are, in a sense, the most general natural form of p-adic statements. Hence, it could turn out to be useful to have the tools to go both ways: from motivic statements to p-adic ones, and back.

Presently there are a few versions of motivic integration. None of them gives the motivic Haar measure on the groups that we are considering directly, for reasons discussed in the next section. Motivic integration on rigid analytic spaces [13] seems

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to be best suited for this purpose. However, at the time when our construction was carried out, it was not available, and so here we construct the Haar measure "from scratch": first on the affine space, and then on an affine chart (a big cell) in the group. The advantage of the present approach (if any) is in the explicitness and simplicity of the construction, and, again, in the technique developed for the proof of Proposition 22.

1 Preliminaries

1.1 Motivation

In the original theory of motivic integration, the motivic measures live on arc spaces of (smooth) varieties and take values in a certain completion of the Grothendieck ring of the category of all algebraic varieties over k. The arc spaces are defined as follows: For an algebraic variety X over k, *the space of formal arcs on X* is denoted by $\mathfrak{L}(X)$. It is the inverse limit $\varprojlim \mathfrak{L}_n(X)$ in the category of k-schemes of the schemes $\mathfrak{L}_n(X)$ representing the functors defined on the category of k-algebras by

$$R \mapsto \mathrm{Mor}_{k\text{-}schemes}(\mathrm{Spec}\, R[t]/t^{n+1}R[t], X).$$

The set of k-rational points of $\mathfrak{L}(X)$ can be identified with the set of points of X over $k[[t]]$, that is,

$$\mathrm{Mor}_{k\text{-}schemes}(\mathrm{Spec}\, k[[t]], X).$$

There are canonical morphisms $\pi_n \colon \mathfrak{L}(X) \to \mathfrak{L}_n(X)$. On the set of points, they correspond to truncation of arcs. In particular, when $n = 0$, we get the natural projection $\pi_X \colon \mathfrak{L}(X) \to X$. The *canonical motivic measure* is an additive function (whose values are, roughly speaking, equivalence classes of k-varieties) on a certain algebra of subsets of the space $\mathfrak{L}(X)$ (see [5]). In the case when X is a smooth variety over k, this function assigns to the sets of the form $\pi_X^{-1}(\pi_X(A))$ with A a subvariety of X, the equivalence class of A. Loosely speaking, the canonical motivic measure *projects under π_X to the tautological measure on X* (see §§1.3, 1.4). Such a normalization makes the motivic measure on $\mathfrak{L}(X)$ unique, [5] (hence the term *canonical*).

For an algebraic group G, uniqueness implies that the canonical motivic measure on $\mathfrak{L}(G)$ is automatically invariant under translations by the elements of $\mathfrak{L}(G)$. We observe that by definition of an arc space, the set of k-points of $\mathfrak{L}(G)$ is in bijection with the set of $k[[t]]$-points of G, that is, with the set of integral points in $G(F)$ (In the p-adic analogy, $\mathfrak{L}(G)$ corresponds to a maximal compact subgroup inside a p-adic group). Our task is to extend the motivic measure beyond the integral points of $G(F)$ in such a way that it would be invariant under the translations by *all* elements of $G(F)$.

For our construction, the arc spaces will not quite suffice because $G(F)$ is not in bijection with the set of k-points of any arc space. We will need a slightly more general setup, described by E. Looijenga [14], and also the language of *ind-schemes*, needed to handle objets that are "bigger" than arc spaces. We review all the necessary definitions and theorems in the next few subsections. In Section 2, we first extend the motivic measure on $\mathfrak{L}(\mathbb{A}^n)$ to the ind-scheme over k whose set of k-points coincides with the F-points of $\mathbb{A}^n$. We then transport the motivic measure from the affine space to a full measure subset of $G(F)$ (namely, the big cell), using the translation-invariant differential form on G.

Shortly before the work on this paper was completed, F. Loeser and J. Sebag developed a variant of motivic integration on formal schemes [13]. Thanks to the use of Néron models built into their construction, the main difficulty that arises in our approach, namely, the singularity in the closed fiber of the auxilliary schemes that we have to introduce, is avoided. In this way, the theory of motivic integration described in [13] seems better suited for the purpose of defining motivic Haar measure (see also Remark 23 below). However, it has not yet been developed in quite the generality required to deduce the existence of motivic Haar measure on $G(F)$ automatically: one needs to be able to pass from the theory that works with the geometric objects to the theory that works with their k-points.

The rest of Section 1 is devoted to a fairly detailed description of all the ingredients we use, namely, the spaces of sections, motivic measure, and ind-schemes.

1.2 The Space of Sections

Almost everything in the following three subsections is quoted from [14].

We reserve the symbol $\mathbb{D}$ for Spec $k[[t]]$. The term $\mathbb{D}$-variety will mean a separated reduced scheme that is flat and of finite type over $\mathbb{D}$ and whose closed fiber is reduced. For a $\mathbb{D}$-variety, $\mathcal{X}/\mathbb{D}$, with closed fiber X, we consider the set $\mathcal{X}_n$ of sections of its structure morphism up to order n. By sections up to order n we mean morphisms over $\mathbb{D}$ from Spec $k[t]/(t^{n+1})$ to $\mathcal{X}$ which make the following diagram commute

$$\begin{tikzcd} & \mathcal{X} \arrow[d] \\ \operatorname{Spec} k[t]/(t^{n+1}) \arrow[ur] \arrow[r] & \mathbb{D}, \end{tikzcd}$$

where the vertical arrow is the structure morphism of $\mathcal{X}$.

The set $\mathcal{X}_n$ is the set of closed points of a k-variety (which we will also denote by the same symbol $\mathcal{X}_n$), [7, §4.2]. Naturally, $\mathcal{X}_0 = X$. The set $\mathcal{X}_\infty$ of sections of the structure morphism $\mathcal{X} \to \mathbb{D}$ is the projective limit of $\mathcal{X}_n$'s, and therefore it is a set of closed points of a provariety over k (by definition, a provariety is a projective limit of a system of varieties; it is a scheme over k, which in our case is not of finite type). If $\mathcal{X}/\mathbb{D}$ is of the form $X \times \mathbb{D} \to \mathbb{D}$, with X a k-variety, then we get the arc spaces described in the introduction: $\mathcal{X}_n = \mathfrak{L}_n(X)$ and $\mathcal{X}_\infty = \mathfrak{L}(X)$.

As in the case of arc spaces, we have projection morphisms $\pi_n^m\colon \mathcal{X}_m \to \mathcal{X}_n$ and $\pi_n\colon \mathcal{X} \to \mathcal{X}_n$ for all $m \geq n$. (When $n = 0$, we shall write π_X and π_X^m instead of π_0, π_0^m.) A fiber of π_n^{n+1} lies in an affine space over the Zariski tangent space of the base point.

Recall that a *constructible* subset of a variety V is a finite disjoint union of (Zariski) locally closed subvarieties of V.

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Definition 1 A set $A \subset \mathcal{X}_\infty$ is called *weakly stable at level n*, if it is a union of fibers of $\pi_n \colon \mathcal{X} \to \mathcal{X}_n$, and $\pi_n(A)$ is constructible. A subset $A \subset \mathcal{X}_\infty$ is called *stable at level n*, if it is weakly stable at level n and for all $m \geq n$, $\pi_{m+1}(A) \to \pi_m(A)$ is a piecewise trivial fibration over $\pi_m(A)$ with fiber $\mathbb{A}_k^d$, where $d = \dim \mathcal{X}_0$. (For a definition of piecewise trivial fibration, see [5, p. 6].) A set is called *(weakly) stable* if it is (weakly) stable at some level n.

Remark 2 It immediately follows from the definition that a set which is stable at level n is also stable at level m for all m, $m > n$. If $\mathcal{X}/\mathbb{D}$ is smooth and of pure dimension, a weakly stable set is automatically stable (for smooth $\mathcal{X}$, a fiber of the projection from $\mathcal{X}_{n+1}$ to $\mathcal{X}_n$ is an affine space of dimension $d = \dim X$ over the tangent space of the base point, [14, p. 4]). It is also worth mentioning that it is not obvious and not always true that $\mathfrak{L}(X)$ is stable at level 0. The fact that it is stable at some level is a theorem (see *e.g.,* [14, Proposition 3.1]). When X is smooth, it follows from the proof of [14, Proposition 3.1] that $\mathfrak{L}(X)$ is actually stable at level 0.

1.3 The Ring $\hat{\mathcal{M}}$

Now let us describe the ring $\hat{\mathcal{M}}$ where the measure will take values. Let $\mathcal{V}_k$ denote the category of all varieties over k, and let $K_0(\mathcal{V}_k)$ be the Grothendieck ring of this category. Let $\mathbb{L} = [\mathbb{A}^1]$ denote the isomorphism class of the affine line, an element in $K_0(\mathcal{V}_k)$. The notation comes from its motivic interpretation: it corresponds to the so-called Lefschetz motive under the map from $K_0(\mathcal{V}_k)$ to the ring of Chow motives, [15]. Consider the localization of $K_0(\mathcal{V}_k)$ at $\mathbb{L}$: $\mathcal{M} = K_0(\mathcal{V}_k)[\mathbb{L}^{-1}]$. In order to get a measure on an interesting algebra of subsets of $\mathcal{X}_\infty$, we need to complete the ring $\mathcal{M}$. Given $m \in \mathbb{Z}$, let $F_m\mathcal{M}$ be the subgroup of $\mathcal{M}$ generated by the elements of the form $[Z]\mathbb{L}^{-r}$ with $\dim Z \leq -m + r$. This is a filtration of $\mathcal{M}$ as a ring: $F_m\mathcal{M}.F_n\mathcal{M} \subset F_{m+n}\mathcal{M}$. This filtration is called the *dimensional filtration*. Denote by $\hat{\mathcal{M}}$ the separated completion of $\mathcal{M}$ with respect to this filtration, *i.e.,*

$$\hat{\mathcal{M}} = \varprojlim \mathcal{M}/F_m\mathcal{M}.$$

This is called the *dimensional completion*. Our motivic measure will be $\hat{\mathcal{M}}$-valued.

Remark 3 Recent work by F. Loeser and R. Cluckers [12] suggests that in fact the completion is unnecessary. Since motivic integration described in [12] specializes to the motivic integration we are using here, the passage from $\hat{\mathcal{M}}$ to $\mathcal{M}$ should be straightforward, once the results of [12] appear in their final form.

1.4 A Measure on the Space of Sections

Let A be a subset of $\mathcal{X}_\infty$ which is stable at level n. Observe that by definition of stability, the number $(\dim \pi_m(A) - md)$ is independent of the choice of $m \geq n$ (here d is the dimension of the closed fiber X of $\mathcal{X}$). We call this number the *virtual dimension* $\dim A$ of A. The class $[\pi_m(A)]\mathbb{L}^{-md} \in \mathcal{M}$ also does not depend on m; we denote it by $\tilde{\mu}_{\mathcal{X}}(A)$. The collection of stable subsets of $\mathcal{X}_\infty$ is a Boolean ring (*i.e.,*

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it is closed under finite union and difference), on which $\tilde{\mu}_{\mathcal{X}}$ defines a finite additive measure.

Let $\mu_{\mathcal{X}}$ be the composition of $\tilde{\mu}_{\mathcal{X}}$ and the completion map $\mathcal{M} \to \hat{\mathcal{M}}$. We call it the *motivic measure* on $\mathcal{X}$. A subset $A \subset \mathcal{X}_\infty$ is called *measurable* if for every (negative) integer m there exists a stable subset $A_m \subset \mathcal{X}_\infty$ and a sequence $(C_i \subset \mathcal{X}_\infty)_{i=0}^{\infty}$ of stable subsets such that the symmetric difference $A \,\Delta\, A_m$ is contained in $\bigcup_{i \in \mathbb{N}} C_i$ with $\dim C_i < m$ for all i and $\dim C_i \to -\infty$ as $i \to \infty$.

Now we cite the key proposition, which is a generalization of Denef and Loeser's theorem, [5].

Proposition 4 ([14, Proposition 2.2]) *The measurable subsets of $\mathcal{X}_\infty$ make up a Boolean subring and $\mu_{\mathcal{X}}$ extends to a measure on this ring by*

$$\mu_{\mathcal{X}}(A) := \lim_{m \to -\infty} \mu_{\mathcal{X}}(A_m).$$

In particular, the above limit exists in $\hat{\mathcal{M}}$ and its value depends only on A.

Remark 5 Notice that this definition of the measure differs from the one in [4] and [5] by a factor of $\mathbb{L}^d$ (with our normalization, the projection of the motivic measure under π_X is the "tautological" measure on X, as it was described in the introduction).

1.5 The Transformation Rule

The following crucial results from [14] show that the additive function of sets $\mu_{\mathcal{X}}$ possesses the properties expected of a measure in the classical sense.

Proposition 6 ([14, Proposition 3.1]) *For a $\mathbb{D}$-variety $\mathcal{X}/\mathbb{D}$ of pure relative dimension over $\mathbb{D}$, the preimage of any constructible subset under $\pi_n \colon \mathcal{X}_\infty \to \mathcal{X}_n$ is measurable. In particular, $\mathcal{X}_\infty$ is measurable. If $\mathcal{Y} \subset \mathcal{X}$ is nowhere dense, then $\mathcal{Y}_\infty$ is of measure zero.*

For $\mathcal{X}/\mathbb{D}$ of pure relative dimension we have a notion of an integrable function $\Phi \colon \mathcal{X}_\infty \to \hat{\mathcal{M}}$. This requires the fibers of Φ to be measurable and the sum $\sum_a \mu_{\mathcal{X}}(\Phi^{-1}(a))a$ ($a \in \mathcal{M}$) to converge, *i.e.*, at most countably many nonzero terms $(\mu_{\mathcal{X}}(\Phi^{-1}(a_i))a_i)_{i \in \mathbb{N}}$ are allowed, and the condition $\mu_{\mathcal{X}}(\Phi^{-1}(a_i))a_i \in F_{m_i}\hat{\mathcal{M}}$ with $\lim_{i \to \infty} m_i = -\infty$ is required to hold. The motivic integral of Φ is then by definition the value of this series:

$$\int \Phi \, d\mu_{\mathcal{X}} = \sum_i \mu_{\mathcal{X}}(\Phi^{-1}(a_i))a_i.$$

An integrable function of particular interest arises from an ideal $\mathcal{J}$ in the structure sheaf, $\mathcal{O}_{\mathcal{X}}$, of $\mathcal{X}$. Such an ideal defines a function $\mathrm{ord}_{\mathcal{J}} \colon \mathcal{X}_\infty \to \mathbb{N} \cup \{\infty\}$ by assigning to $\gamma \in \mathcal{X}_\infty$ the multiplicity of $\gamma^*\mathcal{J}$ as follows. Let $\gamma(o)$ denote the "constant term of γ", that is, the image of the closed point, o, of $\mathbb{D}$ in the closed fiber of $\mathcal{X}$. The map γ^* is the map of rings $\mathcal{O}_{\mathcal{X},\gamma(o)} \to k[[t]]$ that induces γ. Then γ^* applied to $\mathcal{J}$ means the *base change* of $\mathcal{I}$ to $k[[t]]$. That is, $\gamma^*\mathcal{I}$ is a sheaf on Spec $k[[t]]$; if we denote by M the $\mathcal{O}_{\mathcal{X},\gamma(o)}$-module that corresponds, in the world of rings, to the stalk of $\mathcal{I}$ at $\gamma(o)$, then the stalk of $\gamma^*\mathcal{I}$ over the closed point is the $k[[t]]$-module $k[[t]] \otimes_{\mathcal{O}_{\mathcal{X},\gamma(o)}} M$ (to form the tensor product, we view $k[[t]]$ as an $\mathcal{O}_{\mathcal{X},\gamma(o)}$-module via the map γ^*). An example of the function $\operatorname{ord}_{\mathcal{I}}$ when $\mathcal{X} = \mathfrak{L}(X)$, and $\mathcal{I}$ is the sheaf corresponding to a divisor D, is considered in detail in [4, §2.2] (where $\gamma^*\mathcal{I}$ is denoted $\gamma \cdot D$). The condition $\operatorname{ord}_{\mathcal{I}} \gamma = n$ only depends on the n-jet of γ, and it defines a constructible subset $C_n \subset \mathcal{X}_n$. It turns out that the set defined by $\operatorname{ord}_{\mathcal{I}} \gamma = \infty$ is of measure zero, and the function $\mathbb{L}^{-\operatorname{ord}_{\mathcal{I}}}$ is integrable.

We can now state the theorem that is key for all applications — the transformation rule. Let $H\colon \mathcal{Y} \to \mathcal{X}$ be a morphism of $\mathbb{D}$-varieties of pure relative dimension d. We define the *Jacobian ideal* $\mathcal{J}_H \subset \mathcal{O}_{\mathcal{Y}}$ of H as the 0-th Fitting ideal of the sheaf of relative differentials $\Omega_{\mathcal{Y}/\mathcal{X}}$ (for definitions, see [6, §16.1, §20.2] and [8, II.8.9.2]).

Theorem 7 ([14, Theorem 3.2]) *Let $H\colon \mathcal{Y} \to \mathcal{X}$ be a $\mathbb{D}$-morphism of pure dimensional $\mathbb{D}$-varieties with $\mathcal{Y}/\mathbb{D}$ smooth. If A is a measurable subset of $\mathcal{Y}_\infty$ with $H|_A$ injective, then HA is measurable and $\mu_{\mathcal{X}}(HA) = \int_A \mathbb{L}^{-\operatorname{ord}\mathcal{J}_H}\, d\mu_{\mathcal{Y}}$.*

Example 8 Suppose $H\colon \mathfrak{L}(Y) \to \mathfrak{L}(X)$ is induced by an isomorphism $h\colon Y \to X$. Then H preserves the measure: $\mu_{\mathfrak{L}(X)}(HA) = \mu_{\mathfrak{L}(Y)}(A)$ for any measurable subset $A \subset \mathfrak{L}(Y)$.

Proof An isomorphism of algebraic varieties induces an isomorphism on their tangent bundles. Hence, $\mathcal{J}_H$ is trivial (*i.e.,* it is the ideal sheaf that coincides with the structure sheaf of $\mathfrak{L}(Y)$). The function $\mathbb{L}^{-\operatorname{ord}\mathcal{J}_H}$ is identically equal to 1 on $\mathfrak{L}(Y)$ in this case. This, of course, agrees with the statement about the uniqueness of motivic measure on $\mathfrak{L}(X)$. ■

We will need to use the transformation rule in a slightly more general situation, when $\mathcal{Y}$ is not smooth over $\mathbb{D}$, but is allowed to have a singularity in the closed fiber. In this case, however, the set A will be assumed to be away from the singularity.

For a $\mathbb{D}$-variety $\mathcal{X}$ of pure relative dimension d, we denote by $\mathcal{J}(\mathcal{X}/\mathbb{D})$ the d-th Fitting ideal of $\Omega_{\mathcal{X}/\mathbb{D}}$. It defines the locus where $\mathcal{X}$ fails to be smooth over $\mathbb{D}$, see [14, §9].

Lemma 9 *Let $H\colon \mathcal{Y} \to \mathcal{X}$ be a $\mathbb{D}$-morphism of pure dimensional $\mathbb{D}$-varieties; assume that the generic fiber of $\mathcal{Y}$ is smooth. Let A be a measurable subset of $\mathcal{Y}_\infty$ with $H|_A$ injective and such that for all $\gamma \in A$, $\gamma(o)$ is in the regular locus of $\mathcal{Y}$. Then the transformation rule holds for the set A: $\mu_{\mathcal{X}}(HA) = \int_A \mathbb{L}^{-\operatorname{ord}\mathcal{J}_H}\, d\mu_{\mathcal{Y}}$.*

Proof We follow the proof of the transformation rule in [14]. The proof rests on the key Lemma 9.2, and that is where the assumption that $\mathcal{Y}$ is smooth appears first. Here is the statement of Lemma 9.2, [14]:

> Suppose $\mathcal{Y}/\mathbb{D}$ is smooth and let $A \subset \mathcal{Y}_\infty$ be a stable subset of level l. Assume that $H|_A$ is injective and that $\operatorname{ord} \mathcal{J}_H|_A$ is constant equal to $e < \infty$. Then for

$n \geq \sup\{2e, l+e, \operatorname{ord}_{\mathcal{J}(\mathcal{X}/\mathbb{D})}|_{HA}\}$, $H_n \colon \pi_n A \to H_n \pi_n A$ has the structure of an affine-linear bundle of dimension e. (Here H_n is the truncation of the map H, that is, the map induced by H on $\mathcal{Y}_n$.)

We claim that the same statement holds if the assumption that $\mathcal{Y}$ is smooth is replaced by the weaker assumption from the statement of our lemma.

There are two implications of smoothness of $\mathcal{Y}$ that are used in the proof of Lemma 9.2. The first one is that for all points $\gamma \in A$, the $\mathcal{O}$-module $\gamma^*\Omega_{\mathcal{Y}/\mathbb{D}}$ is torsion-free, where $\mathcal{O} = k[[t]]$ (recall the definition of γ^* applied to an ideal sheaf: it is basically the base change to $k[[t]]$ using the map of rings γ^*). For this statement to hold for all $\gamma \in A$, it is not necessary for $\mathcal{Y}$ to be smooth over $\mathbb{D}$. It is sufficient that $\gamma(o)$ is in $\mathcal{Y}_{\text{reg}}$ and the generic fiber of $\mathcal{Y}$ is smooth. We show this by computing the d-th Fitting ideal of the $k[[t]]$-module $\gamma^*\Omega_{\mathcal{Y}/\mathbb{D}}$ in the same way as it is done in [14, §9]. Recall [14] that $\mathcal{J}(\mathcal{Y}/\mathbb{D})$ stands for the d-th Fitting ideal of $\Omega_{\mathcal{Y}/\mathbb{D}}$, where d is the relative dimension of $\mathcal{Y}$. Since Fitting ideals commute with base change, $\gamma^*(\mathcal{J}(\mathcal{Y}/\mathbb{D})) = \operatorname{Fitt}_d(\gamma^*\Omega_{\mathcal{Y}/\mathbb{D}})$. The latter Fitting ideal measures the length of torsion of $\gamma^*\Omega_{\mathcal{Y}/\mathbb{D}}$: if a $k[[t]]$-module of rank d has torsion of length e, its d-th Fitting ideal is (t^e). It remains to observe that the order with respect to t of the ideal $\gamma^*(\mathcal{J}(\mathcal{Y}/\mathbb{D}))$ is the multiplicity of γ along the locus defined by $\mathcal{J}(\mathcal{Y}/\mathbb{D})$, that is, the singular locus of $\mathcal{Y}$ (see [14, §9]). By assumption, γ maps $\mathbb{D}$ to the regular part of $\mathcal{Y}$, thus $\operatorname{ord}_t \gamma^*\mathcal{J}(\mathcal{Y}/\mathbb{D})$ is equal to 0.

The second implication of the smoothness of $\mathcal{Y}$ that is implicitly used in the proof is that [14, Lemma 9.1] can be used with $e = 0$ in the notation of that lemma (in which e stands for the order of $\mathcal{J}(\mathcal{Y}/\mathbb{D})$ along γ). This property holds for any γ if $\mathcal{Y}$ is smooth; in our case it still holds for all $\gamma \in A$ by the assumption on A, as discussed above. ■

1.6 *k*-Spaces

Let G be a linear algebraic group. As noted in the introduction, the set of k-points of $\mathfrak{L}(G)$ is in bijection with $G(k[[t]])$. With the use of the framework of k-spaces [1], more can be said. The following definitions are quoted from [1].

Let k, as above, be an algebraically closed field of characteristic 0. By definition, a k-space (resp., k-group) is a functor from the category of k-algebras to the category of sets (resp., of groups) which is a sheaf for the faithfully flat topology (see [1] for the details of the definition). The category of schemes can be viewed as a full subcategory in the category of k-spaces. Direct limits exist in the category of k-spaces; we shall say that a k-space (resp., a k-group) is an *ind-scheme* (resp., *ind-group*) if it is the direct limit of a directed system of schemes. Note that an ind-group is not necessarily a limit of a directed system of algebraic groups. Let $(X_\alpha)_{\alpha \in I}$ be a directed system of schemes, X its limit in the category of k-spaces, and S a scheme. The set $\operatorname{Mor}(S, X)$ of morphisms of S into X is the direct limit of the sets $\operatorname{Mor}(S, X_\alpha)$, and the set $\operatorname{Mor}(X, S)$ is the inverse limit of the sets $\operatorname{Mor}(X_\alpha, S)$.

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1.7 The Ind-Scheme $G((t))$

In [1], the k-group $GL_r\,(k((t)))$ is the functor on the category of k-algebras defined by $R \mapsto GL_r\,(R((t)))$, and the "maximal compact subgroup" $GL_r\,(k[[t]])$ is the subfunctor $R \mapsto GL_r\,(R[[t]])$. In order to avoid confusion between the functor and the set of $k((t))$-points of GL_r, we will change the notation and denote the functors defined above by $GL_r((t))$ and $GL_r[[t]]$, respectively.

There is a filtration of the k-group $GL_r((t))$ by the subfunctors $GL_r^{(N)}$, where $GL_r^{(N)}(R)$ is the set of matrices $A(t)$ in $GL_r\,(R((t)))$ such that both $A(t)$ and $A(t)^{-1}$ have no poles of order greater than N, that is, all their entries can be written as $\sum_{i=-N}^{\infty} a_i t^i$ with $a_i \in R$.

The construction of the previous paragraph applies to any affine variety. Indeed, let $X = \operatorname{Spec} k[x_1, \dots, x_d]/I$. For a k-algebra R, define $X^{(N)}(R)$ to be the set of elements of $\mathbb{A}^d(R)$ satisfying the equations in I and having poles of order not greater than N in the sense defined above. By $X((t))$ we will denote the direct limit of $X^{(N)}$; naturally, $X((t))$ is a subfunctor of $\mathbb{A}^d((t))$.

Proposition 1.2 of [1] states that the k-group $GL_r[[t]]$ ($GL_r(k[[t]])$ in the notation of the authors) is represented by an affine group scheme and that $(GL_r^{(N)})_{N\geq 0}$ are represented by schemes, making the k-group $GL_r((t))$ an ind-group. The proof uses only the fact that GL_r is an affine variety: to show that $GL_r^{(N)}$ is represented by a scheme, one needs to think of GL_r as the closed subset of the affine space $M_r \times M_r$ (M_r being the space of all $r \times r$-matrices) defined by the equation $AB = \mathrm{Id}$. The equation $AB = \mathrm{Id}$ (which is, in fact, the system of r^2 equations in r^4 variables) can be substituted with any finite number of polynomial equations in d variables, and the proof will carry over to any closed subvariety of $\mathbb{A}^d$. Thus if X is closed in $\mathbb{A}^d$, the k-space $X((t))$ is represented by the ind-scheme that is the direct limit of schemes representing the functors $X^{(N)}$. We will denote these schemes by the same symbol $X^{(N)}$. The affine space $\mathbb{A}^d((t))$ itself and its filtration by $(\mathbb{A}^d)^{(N)}$ are discussed in detail in the next section.

In the case $X = G$, a reductive algebraic group, $G((t))$ is an ind-group.

All of the above is summarized in the following proposition; we omit its rigorous proof.

Proposition 10 *Let G be a reductive algebraic group. Then $\mathfrak{L}(G)$ is embedded in the ind-group $G((t))$, and $G((t))$ is a direct limit of affine schemes $(G^{(N)})_{N\geq 0}$ in the category of k-spaces, with $G^{(0)} = \mathfrak{L}(G)$ representing $G[[t]]$.*

1.8 The Space $\mathbb{A}^d((t))$

We first focus our attention on affine space since we used it above to define $X((t))$ for X an affine variety, and all the subsequent constructions will also be based upon it.

1.8.1

We begin with the arc space of the affine line $\mathfrak{L}(\mathbb{A}^1)$.

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By definition, $\mathfrak{L}_n(\mathbb{A}^1)$ represents the functor

$$R \to \mathrm{Mor}(\mathrm{Spec}\, R[t]/t^{n+1}R[t], \mathbb{A}^1) = \mathrm{Mor}(k[x], R[t]/t^{n+1}R[t]) \cong R[t]/t^{n+1}R[t] \cong R^{n+1}.$$

Hence, $\mathfrak{L}_n(\mathbb{A}^1) \cong \mathbb{A}^{n+1}$, and the natural projection $\mathfrak{L}_{n+1}(\mathbb{A}^1) \to \mathfrak{L}_n(\mathbb{A}^1)$ corresponds to the map $R[t]/t^{n+2}R[t] \to R[t]/t^{n+1}R[t]$ that takes $P \in R[t]/t^{n+2}R[t]$ to $(P \bmod t^{n+1})$, which, in turn, corresponds to the map $(T_0, \dots, T_{n+1}) \mapsto (T_0, \dots, T_n)$ from $\mathbb{A}^{n+2}$ to $\mathbb{A}^{n+1}$. We conclude that the inverse limit of the system $\mathfrak{L}_n(\mathbb{A}^1)$ coincides with the inverse limit of the spaces $\mathbb{A}^n$ with natural projections. The latter is the scheme $\mathbb{A}^\infty = \mathrm{Spec}\, k[T_1, T_2 \dots]$ (see *e.g.,* [10] and references therein for a detailed treatment of $\mathbb{A}^\infty$, but note that all we will use here is its existence as a k-scheme).

1.8.2

We can also consider $\mathbb{A}^1$ with its additive group structure, that is, the group $\mathbb{G}_a$. Let $\mathbb{G}_a^{(N)}$ be the functor

$$R \to \{\text{elements of } R((t)) \text{ with poles of order } \leq N\}.$$

An element of $R((t))$ with poles of order not greater than N is nothing but a sequence of coefficients $(a_{-N}, \dots, a_0, a_1, \dots)$, where $a_i \in R$, $i = 1, 2, \dots$; thus

$$\mathbb{G}_a^{(N)} \cong \mathrm{Spec}\, k[T_{-N}, \dots, T_0, \dots] \cong \mathrm{Spec}\, k[T_0, T_1, \dots] = \mathbb{G}_a^{(0)} \cong \mathfrak{L}(\mathbb{G}_a).$$

An analogous argument works for $\mathbb{A}^d((t))$ with $d \in \mathbb{N}$. In particular, $(\mathbb{A}^d)^{(N)}$ is isomorphic over k to $\mathfrak{L}(\mathbb{A}^d)$ for all $N \in \mathbb{N}$. Denote this isomorphism by S_N.

1.8.3

Recall the notations, $F = k((t))$, $\mathbb{D} = \mathrm{Spec}\, k[[t]]$. If R is a k-algebra, by R-points of a k-space we will simply mean the set which is an image of R (recall that a k-space is a functor from k-algebras to sets). In all that follows we will be mostly concerned with the set of k-points of $\mathbb{A}^d((t))$, because this set is in bijection with $\mathbb{A}^d(F)$.

So far, we have described one way of thinking of $\mathbb{A}^d((t))(k)$: as a union of the sets of k-points of the schemes over k forming the directed system $(\mathbb{A}^d)^{(N)}$. Each isomorphism S_N between $(\mathbb{A}^d)^{(N)}$ and $(\mathbb{A}^d)^{(0)} = \mathfrak{L}(\mathbb{A}^d)$ induces a bijection on the sets of their k-points, shifting the indices of a power series corresponding to a given point by N to the right. We recall that $\mathfrak{L}(\mathbb{A}^d) = (\mathrm{Spec}\, k[T_0, \dots, T_n, \dots])^d$. Now observe that *the set of k-points of $\mathfrak{L}(\mathbb{A}^d)$ is in natural bijection with the set of $k[[t]]$-points of the affine space $\mathbb{A}^d$ as a scheme over $k[[t]]$, that is, of* $\mathrm{Spec}\, k[[t]][x_1, \dots, x_d]$. This gives another, sometimes more convenient, way of looking at k-points of $\mathbb{A}^d((t))$.

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Fix a positive integer N and consider the $k[[t]]$-morphism $\tilde{S}_N$ from $\operatorname{Spec} k[[t]][u_1,\dots,u_d]$ to $\operatorname{Spec} k[[t]][x_1,\dots,x_d]$ (*i.e.,* to itself), induced by the map of rings $x_i \mapsto t^N u_i$, $i = 1,\dots,d$. On $k[[t]]$-points (which are again viewed as d-tuples of power series with coefficients in k), this map induces multiplication by t^N, that is, a shift of all indices to the right by N. Observe that, even though it is *not* an injective map of $k[[t]]$-schemes, on $k[[t]]$-points, it is an injection. Thus, if we take two copies of $\mathbb{A}^d$ over $k[[t]]$ and the morphism $\tilde{S}_N$ between them, we can identify the set of $k[[t]]$-points of the image of $\tilde{S}_N$ with k-points of $\mathfrak{L}(\mathbb{A}^d)$, and then the set of $k[[t]]$-points of the source copy of $\mathbb{A}^d$ will be naturally identified with the set of k-points of $(\mathbb{A}^d)^{(N)}$. This is an alternative description of the map induced on k-points of $(\mathbb{A}^d)^{(N)}$ by the isomorphism $S_N\colon (\mathbb{A}^d)^{(N)} \to \mathfrak{L}(\mathbb{A}^d)$.

1.9 Morphisms

By definition, $G((t))$ is a k-space, that is, a functor. Then a morphism between two such objects is a morphism of functors (a natural transformation). However, we can use the fact that $G((t))$ is represented by an ind-scheme. By a morphism between two affine ind-schemes $X = \varinjlim X_i$ and $Y = \varinjlim Y_i$ we shall mean a map of sets $\phi\colon X \to Y$ such that each $\phi(X_i)$ is contained in some Y_j, and the induced map $X_i \to Y_j$ is a morphism of schemes.

Let G be an algebraic group. Then we can define an action of $G(F)$ (the group of $k((t))$-points of G) on the ind-group $G((t))$ by left or right translations in the same way as it is done for group schemes, see e.g. Section 4.2 of [2].

2 A Construction of the Motivic Measure on $G((t))$

We begin with a construction of an additively invariant motivic measure on the affine space $\mathbb{A}^d((t))$. Then we use the structure theory of G to reduce the problem of constructing a measure on $G((t))$ to the construction on $\mathbb{A}^d((t))$.

2.1 Haar Measure on the Affine Space

The algebra of measurable subsets of the space $\mathfrak{L}(X)$ was defined in [5, Appendix] for any variety X. In particular, we have an algebra of measurable sets in $\mathfrak{L}(\mathbb{A}^d)$. However, notice that in [5], the expression "a subset of a scheme" means a subset of the underlying topological space, whereas for us (as well as in [14]) a subset of $\mathfrak{L}(X)$ is a subset of the set of *closed* points of $\mathfrak{L}(X)$, since it is the set of *closed* points of $\mathfrak{L}(X)$ that is in bijection with the set of sections of the structure morphism $X \times_{\operatorname{Spec} k} \mathbb{D} \to \mathbb{D}$. We obtain the algebra of measurable subsets (in our sense) of $\mathfrak{L}(X)$ by taking the intersection of all elemets of the algebra of sets defined in [5] with the set of closed points of $\mathfrak{L}(X)$. In general, by a subset of an ind-scheme X which is a direct limit of k-schemes $X^{(N)}$ we shall mean an increasing union of subsets of the sets of closed points of the schemes $X^{(N)}$.

Definition 11 We call a subset of $\mathbb{A}^d((t))$ *bounded measurable* if it is contained in $(\mathbb{A}^d)^{(N)}$ for some N and its image under the isomorphism $(\mathbb{A}^d)^{(N)} \to \mathfrak{L}(\mathbb{A}^d)$ defined in §1.8.2 is a measurable subset of $\mathfrak{L}(\mathbb{A}^d)$ as defined in §1.4.

2.1.1

Bounded measurable subsets form an algebra of sets (closed only under finite unions, though). In order to define a measure on this algebra, we need to calculate the volumes of some special subsets of $\mathfrak{L}(\mathbb{A}^d)$. We do it in the case $d = 1$ first.

Example 12 Let $\mathcal{X} = \mathfrak{L}(\mathbb{G}_a)$, and denote the corresponding motivic measure (from Proposition 4) by $\tilde{\mu}_a$. Consider a decreasing filtration of $\mathbb{G}_a(k[[t]])$ by the subsets $t^n k[[t]]$, $n = 0, 1, \dots$. Denote the corresponding algebraic subsets of $\mathfrak{L}(\mathbb{G}_a)$ by B_n, so that the set of k-points of B_n is $t^n k[[t]]$. Let us calculate their volumes. The set B_n ($n \in \mathbb{N}$) is precisely the fiber of $\mathfrak{L}(\mathbb{G}_a)$ over the point $\mathbf{0}_{n-1} = (0, \dots, 0) \in \mathfrak{L}_{n-1}(\mathbb{G}_a)$. Hence, by definition,

$$\begin{aligned}\tilde{\mu}_a(B_n) &= \mathbb{L}^{-n+1}[\pi_{n-1}(B_n)] = \mathbb{L}^{-n+1}[\{\mathrm{pt}\}]\\ &= \mathbb{L}^{-n+1}\mathbf{1} = \mathbb{L}^{-n+1}.\end{aligned}$$

The total volume $\tilde{\mu}_a(\mathfrak{L}(\mathbb{G}_a))$ is by definition $[\mathbb{A}^1]\mathbb{L}^0 = \mathbb{L}$, so we have

$$\tilde{\mu}_a(B_n) = \mathbb{L}^{-n}\tilde{\mu}_a(\mathfrak{L}(\mathbb{G}_a)). \tag{1}$$

2.1.2

Now we can define a motivic measure on $\mathbb{G}_a((t))$. We keep the notation of the previous example. Let A be a measurable subset of $\mathbb{G}_a^{(N)}$, *i.e.*, its image $B = S_N(A)$ in $\mathbb{G}_a^{(0)} = \mathfrak{L}(\mathbb{A}^1)$ is measurable. Then we set

$$\mu_a(A) = \mathbb{L}^N \tilde{\mu}_a(B). \tag{2}$$

On the level of rings, the inclusion $\mathbb{G}_a^{(N-1)} \hookrightarrow \mathbb{G}_a^{(N)}$ corresponds to the map induced by $T_{-N} \mapsto 0$ from $k[T_{-N}, T_{-N+1}, \dots]$ to $k[T_{-N+1}, \dots]$. The map S_N identifies the scheme $\mathbb{G}_a^{(N)}$ with $\mathfrak{L}(\mathbb{G}_a)$, and therefore the image of its subset $\mathbb{G}_a^{(N-1)}$ maps isomorphically onto the fiber of $\mathfrak{L}(\mathbb{G}_a)$ over 0, that is, the set B_1. Similarly, for $M < N$, $S_N\left(\mathbb{G}_a^{(M)}\right) = B_{N-M}$. Then the relation (1) guarantees that the volume $\mu_a(\mathbb{G}_a^{(N)})$ is well defined. A similar calculation applied to an arbitrary measurable subset of $\mathbb{G}_a^{(N)}$ would show that the measure μ_a is well defined.

Remark 13 It is possible to arrive at the same conclusions without writing down the sets B_n and their volumes explicitly, but by using the transformation rule and the following lemma.

Lemma 14 *The order of Jacobian* $\mathrm{ord}_t\, \mathfrak{J}_{\tilde{S}_N}(\gamma)$ *of the map* $\tilde{S}_N \colon \mathfrak{L}(\mathbb{A}^d) \to \mathfrak{L}(\mathbb{A}^d)$ *is equal to* Nd *for all* $\gamma \in \mathfrak{L}(\mathbb{A}^d)$.

Proof As in §1.8.3, we think of the closed points of $\mathfrak{L}(\mathbb{A}^d)$ as sections of the structure morphism of the scheme $\operatorname{Spec} k[[t]][x_1,\dots,x_d]$ over $k[[t]]$. We have the map $\tilde{S}_N\colon \operatorname{Spec} B \to \operatorname{Spec} A$, where $A = k[[t]][x_1,\dots,x_d]$, $B = k[[t]][u_1,\dots,u_d]$, $x_i \mapsto t^N u_i$. There is an exact sequence of modules of differentials [6, §16.1]:

$$\Omega_{A/k[[t]]} \otimes_A B \longrightarrow \Omega_{B/k[[t]]} \longrightarrow \Omega_{B/A} \longrightarrow 0.$$

We see that $\Omega_{B/A}$ is a torsion B-module, and the above exact sequence is its free presentation. Hence $\operatorname{Fitt}_0(\Omega_{B/A}) = (\det(t^N \operatorname{Id})) = (t^{Nd}) \subset B$ by [6, §20.2]. Therefore the Jacobian ideal of the map $\tilde{S}_N$ is the ideal sheaf (t^{Nd}) on $\operatorname{Spec} B$. Let

$$\gamma\colon \operatorname{Spec} k[[t]] \to \operatorname{Spec} k[[t]][u_1,\dots,u_d]$$

be a section. The stalk of $\mathcal{J}_{\tilde{S}_N}$ at every point is the ideal (t^{Nd}) in the local ring of that point, *i.e.*, it is an ideal of $k[[t]]$ embedded into the local ring of the point. Any section γ fixes $k[[t]]$ by definition, so the pullback of $\mathcal{J}_{\tilde{S}_N}$ to $k[[t]]$ by γ is the ideal (t^{Nd}) itself. Thus $\operatorname{ord}_t \mathcal{J}_{\tilde{S}_N}(\gamma) = Nd$. ∎

2.1.3

Recall the notation, $\tilde{\mu}_a$ is the canonical measure on $\mathfrak{L}(\mathbb{A}^d)$ (see Proposition 4).

Definition 15 Let $A \subset (\mathbb{A}^d)^{(N)}$ be a bounded measurable subset. Then define

$$\mu_a(A) = \mathbb{L}^{Nd}\tilde{\mu}_a(S_N(A)).$$

Lemma 16 *The measure μ_a is well defined and additively invariant.*

Proof The first statement is proved exactly the same way as in §2.1.2. The invariance follows from the transformation rule, but it is also easy to check this statement by hand, using the explicit definition of the measure μ_a and the fact that translations are isomorphisms. ∎

Remark 17 By invariance, here we mean that the translates of bounded measurable subsets are again bounded measurable, of the same measure.

It is now possible to define the full algebra of measurable sets in $\mathbb{A}^d((t))$.

Definition 18 We call a subset $B \subset \mathbb{A}^d((t))$ measurable if it can be represented as a disjoint countable union of bounded measurable subsets $B = \bigcup_{n\in\mathbb{N}} B_n$, such that the series of their measures $\sum_{i=1}^{\infty} \mu_a(B_n)$ converges in the ring $\hat{\mathcal{M}}$. The measure of B is defined as $\mu_a(B) = \sum_{n=1}^{\infty} \mu_a(B_n)$.

The proof that $\mu_a(B)$ does not depend on a particular collection B_n mimics standard measure theory, with the use of a norm on $\hat{\mathcal{M}}$ introduced in [5, Appendix]. It is easy to see that the measure μ_a extended to the σ-algebra of measurable sets is still translation-invariant.

2.2 Notation

Let X be an affine variety, $X((t))$, the ind-scheme defined as in §1.7, and U, a Zariski open subset of X with $Z = X \setminus U$ closed. Then $Z((t))$ is a subfunctor of $X((t))$. By $\mathfrak{C}_X(U)$ we will denote the ind-scheme which is the direct limit of the schemes $X^{(N)} \setminus Z^{(N)}$, that is, the "complement of $Z((t))$ in $X((t))$". We shall denote by $\mathfrak{C}_X^0(U)$ the complement of $Z[[t]]$ in $X[[t]]$. Notice that $\mathfrak{C}_X^0(U)$ is not the same as $U[[t]]$, in general, it is much larger. By the construction, there is an inclusion morphism of ind-schemes $\mathfrak{C}_X(U) \hookrightarrow X((t))$. Later we will slightly abuse the terminology by thinking of $\mathfrak{C}_X(U)$ as a measurable subset of $X((t))$, meaning that the set of closed points of $\mathfrak{C}_X(U)$ can be thought of as a subset of the set of closed points of $X((t))$.

Example 19 $X = \mathbb{A}^1$, $Z = \{0\}$, $U = X \setminus Z$. Then $\mathfrak{L}(U)$ is the set B_1 from Example 12, that is, the fiber of π_X over $0 \in \mathfrak{L}_0(X)$, so its motivic volume is different from the volume of X. However, $\mathfrak{C}_X^0(U)$ is the complement of $\mathfrak{L}(Z)$ in $\mathfrak{L}(\mathbb{A}^1)$, that is, a complement of a single point, so the motivic volume of $\mathfrak{C}_X^0(U)$ coincides with the motivic volume of $\mathbb{A}^1$.

In this example, $\mathfrak{C}_X(U) = U((t))$ is the functor that assigns to every ring R the set of Laurent series with coefficients in R such that at least one of the coefficients is a unit in R. Also, notice that $U((t)) \cap \mathfrak{L}(X) = \mathfrak{C}_X^0(U)$.

2.3

Once and for all, we choose the standard coordinates $x_1, \ldots, x_d$ on $\mathbb{A}^d$. Let ω be a top degree differential form $\omega = g dx_1 \wedge \cdots \wedge dx_d$ defined on a Zariski open subset $U \subset \mathbb{A}^d$, where g is a regular function on U. Then define the measure $\mu_{|\omega|}$ on $\mathfrak{C}_{\mathbb{A}^d}(U)$ by

$$\mu_{|\omega|}(A) = \int_A \mathbb{L}^{-\operatorname{ord}_t(g\circ\gamma)}\, d\mu_a(\gamma), \tag{3}$$

where μ_a is the motivic measure on $\mathbb{A}^d((t))$, A is a bounded measurable set contained in $\mathfrak{C}_{\mathbb{A}^d}(U)$; $\operatorname{ord}_t(g \circ \gamma)$ for $\gamma \in (\mathbb{A}^d)^{(N)}$ is the order of vanishing of the formal power series $g(\gamma)$ at $t = 0$ (if the series has a pole at $t = 0$, the order is negative).

In this notation, the measure on $\mathbb{A}^d((t))$ defined in §2.1.3 is the one that corresponds to the form $dx_1 \wedge \cdots \wedge dx_d$.

By definition of the measure μ_a, the integral in (3) can be written as

$$\mu_{|\omega|}(A) = \int_A \mathbb{L}^{-\operatorname{ord}_t(g\circ\gamma)}\, d\mu_a(\gamma) = \int_{S_N(A)} \mathbb{L}^{-\operatorname{ord}_t(\tilde{g}\circ\gamma)+Nd}\, d\mu_a(\gamma) \tag{4}$$

for any $N \geq 0$, where $\tilde{g}(t^N x_1, \ldots, t^N x_d) = g(x_1, \ldots, x_d)$. In particular, since for a bounded set A the number N can be chosen big enough to ensure $S_N(A) \subset \mathfrak{L}(\mathbb{A}^d)$, the motivic integral in the right-hand side of (4) exists (see [4]), and therefore the integral in (3) is also defined (we can use the right-hand side of (4) as its definition).

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2.4 A Coordinate System on the Big Cell

Let G be a connected reductive algebraic group defined over k. Let $T \subset G$ be a maximal torus (recall that the field k is assumed algebraically closed, so T is automatically split), where m is its dimension, Δ a choice of simple roots of the Lie algebra of G, n the cardinality of Δ, $B \supset T$ the Borel subgroup corresponding to Δ, U its unipotent radical, B^- the opposite Borel subgroup with respect to T, and U^- its unipotent radical. Then [9, p. 174], the product morphism is an isomorphism of algebraic varieties

$$U^- \times T \times U \to \Omega',$$

where $\Omega' \subset G$ is a Zariski open subset (a big cell). For our purposes, it is more convenient to consider its conjugate, the set $\Omega = U^- \times U \times T$. The unipotent subgroup U (resp., U^-) is isomorphic to a cartesian product of root subgroups U_α corresponding to positive (resp., negative) roots. Choose a generator for each U_α, and denote it by x'_α if α is positive, and by y'_α if α is negative. Each U_α can be identified with a one-dimensional subspace $\mathfrak{g}_\alpha$ in the Lie algebra of G. Denote by x_α (resp., y_α) the generator of $\mathfrak{g}_\alpha$ that corresponds to x'_α (resp., y'_α) under this isomorphism. This defines a coordinate system on $U^- \times U$. Next, choose a coordinate system $s_1, \dots, s_m$ on T by representing it as a product of m copies of $\mathbb{G}_m$ and choosing a coordinate s_j on each of them. Hence we have defined a coordinate map $i \colon \Omega \to \mathbb{A}^d$, $d = 2n + m$. It is defined over k. The image of this map is the Zariski open subset of $\mathbb{A}^d$ defined by $s_1 \cdots s_m \neq 0$.

2.5

Let Ω be the big cell of G, as in the previous subsection. Denote by Z the complement of Ω in G, a constructible subset which is a union of a finite number of closed subvarieties of G defined over k. Recall from §1.7 that the set of F-points of G can be identified with the set of k-points of the ind-group $G((t))$, which is a direct limit of the system $(G^{(N)})_{N \geq 0}$. Under this bijection the set $\Omega(F)$ is identified with the set of k-points of $\mathfrak{C}_G(\Omega)$. We recall from §2.2 that by definition $G((t)) = \mathfrak{C}_G(\Omega) \cup Z((t))$. Observe that the map i from the previous subsection extends to a map from $\mathfrak{C}_G(\Omega)$ to $\mathbb{A}^d((t))$; it is still a map over k, and we will denote it by the same letter i.

Let ω be a 1-form on Ω that is defined by the following expression in the coordinates (x,y,s) defined in §2.4:

$$(5) \quad \omega = dx_1 \wedge \cdots \wedge dx_n \wedge dy_1 \wedge \cdots \wedge dy_n \wedge \frac{ds_1}{s_1} \wedge \cdots \wedge \frac{ds_m}{s_m} =: dx \wedge dy \wedge \frac{ds}{s}.$$

Lemma 20 *The form ω is invariant under left and right translations on G.*

We omit the proof.

Recall that by a subset of the ind-scheme $G((t))$ we mean a union of subsets of closed points of the schemes $G^{(N)}$. Now we are ready to define a motivic measure on $G((t))$.

Definition 21 Let B be a subset of $G((t))$. We say that B is Ω-measurable if B can be represented as a (disjoint) union $B = C \cup A$, where $C \subset Z((t))$ and A is a measurable subset of $\mathfrak{C}_G(\Omega)$. Here we say that a subset A of $\mathfrak{C}_G(\Omega)$ is measurable if its image $i(A)$ is a measurable subset of $\mathbb{A}^d((t))$. For $B = C \cup A$ measurable, set

$$\mu_\Omega(B) = \mu_{|(i^{-1})^*(\omega)|}(i(A)). \tag{6}$$

We call a measurable subset bounded, if it is contained in $\mathfrak{C}_G(\Omega)$ and its image under the map i is a bounded measurable subset of $\mathbb{A}^d((t))$.

The following proposition is the cornestone of the proof that μ_Ω is the Haar measure.

Proposition 22 *Let g be an element of $G(F)$, and let A be a bounded Ω-measurable set in $\mathfrak{C}_G(\Omega)$ such that $g^{-1}A$ is also contained in $\mathfrak{C}_G(\Omega)$ and bounded. Then $\mu_\Omega(A) = \mu_\Omega(g^{-1}A)$.*

Proof Let us denote by L_g the left translation by g viewed as an automorphism of G defined *over the field* F. On the open subset $\Omega \cap g^{-1}\Omega$ it can be represented as a rational map in the coordinates x, y, s defined in §2.4. We denote this map by $h(x, y, s)$, and its Jacobian matrix by J. More precisely, h is a birational map from $\mathbb{A}^d$ to $\mathbb{A}^d$ over F defined by the formula $h(x, y, s) = i(L_g(i^{-1}(x, y, s)))$. Thus $\det J$ is an F-valued regular function on Ω, and by Lemma 20, we have

$$p(h(x, y, s)) \cdot \det J \cdot dx \wedge dy \wedge ds = p(x, y, s) dx \wedge dy \wedge ds, \tag{7}$$

where $p(x, y, s) = 1/s_1 \cdots s_m$. Now the goal is to represent the restriction of the map L_g to the given set $g^{-1}A$ as a restriction of a $k[[t]]$-morphism of $\mathbb{D}$-varieties, so that the transformation rule for motivic measures can be applied to it.

The sets A and $g^{-1}A$ are both contained in $\mathfrak{C}_G(\Omega)$ and are bounded by assumption. By definition, this means that $i(A)$ is a measurable subset of $(\mathbb{A}^d)^{(N)}$ for some $N \geq 0$, and that $i(g^{-1}A)$ is defined and is contained in $(\mathbb{A}^d)^{(M)}$ for some $M \geq 0$. We choose both integers M, N to be minimal possible. Also, we can assume without loss of generality that A is stable.

We will need the expression $h(t^{-M}x, t^{-M}y, t^{-M}s)$. We write it in the form

$$h(t^{-M}x, t^{-M}y, t^{-M}s) = \left(\frac{\tilde{f}_1(x, y, s)}{\Delta(x, y, s)}, \dots, \frac{\tilde{f}_d(x, y, s)}{\Delta(x, y, s)} \right), \tag{8}$$

where $\tilde{f}_i$, $i = 1, \dots, d$ and Δ are in $k[[t]][x, y, s]$, and $\gcd(\tilde{f}_1, \dots, \tilde{f}_d, \Delta) = 1$.

Let us break up the set A according to the order of vanishing of Δ on $S_M(i(g^{-1}A))$:

$$A = \bigcup_{e \geq 0} A_e,$$

$$A_0 = \{\gamma \in A \mid \operatorname{ord}_t \Delta(S_M \circ i \circ g^{-1}\gamma) \leq 0\},$$

$$A_e = \{\gamma \in A \mid \operatorname{ord}_t \Delta(S_M \circ i \circ g^{-1}\gamma) = e\} \text{ for } e \geq 1.$$

Now we are ready to construct, for each $e = 0, 1, \dots$, a scheme $\mathcal{X}_e$ over $\mathbb{D}$ and two $\mathbb{D}$-morphisms H_1 and H_2 from $\mathcal{X}_e$ to $\mathbb{A}^d[[t]]$, such that the following conditions hold:

(i) There exists a measurable subset B of $(\mathcal{X}_e)_\infty$ such that H_1 induces a bijection between B and $S_M(i(g^{-1}A_e))$.

(ii) The morphism H_2 induces a bijection between B and $S_e(i(A_e))$.

(iii) The following diagram (of maps of sets) commutes:

$$\begin{tikzcd} S_M(i(g^{-1}A_e)) & B \arrow[l, "H_1"'] \arrow[r, "H_2"] & S_e(i(A)) \\ i(g^{-1}A_e) \arrow[u, "S_M"] \arrow[rr, "h"'] & & i(A_e) \arrow[u, "S_e"'] \end{tikzcd}$$

Define the scheme $\mathcal{X}_e$ to be

$$\mathcal{X}_e = \operatorname{Spec} k[[t]][x_1, \dots, x_n, y_1, \dots, y_n, s_1, \dots, s_m, z]/(z\Delta - t^e).$$

Let $H_1 \colon \mathcal{X}_e \to \mathbb{A}^d[[t]] = \operatorname{Spec} k[[t]][u_1, \dots, u_d]$ be the morphism of schemes induced by the identity map on the first d variables:

$$(9) \qquad u_i \mapsto x_i, 1 \le i \le n; \quad u_i \mapsto y_{i-n}, n+1 \le i \le 2n; \quad u_i \mapsto s_{i-2n}, 2n < i \le d.$$

When $e = 0$, the map H_1 is nothing but the inclusion morphism of the open subset of $\mathbb{A}^d[[t]]$ defined by $\Delta \neq 0$ into $\mathbb{A}^d[[t]]$.

Let $H_2 \colon \mathcal{X}_e \to \mathbb{A}^d[[t]] = \operatorname{Spec} k[[t]][u_1, \dots, u_d]$ be the morphism defined by

$$(10) \qquad u_i \mapsto z\tilde{f}_i(x, y, s), \quad i = 1, \dots, d.$$

Naturally, H_1 induces a bijection on $k[[t]]$-points. Let $B \subset (\mathcal{X}_e)_\infty$ be the preimage of the set $S_M(i(g^{-1}A_e))$ under this bijection. Then it immediately follows from the definition of H_2 that property (iii) holds (recall that S_M is a bijection between the sets $i(g^{-1}A_e)$ and $S_M(i(g^{-1}A_e))$; $h(t^{-M}x, t^{-M}y, t^{-M}s) = \left(\frac{\tilde{f}_1}{\Delta}, \dots, \frac{\tilde{f}_d}{\Delta}\right)$, and "$z = \frac{t^e}{\Delta}$"). Since the map h is a coordinate expression of a translation by a group element, it is a bijection; thus the commutativity of the diagram implies that H_2 induces a bijection between the set B and the set $S_e(i(A_e))$.

In the case $e = 0$ the scheme $\mathcal{X}_e$ is smooth over $\mathbb{D}$. For $e > 0$, $\mathcal{X}_e$ has smooth generic fiber, and the singular locus in its closed fiber is defined by the equations $\Delta(x, y, s) = z = 0$. We observe that the z-coordinate of $\gamma(o)$ (the image of the closed point of $\mathbb{D}$) is not equal to zero for any element γ of the set B since Δ is assumed to vanish exactly up to order e on the image of γ in $S_M(i(g^{-1}A_e))$. That is, $\gamma(o)$ does not lie in the singular locus of the closed fiber of $\mathcal{X}_e$. Since A is assumed to be stable, the set B is also stable: the condition $\operatorname{ord}_t \Delta(\gamma) = e$ depends only on the e-jet of γ. Indeed, the stability of A implies the stability of all the sets A_e. The only formal difference between B and A_e is that the points in B have an extra coordinate $z = z_0 + z_1 t + \cdots + z_n t^n + \cdots$ and satisfy an extra equation $z\Delta(x, y, s) = t^e$. By our

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assumption on A_e and by the definition of B, the order of Δ/t^e is equal to 0. Hence, if $n > e$, each equation in z_{n+1} of the form $(z_0 + \cdots + z_{n+1}t^{n+1} + \cdots)\Delta(x, y, s) = t^e + t^{n+1}g(t)$, $g(t) \in k[[t]]$ with fixed x, y, s and fixed $z_0, \ldots, z_n$ has a unique solution. Therefore, the set B is stable at level e or the level of A, whichever is greater.

It follows now from Lemma 9 that the transformation rule can be applied to the restriction of the morphisms H_1 and H_2 to the set B. Let us denote the motivic measure on $\mathfrak{X}_e$ that was defined in §1.4 by $d\mu_e$, and the motivic measure on $\mathbb{A}^d[[t]]$ by $d\mu_a$, as before. By the transformation rule, we have

$$\begin{aligned} d\mu_a|_{S_M(i(g^{-1}A_e))} &= \mathbb{L}^{-\operatorname{ord} \mathfrak{J}_{H_1}} d\mu_e|_B, \\ d\mu_a|_{S_e(i(A_e))} &= \mathbb{L}^{-\operatorname{ord} \mathfrak{J}_{H_2}} d\mu_e|_B. \end{aligned} \tag{11}$$

It remains to calculate $\mathfrak{J}_{H_1}$ and $\mathfrak{J}_{H_2}$. We start with the Jacobian of H_1. Let R_1 be the ring $R_1 = k[[t]][u_1, \ldots, u_d]$, and let

$$R_2 = k[[t]][x_1, \ldots, x_n, y_1, \ldots, y_n, s_1, \ldots, s_m, z]/(z\Delta - t^e).$$

By definition, $\mathfrak{J}_{H_1}$ is the 0-th Fitting ideal of the module Ω_{R_2/R_1}, where the map $R_1 \to R_2$ is given by formula (9). We have the exact sequence

$$\Omega_{R_1/k[[t]]} \otimes_{R_1} R_2 \longrightarrow \Omega_{R_2/k[[t]]} \longrightarrow \Omega_{R_2/R_1} \longrightarrow 0. \tag{12}$$

Hence, Ω_{R_2/R_1} is in this case a torsion R_2-module isomorphic to $R_2[\sigma]/\sigma\Delta$. Its 0-th Fitting ideal is (Δ). Notice that by the remark on definition of the set B earlier in this proof, $\operatorname{ord}_t(\gamma^*\Delta) = e$ for all $\gamma \in B$.

Let us now calculate the Jacobian of H_2. The rings R_1 and R_2 remain the same, but the map $R_1 \to R_2$ is given by formula (10) now. Then Ω_{R_2/R_1} is the R_2-module generated over R_2 by the formal symbols $dx_1, \ldots, dx_n, dy_1, \ldots, dy_n, ds_1, \ldots, ds_m, dz$ with the relations obtained by setting to zero the formal derivatives of the polynomials $z\Delta$ and $z\tilde{f}_i(x, y, s)$, $i = 1, \ldots, d$. Hence, by definition of the Fitting ideal, the 0-th Fitting ideal of this module is generated by the following $(d + 1) \times (d + 1)$-determinant:

$$\begin{vmatrix} z\frac{\partial \tilde{f}_1}{\partial x_1} & \cdots & z\frac{\partial \tilde{f}_d}{\partial x_1} & z\frac{\partial \Delta}{\partial x_1} \\ z\frac{\partial \tilde{f}_1}{\partial x_2} & \cdots & z\frac{\partial \tilde{f}_d}{\partial x_2} & z\frac{\partial \Delta}{\partial x_2} \\ \vdots & \cdots & \vdots & \vdots \\ z\frac{\partial \tilde{f}_1}{\partial s_m} & \cdots & z\frac{\partial \tilde{f}_d}{\partial s_m} & z\frac{\partial \Delta}{\partial s_m} \\ \tilde{f}_1 & \cdots & \tilde{f}_d & \Delta \end{vmatrix} = z^d \begin{vmatrix} \frac{\partial \tilde{f}_1}{\partial x_1} - \frac{\partial \Delta}{\partial x_1}\frac{\tilde{f}_1}{\Delta} & \cdots & \frac{\partial \tilde{f}_d}{\partial x_1} - \frac{\partial \Delta}{\partial x_1}\frac{\tilde{f}_d}{\Delta} & \frac{\partial \Delta}{\partial x_1} \\ \frac{\partial \tilde{f}_1}{\partial x_2} - \frac{\partial \Delta}{\partial x_2}\frac{\tilde{f}_1}{\Delta} & \cdots & \frac{\partial \tilde{f}_d}{\partial x_2} - \frac{\partial \Delta}{\partial x_2}\frac{\tilde{f}_d}{\Delta} & \frac{\partial \Delta}{\partial x_2} \\ \vdots & \cdots & \vdots & \vdots \\ \frac{\partial \tilde{f}_1}{\partial s_m} - \frac{\partial \Delta}{\partial s_m}\frac{\tilde{f}_1}{\Delta} & \cdots & \frac{\partial \tilde{f}_d}{\partial s_m} - \frac{\partial \Delta}{\partial s_m}\frac{\tilde{f}_d}{\Delta} & \frac{\partial \Delta}{\partial s_m} \\ 0 & \cdots & 0 & \Delta \end{vmatrix}.$$

By formula (8), the latter determinant is equal to $z^d\Delta^d(t^{-Md} \det J)\Delta$, where $\det J$ is the Jacobian determinant of the map h that was defined in the beginning of the proof (we are using the equality $\frac{\partial f}{\partial x} - \frac{\partial \Delta}{\partial x}\frac{f}{\Delta} = \Delta\frac{\partial(f/\Delta)}{\partial x}$). Finally, we see that the Jacobian ideal of the map H_2 is the ideal $(t^{(e-M)d} \det J\Delta)$.

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Let $\tilde{p}(t^M x, t^M y, t^M s) = p(x, y, s) = 1/s_1 \cdots s_m = \bar{p}(t^e x, t^e y, t^e s)$. With these notations, by (4) and (11), get:

$$\mu_\Omega(g^{-1}A_e) = \mathbb{L}^{Md} \int_{S_M(i(A_e))} \mathbb{L}^{-\operatorname{ord}_t \tilde{p}\circ\gamma}\, d\mu_a(\gamma)$$

$$= \mathbb{L}^{Md} \int_B \mathbb{L}^{-\operatorname{ord}_t \tilde{p}\circ H_1(\gamma) - \operatorname{ord}_t \mathcal{J}_{H_1}(\gamma)}\, d\mu_e(\gamma);$$

$$\mu_\Omega(A_e) = \mathbb{L}^{ed} \int_B \mathbb{L}^{-\operatorname{ord}_t \bar{p}\circ H_2(\gamma) - \operatorname{ord}_t \mathcal{J}_{H_2}(\gamma)}\, d\mu_e(\gamma).$$

It remains to compare the subintegral expressions. We need to show that

$$M - (\operatorname{ord}_t \tilde{p} \circ H_1(\gamma) + \operatorname{ord}_t \mathcal{J}_{H_1}(\gamma)) = e - (\operatorname{ord}_t \bar{p} \circ H_2(\gamma) + \operatorname{ord}_t \mathcal{J}_{H_2}(\gamma))$$

for $\gamma \in B$. This equality immediately follows from (7) and the formulas for $\mathcal{J}_{H_1}$ and $\mathcal{J}_{H_2}$.

We have shown that $\mu_\Omega(A_e) = \mu_\Omega(g^{-1}A_e)$ for $e = 0, 1, \ldots$. Hence, by the additivity of the measure, $\mu_\Omega(A) = \mu_\Omega(g^{-1}A)$. ■

It is relevant to mention the alternative approach to motivic measure, based on motivic integration for formal schemes [13] here. The following remark is due to discussions with J.-K. Yu.

Remark 23 In our construction, we interpret $G(k((t)))$ as the set of k-points of an ind-scheme. We are using the invariant differential form on G, whose explicit expression in terms of the coordinates on the big cell is known, to define the motivic measure. The main difficulty, as we just saw in the proof of the above proposition, is that we naturally encounter schemes over $\mathbb{D}$ that have a singularity in the closed fiber. If the same construction were carried out in the context of formal schemes and rigid analytic spaces, this difficulty would have been avoided: thanks to the use of Néron models in [13], one could define the motivic measure on the analogues of our auxilliary schemes $\mathfrak{X}_e$ without having to deal with singular objects. However, to carry this out precisely, the results of [13] would have to be extended to allow us to work on the level of k-points rather than with the objects themselves (a procedure similar to Looijenga's generalization of the original work of Denef and Loeser).

Now we return to our construction and extend Proposition 22 to all measurable sets, not only bounded ones.

Theorem 24 *The measure μ_Ω is translation-invariant (both on the left and on the right).*

Proof We will prove left-invariance; right-invariance is proved identically. Let A be an Ω-measurable subset of $G((t))$, and $g \in G(F)$. We need to show that $\mu_\Omega(A) =$

$\mu_\Omega(g^{-1}A)$. We can assume that A is *bounded* Ω-measurable without loss of generality, since any unbounded Ω-measurable set by definition can be represented as a countable disjoint union of bounded Ω-measurable sets.

Let us break up the set $g^{-1}A$ according to the maximal order of pole of the coordinates of its points: $g^{-1}A = \bigcup_{n=0}^{\infty} B_n \cup B_\infty$, where

$$B_0 = g^{-1}A \cap \mathfrak{L}(G),$$
$$B_n = \{\gamma \in g^{-1}A \mid \gamma \in \mathfrak{C}_G(\Omega); i(\gamma) \in (\mathbb{A}^d)^{(N)} \setminus (\mathbb{A}^d)^{(N-1)}\}, \quad n \geq 1,$$
$$B_\infty = \{\gamma \in g^{-1}A \mid \gamma \notin \mathfrak{C}_G(\Omega)\}.$$

Then $\mu_\Omega(gB_n) = \mu_\Omega(B_n)$ for $n \geq 0$ by Proposition 22; $\mu_\Omega(B_\infty) = 0$ by definition. It remains to show that $\mu_\Omega(gB_\infty) = 0$. Then we will have

$$\mu_\Omega(g^{-1}A) = \sum_{n=1}^{\infty} \mu_\Omega(B_n) + \mu_\Omega(B_\infty) = \sum_{n=1}^{\infty} \mu_\Omega(gB_n) + \mu_\Omega(gB_\infty) = \mu_\Omega(A).$$

The set gB_∞ is contained in the set $E = gZ((t)) \cap \mathfrak{C}_G(\Omega)$, so it suffices to show that the set E has measure 0. We can represent it as a disjoint union of bounded subsets of $\mathfrak{C}_G(\Omega)$: $E = \bigcup_{N=0}^{\infty} E_N$ with $E_0 = E \cap \mathfrak{L}(G)$ and $E_N = E \cap (\Omega^{(N)} \setminus \Omega^{(N-1)})$ for $N \geq 1$. It remains to observe that $S_N(i(E_N))$ is well defined and it is a locally closed subscheme of $\mathfrak{L}(\mathbb{A}^d)$. Its relative dimension over $k[[t]]$ is less than d, and therefore by definition of the measure on the affine space we have $\mu_a(S_N(i(E_N))) = 0$. This implies $\mu_\Omega(E_N) = 0$ for all $N \geq 0$; hence $\mu_\Omega(E) = 0$. ■

Corollary 25 *The algebra of Ω-measurable sets and the measure μ_Ω itself do not depend on the choice of the torus T or the set of positive roots (that is, Ω can be dropped from the notation).*

Proof This follows from the theorem and the fact that all the big cells are conjugate in G over k (recall that we are assuming k to be algebraically closed). ■

2.6

As stated in the introduction, the goal was to define a motivic measure on $G((t))$ that would extend the canonical motivic measure on $\mathfrak{L}(G)$. The following theorem shows that we have achieved it.

Theorem 26 *Let Ω be any big cell in the group G. Then $\mathfrak{L}(G)$ is Ω-measurable, and the restriction of μ_Ω to $\mathfrak{L}(G)$ coincides with the canonical motivic measure on $\mathfrak{L}(G)$.*

Proof Let us denote the canonical motivic measure on $\mathfrak{L}(G)$ by μ_G. Denote the complement of Ω in G by Z, as before. First, notice that $\mu_G(\mathfrak{L}(Z)) = 0$ by the axioms of the canonical measure; $\mathfrak{L}(G) = \mathfrak{L}(Z) \cup (\mathfrak{C}_G(\Omega) \cap \mathfrak{L}(G))$, and $\mu_\Omega(Z) = 0$ by definition of μ_Ω. Therefore, we only need to show that the restrictions of μ_Ω and μ_G to $\mathfrak{C}_G(\Omega) \cap \mathfrak{L}(G)$ coincide (and are defined on the same algebra of sets).

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Consider the multiplication map $U^- \times U \times T \to G$ over k. This map is an isomorphism between $\mathbb{A}^n \times \mathbb{A}^n \times \mathbb{G}_m^m$ and Ω over k. It induces an isomorphism (over $k[[t]]$) of the arc spaces: $\mathfrak{L}(\mathbb{A}^n) \times \mathfrak{L}(\mathbb{A}^n) \times \mathfrak{L}(\mathbb{G}_m^m) \to \mathfrak{L}(\Omega)$. If we apply the transformation rule to this isomorphism, we immediately obtain that the restrictions of μ_Ω and μ_G to $\mathfrak{L}(\Omega)$ coincide, by Example 8 and the observation that $\operatorname{ord}_t(s_1 \dots s_m) = 1$ on $\mathbb{G}_m^m[[t]]$.

Since $\mathfrak{L}(\Omega)$ is a smaller set than $\mathfrak{C}_G(\Omega) \cap \mathfrak{L}(G)$, the equality between the two measures restricted to $\mathfrak{L}(\Omega)$ is not enough. However, we claim that a finite number of translates of $\mathfrak{L}(\Omega)$ cover the whole arc space of G, and then the theorem follows immediately.

The claim can be proved, for example, as follows. At first consider the situation over k. All possible big cells cover the group $G(k)$ (recall that k is assumed algebraically closed, and hence even Borel subgroups cover $G(k)$). Since G is quasicompact in Zariski topology, and the big cells are Zariski open, there exists a finite subcover by some big cells $\Omega_1(k), \dots, \Omega_n(k)$. The arc space $\mathfrak{L}(G)$ itself is stable at level 0 since G is a smooth variety, and so are $\mathfrak{L}(\Omega_1), \dots, \mathfrak{L}(\Omega_n)$, by Remark 2. In particular, $\mathfrak{L}(G) = \pi_0^{-1}(G)$, $\mathfrak{L}(\Omega_i) = \pi_0^{-1}(\Omega_i)$, $i = 1, \dots, n$. It follows that $\bigcup_{i=1}^n \mathfrak{L}(\Omega_i) = \mathfrak{L}(G)$. Hence, any μ_G-measurable subset A of $\mathfrak{L}(G)$ can be broken up into a disjoint union $A = \bigcup_{i=1}^n A_i$ with $A_i \subset \mathfrak{L}(\Omega_i)$, Ω_i-measurable. By Corollary 25, any Ω_i-measurable set is also Ω-measurable, and $\mu_\Omega(A_i) = \mu_{\Omega_i}(A_i)$ for any $i = 1, \dots, n$. On the other hand, we have shown in the beginning of this proof that $\mu_{\Omega_i}(A_i) = \mu_G(A_i)$. Hence, $\mu_\Omega(A) = \sum_{i=1}^n \mu_G(A_i) = \mu_G(A)$. ■

Remark 27 1. It is possible to construct explicitly the finite number of translates of the given big cell Ω that cover $\mathfrak{L}(G)$. It can be done by means of Bruhat decomposition and the following statement [3, §2.1, p. 43]: if w, s are elements of the Weyl group of G satisfying $l(s) = 1$ and $l(sw) = l(w) + 1$, and $n \in G$ is a representative of s, then $nBwB$ is contained in $B(sw)B$ (here B is a fixed Borel subgroup, and $l(w)$ stands for length of w).

2. The statement of the last theorem can be proved directly by a Jacobian calculation in a way similar to the proof of Proposition 22. Namely, after having established the equality of the two measures on $\mathfrak{L}(\Omega)$, we could subdivide the remaining part of $\mathfrak{L}(G) \cap \Omega((t))$ into a disjoint union of subsets according to the order of pole of Ω-coordinates of its elements, and then repeat the procedure described in Proposition 22: construct an auxilliary $\mathbb{D}$-variety corresponding to each piece with a given order of pole and a $k[[t]]$-morphism from it to $\mathfrak{L}(G)$ which corresponds to the natural inclusion of the big cell into G. A complicated calculation shows that the Jacobian ideal of this morphism coincides with the principal ideal generated by $(s_1 \cdots s_m)$ (recall that $s_1, \dots, s_m$ are the coordinates of the torus component of the given element of the big cell). Then the statement follows from the Jacobian transformation rule applied to this morphism.

2.7 Concluding Remarks

Finally, I would like to mention briefly a few closely related questions which have not been discussed so far, and which hopefully will be addressed in the future.

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2.7.1 Uniqueness

The classical Haar measure is unique up to a scalar multiple. The canonical motivic measure on $\mathfrak{L}(G)$ is unique because it is normalized in such a way that it projects to the tautological measure on the variety G. Our construction of the motivic measure on $G((t))$ gives an answer that does not depend on the choice of the big cell (Corollary 25) and coincides with the unique motivic measure on $\mathfrak{L}(G)$. Since we do not have an axiomatic measure theory for motivic measures on ind-schemes in general, it is hard to formulate a uniqueness statement for motivic Haar measure. The most general approach to motivic Haar measure probably requires the context of rigid analytic spaces and formal schemes, and so we will not address the issue of uniqueness here.

2.7.2

The assumptions that the ground field k is algebraically closed and has characteristic 0 were adopted because we followed the exposition of [14] where these assumptions were made. However, it should be possible to extend our result without any difficulty to the case when k is not algebraically closed but the group G is assumed split over F. It would also be interesting to construct a motivic Haar measure for reductive groups that are not split over F, which would probably require the context of [13].

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