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Einstein-Like Lorentz Metrics and Three-Dimensional Curvature Homogeneity of Order One

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Abstract. We completely classify three-dimensional Lorentz manifolds, curvature homogeneous up to order one, equipped with Einstein-like metrics. New examples arise with respect to both homogeneous examples and three-dimensional Lorentz manifolds admitting a degenerate parallel null line field.

1 Introduction

A pseudo-Riemannian manifold (M, g) is said to be *curvature homogeneous up to order k* if, for any points $p, q \in M$, there exists a linear isometry $\phi\colon T_pM \to T_qM$ such that $\phi * (\nabla^i R(q)) = \nabla^i R(p)$ for all $i \leq k$. A locally homogeneous space is curvature homogeneous of any order k. Conversely, if k is sufficiently high, curvature homogeneity up to order k implies local homogeneity. This result was proved by Singer [15] for Riemannian manifolds. Through the equivalence theorem for G-structures due to Cartan and Sternberg [16], Singer's result extends to the pseudo-Riemannian case.

Given a pseudo-Riemannian manifold (M, g), its *Singer index* k_M is the smallest integer such that curvature homogeneity up to order $k > k_M$ implies local homogeneity. If $\dim M = 2$, then curvature homogeneity (up to order 0) already implies local homogeneity. In [14], K. Sekigawa proved that a three-dimensional Riemannian manifold, which is curvature homogeneous up to order one, is locally homogeneous. Bueken and Djorić [3] determined all three-dimensional Lorentzian manifolds which are curvature homogeneous up to order one and showed that only curvature homogeneity up to order two is sufficient for a three-dimensional Lorentzian manifold to be locally homogeneous.

Interesting relationships have been showed in the Riemannian case between Einstein-like metrics and homogeneity properties. *Einstein-like* metrics were first introduced by A. Gray in [11]. They are defined through conditions on the Ricci tensor, and their definition extends at once to pseudo-Riemannian manifolds. Precisely, we have:

Class $\mathcal{A}$: A pseudo-Riemannian manifold (M, g) belongs to class $\mathcal{A}$ if and only if its Ricci tensor ϱ is *cyclic-parallel*, that is,

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$$(1.1) \qquad (\nabla_X \varrho)(Y,Z) + (\nabla_Y \varrho)(Z,X) + (\nabla_Z \varrho)(X,Y) = 0,$$

for all vector fields X, Y, Z tangent to M. (1.1) is equivalent to requiring that ϱ is a *Killing tensor*, that is,

$$(1.2) \qquad (\nabla_X \varrho)(X,X) = 0.$$

Class $\mathcal{B}$: (M,g) belongs to class $\mathcal{B}$ if and only if its Ricci tensor is a *Codazzi tensor*, that is,

$$(1.3) \qquad (\nabla_X \varrho)(Y,Z) = (\nabla_Y \varrho)(X,Z).$$

Any manifold belonging to either class $\mathcal{A}$ or $\mathcal{B}$ has constant scalar curvature. Moreover, for the class $\mathcal{P}$ of Ricci-parallel manifolds and the class $\mathcal{E}$ of Einstein spaces, one has $\mathcal{A} \cap \mathcal{B} = \mathcal{P} \supset \mathcal{E}$. However, $\mathcal{P} \neq \mathcal{E}$. In particular, in the pseudo-Riemannian settings there exist plenty of manifolds with parallel Ricci tensor that are neither Einstein nor locally decomposable. More details and some interesting Riemannian examples can be found in [11].

Several authors have studied Einstein-like metrics in different classes of Riemannian manifolds [1, 2, 4, 5, 9, 13]. Recently, Einstein-like metrics have been studied in some classes of three-dimensional Lorentz manifolds including manifolds admitting a parallel null vector field in [10] and homogeneous manifolds in [7]. Note that three-dimensional manifolds are natural candidates for a deep investigation of Einstein-like metrics, because in dimension three the curvature is completely determined by the Ricci tensor.

In [2] it was proved that three-dimensional (connected, simply connected) homogeneous Riemannian manifolds belong to class $\mathcal{A}$ (respectively, class $\mathcal{B}$), if and only if they are *naturally reductive* (respectively, *symmetric*). Since three-dimensional Riemannian spaces with curvature homogeneous up to order one, are locally homogeneous [14], the first natural extension to consider was given by curvature homogeneous spaces (up to order zero). This was studied in [4], but did not lead to new examples. In fact, a three-dimensional curvature homogeneous Riemannian space, equipped with an Einstein-like metric belonging to class $\mathcal{A}$ (respectively, class $\mathcal{B}$), is locally isometric to a naturally reductive space (respectively, to a locally symmetric space). In particular, it is locally homogeneous. An even stronger result holds for class $\mathcal{A}$, since any Riemannian three-manifold inside this class is locally homogeneous [13].

The situation appears quite different in the Lorentz framework. In fact, the author classified Einstein-like Lorentz metrics on three-dimensional homogeneous manifolds in [7] and found that, besides naturally reductive and symmetric examples, some exceptional cases arise. Since curvature homogeneity up to order two is needed to ensure the local homogeneity of a three-dimensional Lorentz manifold [3], curvature homogeneity of order one is the first step below homogeneity to look at. The aim of this paper is to provide the classification of three-dimensional curvature homogeneous up to order one Lorentz manifolds, equipped with Einstein-like metrics. A remarkable difference arises between the Riemannian and Lorentzian cases. In fact, there exist two classes of nonhomogeneous Lorentz three-manifolds, curvature homogeneous up to order one [3], to which we refer here as (M_1, g) and (M_2, g) respectively. For (M_1, g), the assumption that the Lorentz metric is Einstein-like is not sufficient to ensure local homogeneity, and there is a large family of Lorentz three-manifolds of type (M_1, g), depending on a smooth function and a real constant, equipped with Einstein-like Lorentz metrics and different from the previous examples found in [7, 10].

The paper is organized in the following way. In Section 2, we recall some basic facts concerning three-dimensional Lorentz manifolds, and describe (M_1, g) and (M_2, g). The classification of Einstein-like metrics on M_1 and M_2 is provided in Sections 3 and 5, respectively. In Section 4, we classify Lorentz metrics in M_1 admitting a parallel degenerate line field. Section 6 concludes the paper with some remarks about conformally flat metrics on three-dimensional Lorentz manifolds M_1.

2 Curvature Homogeneity of Order One for Lorentz Three-Manifolds

Let (M, g) be a three-dimensional Lorentz manifold, ∇ its Levi-Civita connection and R its curvature tensor, taken with the sign convention

$$R(X,Y)Z = -\nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z + \nabla_{[X,Y]} Z.$$

The curvature of (M, g) is completely determined by the Ricci tensor ϱ defined, for any point $p \in M$ and $X, Y \in T_pM$, by

$$\varrho(X,Y)_p = \sum_{i=1}^{3} \varepsilon_i g(R(X,e_i)Y, e_i), \tag{2.1}$$

where $\{e_1, e_2, e_3\}$ is a pseudo-orthonormal basis of T_pM and $\varepsilon_i = g_p(e_i, e_i) = \pm 1$ for all i. Throughout the paper, if not stated otherwise, we shall assume that e_3 is *timelike*, that is, $g(e_1, e_1) = g(e_2, e_2) = -g(e_3, e_3) = 1$.

Because of the symmetries of the curvature tensor, the Ricci tensor ϱ is symmetric [12]. So, the *Ricci operator* Q, defined by $g(QX, Y) = \varrho(X, Y)$, is self-adjoint. In the Riemannian case, there always exists an orthonormal basis diagonalizing Q, while in the Lorentz case four different cases can occur [3, 12], and there exists a pseudo-orthonormal basis $\{e_1, e_2, e_3\}$, with e_3 timelike, such that Q takes one of the following forms, called *Segre types*:

$$\begin{aligned} &\text{Segre type}\{11,1\}: \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}, \qquad \text{Segre type}\{1z\bar{z}\}: \begin{pmatrix} a & 0 & 0 \\ 0 & b & c \\ 0 & -c & b \end{pmatrix}, \\ &\text{Segre type}\{3\}: \begin{pmatrix} b & a & -a \\ a & b & 0 \\ a & 0 & b \end{pmatrix}, \qquad \text{Segre type}\{21\}: \begin{pmatrix} a & 0 & 0 \\ 0 & b & \eta \\ 0 & -\eta & b-2\eta \end{pmatrix}, \end{aligned} \tag{2.2}$$

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where $\eta = \pm 1$. If (M, g) is curvature homogeneous, then Q has the same Segre type at any point $p \in M$ and, at least locally, there exists a pseudo-orthonormal frame field $\{e_1, e_2, e_3\}$ for which the Ricci operator is described by one of forms in (2.2), for some constants a, b, and c. When the components of $\nabla \varrho$ with respect to $\{e_1, e_2, e_3\}$ are constant, (M, g) is curvature homogeneous up to order one.

The author completely classified homogeneous Lorentz three-manifolds in [6], and in [7] he classified Einstein-like metrics on these manifolds. On the other hand, Lorentz three-manifolds with curvature homogeneous up to order one, have been investigated by Bueken and Djorić in [3]. They proved that there exist exactly two classes of proper (that is, nonhomogeneous) curvature homogeneous of order one Lorentz three-manifolds, corresponding to some special cases of Segre types $\{21\}$ and $\{11, 1\}$. We report their description here.

Segre type $\{21\}$: When $a = b + \eta$, a three-dimensional Lorentz manifold (M_1, g) is curvature homogeneous up to order one if and only if there exists (at least locally) a pseudo-orthonormal basis $\{e_1, e_2, e_3\}$, with e_3 timelike, two constants C and D and a function γ, such that

$$\begin{aligned} [e_1, e_2] &= -(\gamma + D)e_2 + \eta(C - \gamma)e_3, \\ [e_1, e_3] &= \eta(C + \gamma)e_2 + (\gamma - D)e_3, \\ [e_2, e_3] &= 0, \end{aligned} \tag{2.3}$$

and

$$\begin{cases} e_1(\gamma) = \eta - 2\gamma(C + D), \\ e_2(\gamma) + \eta\, e_3(\gamma) = 0. \end{cases} \tag{2.4}$$

In particular, (M_1, g) is locally homogeneous if and only if γ is constant (or $C = D = 0$ and γ satisfies (3.1), as we shall remark on Section 3).

Segre type $\{11, 1\}$: when $c = b \neq a$, a three-dimensional Lorentz manifold (M_2, g) is curvature homogeneous up to order one if and only if there exists (at least locally) a pseudo-orthonormal basis $\{e_1, e_2, e_3\}$, with e_3 timelike, a constant G and a function I, such that

$$\begin{aligned} [e_1, e_2] &= -e_2 - (G + 2)e_3, \\ [e_1, e_3] &= -Ge_2 + e_3, \\ [e_2, e_3] &= 2(G + 1)e_1 - Ie_2 - Ie_3, \end{aligned} \tag{2.5}$$

and

$$\begin{cases} a = -2(G + 1)^2, \\ b = -(e_2 + e_3)(I), \\ e_1(I) = I(G + 1). \end{cases} \tag{2.6}$$

In particular, (M_2, g) is locally homogeneous if and only if I is constant.

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Remark 2.1 In [3], the case of Segre type $\{11,1\}$ described here by (2.5) and (2.6) is written in exactly the same way, even if later on the authors pass to a "null" frame field. The case of Segre type $\{21\}$ described here by (2.3) and (2.4), is presented in [3] directly in terms of a null frame field. Here, we choose to use a pseudo-orthonormal frame field, because such a frame was used in [6] for the description of the homogeneous models. Starting from the null frame field $\{E_1, E_2, E_3\}$ used in [3], the pseudo-orthonormal frame field $\{e_1, e_2, e_3\}$ we used in (2.3) and (2.4) is defined as follows:

$$e_1 = E_1, \quad e_2 = \eta\frac{E_2 + E_3}{\sqrt{2}}, \quad e_3 = \frac{E_2 - E_3}{\sqrt{2}}.$$

In the sequel, we shall denote by (M_1, g) a three-dimensional Lorentz manifold, curvature homogeneous up to order one, described by (2.3) and (2.4), and by (M_2, g) a three-dimensional Lorentz manifold, curvature homogeneous up to order one, described by (2.5) and (2.6).

Let (M, g) be a three-dimensional Lorentz manifold. In order to describe the curvature of (M, g) with respect to a pseudo-orthonormal frame $\{e_i\}$, we put

$$\nabla_{e_i} e_j = \sum_k \varepsilon_j B_{ijk} e_k.$$

Functions B_{ijk} completely determine the Levi-Civita connection of (M, g). Note that from $\nabla g = 0$ it follows that $B_{ikj} = -B_{ijk}$, for all i, j, k. In particular, $B_{ijj} = 0$ for all indices i and j. Functions B_{ijk} are determined by the expression of the Lie brackets of vectors e_1, e_2, e_3 and conversely, since the well-known *Koszul formula* yields

$$2\varepsilon_j\varepsilon_k B_{ijk} = 2g(\nabla_{e_i} e_j, e_k) = g([e_i, e_j], e_k) - g([e_j, e_k], e_i) + g([e_k, e_i], e_j). \tag{2.7}$$

As concerns the covariant derivative of the Ricci tensor, easy calculations show that

$$\nabla_i \varrho_{jk} = -\sum_t \left(\varepsilon_j B_{ijt}\varrho_{tk} + \varepsilon_k B_{ikt}\varrho_{tj}\right) \tag{2.8}$$

for all indices i, j, k. We now treat the cases of (M_1, g) and (M_2, g) separately.

Curvature of $(\mathbf{M_1}, \mathbf{g})$: Consider a three-dimensional Lorentz manifold (M_1, g) described by (2.3) and (2.4). Starting from (2.3), we can use (2.7) to determine the Levi-Civita connection of (G, g). Standard calculations then give

$$\begin{aligned} &\nabla_{e_1} e_1 = 0, && \nabla_{e_2} e_1 = (\gamma + D)e_2 + \eta\gamma e_3, && \nabla_{e_3} e_1 = -\eta\gamma e_2 + (D - \gamma)e_3, \\ &\nabla_{e_1} e_2 = \eta C e_3, && \nabla_{e_2} e_2 = -(\gamma + D)e_1, && \nabla_{e_3} e_2 = \eta\gamma e_1, \\ &\nabla_{e_1} e_3 = \eta C e_2, && \nabla_{e_2} e_3 = \eta\gamma e_1, && \nabla_{e_3} e_3 = (D - \gamma)e_1. \end{aligned} \tag{2.9}$$

Using (2.9) and the definition of the curvature tensor (and taking into account (2.4)), we easily get

$$R(e_1, e_2)e_1 = -(D^2 + \eta)e_2 - e_3, \quad R(e_1, e_3)e_3 = (\eta - D^2)e_1, \quad R(e_2, e_3)e_2 = -a^2 e_3,$$

that is,

$$R_{1212} = -D^2 - \eta, \qquad R_{1313} = D^2 - \eta, \qquad R_{2323} = D^2,$$
$$R_{1213} = 1 \qquad R_{1223} = 0, \qquad R_{1323} = 0.$$

From (2.1) it then follows that the Ricci components are given by

$$\varrho_{ij} = \begin{pmatrix} -2D^2 & 0 & 0 \\ 0 & -2D^2 - \eta & 1 \\ 0 & 1 & 2D^2 - \eta \end{pmatrix} \tag{2.10}$$

(according to Segre type $\{21\}$ for the Ricci operator, and condition $a = b + \eta$). Finally, using (2.9) and (2.10) in (2.8), we find that the only possibly non-vanishing components of $\nabla \varrho$ are

$$\begin{aligned} &\nabla_1 \varrho_{22} = \nabla_1 \varrho_{33} = -2\eta C, && \nabla_1 \varrho_{23} = 2C, \\ &\nabla_2 \varrho_{12} = \eta D, && \nabla_2 \varrho_{13} = -D, \\ &\nabla_3 \varrho_{12} = -D, && \nabla_3 \varrho_{13} = \eta D, \end{aligned} \tag{2.11}$$

and the ones obtained from them using the symmetries of $\nabla \varrho$.

Curvature of $(\mathbf{M_2}, \mathbf{g})$: Let (M_2, g) be described by (2.5) and (2.6). We can proceed exactly as in the previous case. So, from (2.5) and (2.7) we obtain

$$\begin{aligned} &\nabla_{e_1} e_1 = 0, && \nabla_{e_2} e_1 = e_2 + (G+2)e_3, && \nabla_{e_3} e_1 = Ge_2 - e_3, \\ &\nabla_{e_1} e_2 = 0, && \nabla_{e_2} e_2 = -e_1 - Ie_3, && \nabla_{e_3} e_2 = -Ge_1 + Ie_3, \\ &\nabla_{e_1} e_3 = 0, && \nabla_{e_2} e_3 = (G+2)e_1 - Ie_2, && \nabla_{e_3} e_3 = -e_1 + Ie_2. \end{aligned} \tag{2.12}$$

From (2.12) and the definition of the curvature tensor, we get

$$\begin{aligned} R(e_1, e_2)e_1 &= -(G+1)^2 e_2, \\ R(e_1, e_3)e_3 &= -(G+1)^2 e_1, \\ R(e_2, e_3)e_2 &= \big(G+1)^2 - (e_2 + e_3)(I) \big) e_3, \end{aligned}$$

that is, taking into account (2.6),

$$\begin{aligned} &R_{1212} = -(G+1)^2 = \frac{a}{2}, && R_{1213} = 0, \\ &R_{1313} = (G+1)^2 = -\frac{a}{2}, && R_{1223} = 0, \\ &R_{2323} = (G+1)^2 - (e_2 + e_3)(I) = -\frac{a}{2} - b, && R_{1323} = 0. \end{aligned} \tag{2.13}$$

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From (2.1) and (2.13) we have

$$\varrho_{ij} = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & -b \end{pmatrix} \tag{2.14}$$

(according to Segre type $\{11, 1\}$ and condition $c = b \neq a$ for the eigenvalues of the Ricci operator). We can now use (2.12) and (2.14) in (2.8), and we obtain that the only possibly non-vanishing components of $\nabla\varrho$ are

$$\begin{aligned} &\nabla_2\varrho_{12} = b - a, && \nabla_2\varrho_{13} = (a - b)(G + 2), \\ &\nabla_3\varrho_{12} = (b - a)G, && \nabla_3\varrho_{13} = b - a, \end{aligned} \tag{2.15}$$

and the ones obtained by them using the symmetries of $\nabla\varrho$.

3 Einstein-Like Lorentz Metrics on M_1

As is well known, a three-dimensional pseudo-Riemannian manifold (M, g) is locally symmetric if and only if it is Ricci-parallel. As concerns (M_1, g), (2.11) implies at once that (M_1, g) is locally symmetric if and only if $C = D = 0$, *for any function* γ *satisfying*

$$e_1(\gamma) = \eta, \quad e_2(\gamma) + e_3(\gamma) = 0. \tag{3.1}$$

Since a locally symmetric space is locally homogeneous, but a function γ satisfying the first equation in (3.1) cannot be constant, this case is missing in the characterization given in [3] of locally homogeneous spaces of the form (M_1, g). Hence, Corollary 4 in [3] can be corrected in the following way.

Proposition 3.1 *A Lorentz manifold* (M_1, g)*, described by* (2.3) *and* (2.4)*, is locally homogeneous if and only if either* γ *is constant, or* $C = D = 0$ *and* γ *satisfies* (3.1).

Proposition 3.1 also agrees with the general result of [6], where the author proved that any three-dimensional homogeneous Lorentz manifold is either symmetric or is a Lie group equipped with a left-invariant Lorentz metric. More precisely, if $C = D = 0$ and γ satisfies (3.1), then it is easy to check using (2.9) that (M_1, g) admits a parallel null vector field $u = e_2 + \eta e_3$. To our knowledge, locally symmetric Lorentz three-manifolds admitting a parallel null vector field were first studied in [10].

We now determine manifolds of the form (M_1, g) belonging to class $\mathcal{A}$. Expressing (1.1) (equivalently, (1.2)) with respect to the pseudo-orthonormal frame $\{e_i\}$, we find that (M_1, g) belongs to class $\mathcal{A}$ if and only if

$$\nabla_i\varrho_{jk} + \nabla_j\varrho_{ik} + \nabla_k\varrho_{ij} = 0, \tag{3.2}$$

for all i, j, k. Taking into account (2.11), (3.2) holds if and only if

$$\begin{cases} 0 = \nabla_1\varrho_{22} + 2\nabla_2\varrho_{12} = -2\eta(C - D), \\ 0 = \nabla_1\varrho_{33} + 2\nabla_3\varrho_{13} = -2\eta(C - D), \\ 0 = \nabla_1\varrho_{23} + \nabla_2\varrho_{13} + \nabla_3\varrho_{12} = 2(C - D), \end{cases}$$

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that is, $C = D$, for any function γ (satisfying (2.4)). Hence, we have proved the following.

Theorem 3.2 *A Lorentz manifold* (M_1, g)*, described by* (2.3) *and* (2.4)*, belongs to class* $\mathcal{A}$ *if and only if* $C = D$.

Note that, by Proposition 3.1, for any constant $C = D \neq 0$ and non-constant function γ (satisfying (2.4)), (M_1, g) is a nonhomogeneous space. Therefore, in this case (M_1, g) belongs to class $\mathcal{A}$ but is not one of the homogeneous examples described in [7]. As we shall verify in the next section, unless it is symmetric, such a manifold does not even admit a parallel degenerate line field and so, it is not one of the examples already described in [10]. Theorem 3.2 shows a clear difference between the Lorentzian case and the Riemannian one, since Riemannian three-manifolds inside class $\mathcal{A}$ are locally homogeneous [13].

Next, we determine manifolds of the form (M_1, g) belonging to class $\mathcal{B}$. Expressing (1.3) with respect to the pseudo-orthonormal frame $\{e_i\}$, we easily find that (M_1, g) belongs to class $\mathcal{B}$ if and only if

$$\nabla_i \varrho_{jk} = \nabla_j \varrho_{ik}, \tag{3.3}$$

for all i, j, k. By (2.11), one can conclude at once that (3.3) holds if and only if $D = -2C$, for any function γ (satisfying (2.4)). So, we have proved the following.

Theorem 3.3 *A Lorentz manifold* (M_1, g)*, described by* (2.3) *and* (2.4)*, belongs to class* $\mathcal{B}$ *if and only if* $D = -2C$.

For any constant $D = -2C \neq 0$ and non-constant function γ (satisfying (2.4)), (M_1, g) is a nonhomogeneous three-dimensional Lorentz manifold (curvature homogeneous up to order one) belonging to class $\mathcal{B}$. So, it is not an example listed in [7]. Moreover, as we shall see in the next section, unless it is symmetric, this manifold does not admit a parallel degenerate line field.

4 (M_1, g) Admitting a Parallel Degenerate Line Field

A *parallel degenerate line field* over a Lorentz manifold (M, g), is a one-dimensional distribution $\mathcal{D}$, such that $\nabla\mathcal{D} \subset \mathcal{D}$. A parallel degenerate line field is locally spanned by a nonvanishing null vector field u satisfying $\nabla u = \omega \otimes u$, where ω is a (local) 1-form over M. In particular, if $\omega = 0$, then u is a parallel null vector field over M.

Three-dimensional Lorentz manifolds admitting a parallel degenerate line field have been investigated in [10]. Such a manifold admits local coordinates (t, x, y) such that, with respect to the "null" local frame field $\{(\frac{\partial}{\partial t}), (\frac{\partial}{\partial x}), (\frac{\partial}{\partial y})\}$, the Lorentzian metric g and the Ricci operator are given by

$$g = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \varepsilon & 0 \\ 1 & 0 & f \end{pmatrix}, \quad Q = \begin{pmatrix} \frac{1}{2} f_{tt} & \frac{1}{2} f_{tx} & -\frac{1}{2\varepsilon} f_{xx} \\ 0 & 0 & \frac{1}{2\varepsilon} f_{tx} \\ 0 & 0 & \frac{1}{2\varepsilon} f_{tt} \end{pmatrix} \tag{4.1}$$

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for some function $f = f(t,x,y)$, where $\varepsilon = \pm 1$ and $U = (\frac{\partial}{\partial t})$ is a null vector field spanning a parallel degenerate line field. Starting from explicit expressions given in (4.1), many curvature properties of these manifolds have been investigated in [10]. In particular, Lorentz three-manifolds admitting a parallel degenerate line field, which are either locally symmetric or equipped with Einstein-like Lorentz metrics, have been completely described in [10].

It is easy to build a (local) pseudo-orthonormal frame field from $\{(\frac{\partial}{\partial t}), (\frac{\partial}{\partial x}), (\frac{\partial}{\partial y})\}$ and to check that in many cases, the Ricci operator described by (4.1) is of Segre type $\{21\}$. So, it is particularly interesting to check when (M_1, g) admits a parallel degenerate line field. The answer is provided by the following.

Theorem 4.1 *A Lorentz manifold (M_1, g), described by (2.3) and (2.4), admits a parallel degenerate line field if and only if $D = 0$. In this case, (M_1, g) also admits a parallel null vector field.*

Proof The "if" part follows almost immediately from (2.9). In fact, we can consider an arbitrary smooth function $\mu\colon M \to \mathbb{R}$, $\mu \neq 0$, and (at least, locally) the vector field $u = \mu(e_2 + \eta e_3)$. Then, $\|u\|^2 = 0$, that is, u is a null vector field. Moreover, assuming $D = 0$, by (2.9) we get

$$\nabla_{e_1} u = (\frac{1}{\mu} e_1(\mu) + C)u, \quad \nabla_{e_2} u = \frac{1}{\mu} e_2(\mu)u, \quad \nabla_{e_3} u = \frac{1}{\mu} e_3(\mu)u.$$

Therefore, u spans a parallel degenerate line field. In particular, choosing μ as a solution of the system of partial differential equations

$$\begin{cases} e_1(\mu) = -C\mu, \\ e_2(\mu) = 0, \\ e_3(\mu) = 0, \end{cases} \tag{4.2}$$

we obtain that $u = \mu(e_2 + \eta e_3)$ is a parallel null vector field. Note that (2.3) and (4.2) easily imply that integrability conditions for (4.2), namely, $[e_i, e_j](\mu) = e_i(e_j(\mu)) - e_j(e_i(\mu))$, are satisfied for all indeces i, j.

Conversely, suppose now that (M_1, g) admits a parallel degenerate line field. Hence, there exists (locally) a null vector field u and a 1-form ω, such that $\nabla u = \omega \otimes u$. With respect to the pseudo-orthonormal frame $\{e_1, e_2, e_3\}$ and its dual frame $\{\theta_1, \theta_2, \theta_3\}$, we can write $u = x_1 e_1 + x_2 e_2 + x_3 e_3$ and $\omega = \lambda_1 \theta_1 + \lambda_2 \theta_2 + \lambda_3 \theta_3$, for some smooth functions $x_i, \lambda_i : M \to \mathbb{R}$. Note that since u is a (nonvanishing) null vector field, we have $0 = ||u||^2 = x_1^2 + x_2^2 - x_3^2$ and so, $x_3 \neq 0$ at any point. Expressing condition $\nabla u = \omega \otimes u$ in terms of the pseudo-orthonormal frame $\{e_1, e_2, e_3\}$, we have $\nabla_{e_i} u = \lambda_i u$ for all $i = 1, 2, 3$, that is, using (2.9), functions x_i, λ_i must satisfy the following system of partial differential equations:

$$(4.3)\qquad \begin{cases} e_1(x_1) = \lambda_1 x_1, \\ e_1(x_2) = \lambda_1 x_2 - \eta C x_3, \\ e_1(x_3) = \lambda_1 x_3 - \eta C x_2, \\ e_2(x_1) = \lambda_2 x_1 + (\gamma + D) x_2 - \eta\gamma x_3, \\ e_2(x_2) = \lambda_2 x_2 - (\gamma + D) x_1, \\ e_2(x_3) = \lambda_2 x_3 - \eta\gamma x_1, \\ e_3(x_1) = \lambda_3 x_1 - \eta\gamma x_2 + (\gamma - D) x_3, \\ e_3(x_2) = \lambda_3 x_2 + \eta\gamma x_1, \\ e_3(x_3) = \lambda_3 x_3 + (\gamma - D) x_1. \end{cases}$$

Now, we can compute $[e_i, e_j](x_k)$, for all indices i, j, k, using both (4.3) and (2.3). In particular, we get

$$(4.4)\qquad \begin{cases} 0 = [e_2, e_3](x_2) = (e_2(\lambda_3) - e_3(\lambda_2))x_2 - D^2 x_3, \\ 0 = [e_2, e_3](x_3) = (e_2(\lambda_3) - e_3(\lambda_2))x_3 - D^2 x_2. \end{cases}$$

Since $x_3 \neq 0$ at any point, we can compute $e_2(\lambda_3) - e_3(\lambda_2)$ from the second equation in (4.4) and replace it in the first one. In this way, we obtain $D^2(x_2^2 - x_3^2) = 0$, that is, either $D = 0$ or $x_3 = \pm x_2$. We now prove that, even when $x_3 = \pm x_2$, we necessarily have $D = 0$. In fact, from $||u||^2 = 0$ and $x_3 = \pm x_2$ it follows $x_1 = 0$. When $x_3 = x_2$, taking into account $x_1 = 0$, the fourth equation in (4.3) gives at once $-Dx_2 = 0$ and so, $D = 0$ (since $x_2 = x_3 \neq 0$). When $x_3 = -x_2$, then the fourth and seventh equations in (4.3) become

$$(4.5)\qquad \begin{cases} (\gamma + \eta\gamma + D)x_2 = 0, \\ -(\gamma + \eta\gamma - D)x_2 = 0. \end{cases}$$

Since $x_2 = -x_3 \neq 0$, summing the two equations in (4.5), we have that $D = 0$. ■

Remark 4.2 Comparing the results of Theorem 4.1 with those of Theorem 3.2, we can conclude that when (M_1, g) belongs to class $\mathcal{A}$, it admits a parallel degenerate line field if and only if $C = D = 0$, that is, when (M_1, g) is locally symmetric. Therefore, whenever $C = D \neq 0$, the Lorentz manifold (M_1, g) belongs to class $\mathcal{A}$ but does not admit a parallel degenerate line field (and so, it is not an example described in [10]).

A similar argument, starting from Theorems 3.3 and 4.1, leads us to conclude that a Lorentz manifold (M_1, g) belonging to class $\mathcal{B}$ admits a parallel degenerate line field if and only if $D = -2C = 0$ (in particular, it is locally symmetric). Whenever $D = -2C \neq 0$, (M_1, g) belongs to class $\mathcal{B}$ but does not admit a parallel degenerate line field.

We end this section with the classification of Lorentz manifolds (M_2, g) admitting a parallel degenerate line field. As was proved in [10], a Lorentz metric described by (4.1) has constant Ricci eigenvalues and a diagonalizable Ricci operator if and only if

$$f(t, x, y) = kt^2 + tP(y) + x\eta(y) + \xi(y), \tag{4.6}$$

for some smooth functions P, η, ξ. Using (4.6) in (4.1), one easily finds that the Ricci eigenvalues are 0 and k, the latter of multiplicity two. Comparing these with the eigenvalues a, b of (M_2, g) as described in (2.6), we then get $G + 1 = 0$ (and $a = 0 \neq k = b$). Proceeding as in the proof of Theorem 4.1, via standard calculations we then obtain the following.

Theorem 4.3 *A Lorentz manifold (M_2, g) described by (2.5) and (2.6) admits a parallel degenerate line field if and only if $G = -1$.*

5 Einstein-Like Lorentz Metrics on M_2 and Classification Results

We first remark that (2.15) implies at once that (M_2, g) is locally symmetric if and only if $a = b$. In this case, all Ricci eigenvalues coincide, that is, (M_2, g) has constant sectional curvature $\frac{a}{2}$. However, since $a = b$ contradicts $c = b \neq a$, this possibility was correctly excluded in [3].

Next, (M_2, g) belongs to class $\mathcal{A}$ if and only if (3.2) is satisfied. Because of (2.15), (3.2) implies

$$0 = \nabla_1 \varrho_{22} + 2\nabla_2 \varrho_{12} = b - a$$

that is, $a = b$. Therefore, *a Lorentz manifold (M_2, g) described by (2.5) and (2.6) never belongs to class $\mathcal{A}$* (or, if we admit the possibility $a = b$, it belongs to class $\mathcal{A}$ only in the trivial case when (M_2, g) has constant sectional curvature).

In the same way, (M_2, g) belongs to class $\mathcal{B}$ if and only if (3.3) holds. Because of (2.15), we have that $\nabla_1 \varrho_{22} = \nabla_2 \varrho_{12}$ implies at once $0 = b - a$, that is, $a = b$. Hence, *a Lorentz manifold (M_2, g) described by (2.5) and (2.6) never belongs to class $\mathcal{B}$*.

It is not surprising that no exceptional examples arise among (M_2, g), because the case of a Lorentz manifold having a diagonal Ricci operator is the most similar to the Riemannian case. Since (M_1, g) and (M_2, g) are the only nonhomogeneous examples of Lorentz three-manifolds which are curvature homogeneous up to order one [3], taking into account the results of [7] we can state the following classification results.

Theorem 5.1 *The class of three-dimensional Lorentz manifolds curvature homogeneous up to order one and belonging to class $\mathcal{A}$ consists of*

- *Lorentz manifolds locally isometric to a naturally reductive space* [8],
- *locally homogeneous spaces, locally isometric to some homogeneous Lorentz manifolds which are not naturally reductive* [7],
- *M_1 with $C = D \neq 0$ and γ any non-constant function satisfying* (2.4).

Theorem 5.2 *The class of three-dimensional Lorentz manifolds curvature homogeneous up to order one and belonging to class $\mathcal{B}$ consists of*

- *locally symmetric spaces,*

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- *some locally homogeneous spaces which are not locally symmetric* [7],
- M_1 *with* $D = -2C \neq 0$ *and* γ *any non-constant function satisfying* (2.4).

The full classification of *locally homogeneous* Lorentz three-manifolds belonging to either class $\mathcal{A}$ or $\mathcal{B}$ can be found in [7], while three-dimensional naturally reductive Lorentzian spaces have been classified in [8]. Theorems 5.1 and 5.2 show that *Lorentz three-manifolds, curvature homogeneous up to order one and belonging to either class* $\mathcal{A}$ *or* $\mathcal{B}$ *need not even to be locally homogeneous*. Compared with the results of [4] and [13], Theorems 5.1 and 5.2 emphasize how different it is to consider Einstein-like metrics in the Lorentzian framework and in the Riemannian one.

6 Conformally Flat Lorentz Metrics on M_1

As is well known, a three-dimensional pseudo-Riemannian manifold (M, g) is *(locally) conformally flat* if and only if its *Schouten tensor c* vanishes, that is,

$$c(X, Y, Z) = (\nabla_X \varrho)(Y, Z) - (\nabla_Y \varrho)(X, Z) - \tfrac{1}{2}\big(g((\nabla_X \tau)Y, Z) - g((\nabla_Y \tau)X, Z)\big) = 0, \tag{6.1}$$

for all vector fields X, Y and Z, where τ denotes the *scalar curvature* of (M, g). It is also well known that whenever the scalar curvature τ is constant, (6.1) reduces exactly to (1.3), that is, a pseudo-Riemannian three-manifold (M, g) of constant scalar curvature is conformally flat if and only if it belongs to class $\mathcal{B}$. In particular, since the scalar curvature is defined by contraction of the Ricci tensor, it is constant on any curvature homogeneous pseudo-Riemannian manifold. Therefore, from Theorem 5.2 we get the following.

Theorem 6.1 *The class of conformally flat three-dimensional Lorentz manifolds with curvature homogeneous up to order one consists of*

- *locally symmetric spaces,*
- *some locally homogeneous spaces which are not locally symmetric* [7],
- M_1 *with* $D = -2C \neq 0$ *and* γ *a non-constant function satisfying* (2.4).

Theorem 6.1 confirms that conformal flatness is a weaker assumption in Lorentzian geometry than in the Riemannian framework. In fact, *locally symmetric spaces are the only conformally flat curvature homogeneous Riemannian manifolds* [9, 17]. Other interesting differences about conformal flatness in Riemannian and Lorentzian geometries were emphasized in [10].

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