ON THE SPECTRUM OF \( n \)-TUPLES OF \( p \)-HYPONORMAL OPERATORS

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1. Introduction. Let \( B(H) \) denote the algebra of operators (i.e., bounded linear transformations) on the Hilbert space \( H \). \( A \in B(H) \) is said to be \( p \)-hyponormal \((0 < p \leq 1)\), if \((AA^*)^p \leq (A^*A)^p\). (Of course, a 1-hyponormal operator is hyponormal.) The \( p \)-hyponormal property is monotonic decreasing in \( p \) and a \( p \)-hyponormal operator is \( q \)-hyponormal operator for all \( 0 < q \leq p \). Let \( A \) have the polar decomposition \( A = U |A| \), where \( U \) is a partial isometry and \( |A| \) denotes the (unique) positive square root of \( A^*A \). If \( A \) has equal defect and nullity, then the partial isometry \( U \) may be taken to be unitary. Let \( \mathcal{HU}(p) \) denote the class of \( p \)-hyponormal operators for which \( U \) in \( A = U |A| \) is unitary. \( f((1/2) \) operators were introduced by Xia and \( \mathcal{HU}(p) \) operators for a general \( 0 < p < 1 \) were first considered by Aluthge (see [1, 14]); \( \mathcal{HU}(p) \) operators have since been considered by a number of authors (see [3, 4, 5, 9, 10] and the references cited in these papers). Generally speaking, \( \mathcal{HU}(p) \) operators have spectral properties similar to those of hyponormal operators. Indeed, let \( A \in \mathcal{HU}(p) \), \((0 < p < 1/2)\), have the polar decomposition \( A = U |A| \), and define the \( \mathcal{HU}(p+1/2) \) operator \( \tilde{A} \) by \( \tilde{A} = |A|^{1/2} U |A|^{1/2} \). Let \( \tilde{A} = V |\tilde{A}| \) with \( V \) unitary and \( \tilde{A} \) be the hyponormal operator defined by \( \tilde{A} = |\tilde{A}|^{1/2} V |\tilde{A}|^{1/2} \). Then we have the following result.

**Lemma 0.** \( \sigma_\text{s}(A) = \sigma_\text{s}(\tilde{A}) \), where \( \sigma_\text{s} \) denotes either of the following: point spectrum, approximate point spectrum, eigenvalues of finite multiplicity, spectrum, Weyl spectrum, and essential spectrum.

Recall that an \( n \)-tuple \( \mathcal{A} = (A_1, A_2, \ldots, A_n) \) of operators is said to be doubly commuting if \( A_i A_j - A_j A_i = 0 \) and \( A_i^* A_j - A_j A_i^* = 0 \), for all \( 1 \leq i \neq j \leq n \). Doubly commuting \( n \)-tuples \( \mathcal{A} \) of operators in \( \mathcal{HU}(p) \) have been considered by Munee Cho in [3], where it is shown that a weak Putnam theorem holds for \( \mathcal{A} \) and that \( \mathcal{A} \) is jointly normaloid. In this note we study the relationship between the spectral properties of \( \mathcal{A} \) and \( \mathcal{A} = (\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_n) \), and prove that \( \sigma_\text{s}(\mathcal{A}) = \sigma_\text{s}(\tilde{A}) \), where \( \sigma_\text{s} \) is either the joint point spectrum or the joint approximate point spectrum or the joint (Taylor) spectrum. This then leads us to:

(a) \( \| \mathcal{A} \| = \| \tilde{\mathcal{A}} \| \);
(b) if \( \sigma(\mathcal{A}) \in \mathbb{R}^n \), then \( A_i \) is self-adjoint, for all \( 1 \leq i \leq n \).

We show that the (Cho-Takaguchi) joint Weyl spectrum of \( \mathcal{A} \) is contained in the (Taylor) spectrum \( \sigma(\mathcal{A}) \) of \( \mathcal{A} \) minus the set of isolated points of \( \sigma(\mathcal{A}) \) which are joint eigenvalues of finite multiplicity, and that \( \mathcal{A} \) and \( \tilde{\mathcal{A}} \) have the same (Harte) essential spectrum. We conclude this note with a result (in the spirit of Dash [8, Corollary 4.6]) on the joint eigenvalues of \( \mathcal{A} \) in the Calkin algebra.

We assume henceforth, without loss of generality, that \( 0 < p < 1/2 \). Most of the notation that we use in this note is standard (and usually explained at the first instance of

occurrence). The following theorem, the n-tuple version of the Berberian extension theorem, will play an important role in the sequel.

**Theorem B.** If \( \mathcal{A} = (A_1, A_2, \ldots, A_n) \) is an n-tuple of commuting operators on \( H \), then there exists a Hilbert space \( H^0 \supset H \) and an isometric \(*\)-isomorphism \( A_i \to A^0_i \), \( (1 \leq i \leq n) \), preserving order such that \( \sigma_p(A_i) = \sigma_p(A^0_i) = \sigma_A(\mathcal{A}) \) and \( \sigma_A(\mathcal{A}) = \sigma_A(A_1, A_2, \ldots, A_n) = \sigma_p(A_1^0, A_2^0, \ldots, A_n^0) = \sigma_p(\mathcal{A}^0) \). (Here \( \sigma_p(\mathcal{A}) \) and \( \sigma_A(\mathcal{A}) \) denote, respectively, the joint spectrum and the joint approximate point spectrum (defined below) of \( \mathcal{A} \).)

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2. Results. Throughout the following \( \mathcal{A} = (A_1, A_2, \ldots, A_2) \) will denote a doubly commuting (i.e., \( A_i A_j - A_j A_i = 0 \) and \( A_i^* A_j - A_j^* A_i = 0 \), for all \( 1 \leq i \neq j \leq n \)) n-tuple of \( \mathcal{H} U(p) \) operators \( A_i \), \( (1 \leq i \leq n) \). Given \( A_i = U_i |A_i| \), define \( \hat{A}_i \) by \( \hat{A}_i = |A_i|^{1/2} U_i |A_i|^{1/2} \); also, letting \( \hat{A}_i \) have the polar decomposition \( \hat{A}_i = V_i |\hat{A}_i| \), define \( \hat{A}_i \) by

\[
\hat{A}_i = |\hat{A}_i|^{1/2} V_i |\hat{A}_i|^{1/2} \quad (1 \leq i \leq n).
\]

The n-tuples \( \mathcal{A} \) and \( \mathcal{A}_1 \) are then defined by \( \mathcal{A} = (\hat{A}_1, \hat{A}_2, \ldots, \hat{A}_n) \) and \( \mathcal{A}_1 = (\hat{A}_1, \hat{A}_2, \ldots, \hat{A}_n) \).

**Lemma 1.** \( \mathcal{A} \) is doubly commuting \( \Rightarrow \mathcal{A} \) is doubly commuting \( \Rightarrow \mathcal{A}_1 \) is doubly commuting. Also, \( \mathcal{A} \) is doubly commuting \( \Rightarrow [A_i, |\hat{A}_i|] = 0 = [\hat{A}_i, |\hat{A}_j|] = 0 \), for \( 1 \leq i \neq j \leq n \), where \([A, B]\) denotes the commutator \( AB - BA \) of \( A \) and \( B \).

**Proof.** Given \( A_i = U_i |A_i| \) and \( \hat{A}_i = V_i |\hat{A}_i| \), the doubly commuting hypothesis on \( \mathcal{A} \) implies that

\[
[U_i, U_j] = [A_i, A_j] = [A_i, A_j^*] = 0,
\]

for all \( 1 \leq i \neq j \leq n \). (See [11, Theorems 2 and 4].) Consequently, \( \mathcal{A}_1 \) is doubly commuting and so

\[
[V_i, V_j] = [|\hat{A}_i|, |\hat{A}_j|] = [|\hat{A}_i|, V_j] = 0,
\]

for all \( 1 \leq i \neq j \leq n \). This implies that \( \mathcal{A}_1 \) is doubly commuting. The argument above also implies that \([A_i, \hat{A}_j] = [A_i, \hat{A}_j^*] = [\hat{A}_i, \hat{A}_j] = [\hat{A}_i, \hat{A}_j^*] = 0\), for all \( 1 \leq i \neq j \leq n \). Hence, also,

\[
[A_i, |\hat{A}_j|] = [\hat{A}_i, |\hat{A}_j|] = 0,
\]

for all \( 1 \leq i \neq j \leq n \).

In the following we shall denote the Taylor joint spectrum of \( \mathcal{A} \) by \( \sigma(\mathcal{A}) \). (See [13] for the definition of Taylor spectrum of a commuting n-tuple of operators.) We say that \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \), \( (\lambda_i \in \mathbb{C} \text{ for all } 1 \leq i \leq n) \), is in the joint approximate point spectrum \( \sigma_A(\mathcal{A}) \) of \( \mathcal{A} \) if there exists a sequence \( \{x_k\} \) of unit vectors in \( H \) such that

\[
\|(A_i - \lambda_i)x_k\| \to 0 \quad \text{as } \quad k \to \infty,
\]

for all \( 1 \leq i \leq n \); \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \), \( (\lambda_i \in \mathbb{C} \text{ for all } 1 \leq i \leq n) \), is in the joint point spectrum \( \sigma_p(\mathcal{A}) \) of \( \mathcal{A} \) if there exists a non-trivial vector \( x \in H \) such that

\[
(A_i - \lambda_i)x = 0, \quad \text{for all } \quad 1 \leq i \leq n.
\]
We say that $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ is in the normal point spectrum $\sigma_{np}(A)$ of $A$ if there exists a non-trivial vector $x \in H$ such that $(A_i - \lambda_i)x = 0 \iff (A_i - \lambda_i)^*x = 0$, for all $1 \leq i \leq n$.

**Lemma 2.** $\sigma_p(A) = \sigma_{np}(A) = \sigma_{np}(\tilde{A}) = \sigma_p(\tilde{A})$.

**Proof.** Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \sigma_p(A)$ and let $x \in H$ be such that $x \neq 0$ and $(A_i - \lambda_i)x = 0$, for all $1 \leq i \leq n$. It is easily seen that $\tilde{A}_i |\tilde{A}_i|^{1/2} |A_i|^{1/2} = |\tilde{A}_i|^{1/2} |A_i|^{1/2} A_i$; hence $A_i|A_i|^{1/2} |A_i|^{1/2} x = \lambda_i |\tilde{A}_i|^{1/2} |A_i|^{1/2} x$, for all $1 \leq i \leq n$. Let $y = \prod_{i=1}^{n'} |\tilde{A}_i|^{1/2} |A_i|^{1/2} x$, where """" on the product """" $\prod_{i=1}^{n'}$ """" denotes that only those $|A_i|$ s, (and so also $|\tilde{A}_i|$ s), appear in the product for which $\lambda_i$ in $A_i x = \lambda_i x$ does not equal 0. Then $y$ is non-trivial, and $A_i y = \lambda_i y$, for all $i = 1, 2, \ldots, n$ for which $\lambda_i \neq 0$.

If $\lambda_i = 0$, i.e. $A_i x = 0$, then $|A_i|^{1/2} x = 0$. This implies that $\tilde{A}_i x = 0$. Since this in turn implies that $|\tilde{A}_i|^{1/2} x = 0$, we conclude that $\tilde{A}_ix = 0$. Since $[A_i, \tilde{A}_i] = 0$ for all $1 \leq i \neq j \leq n$, we have that $\tilde{A}_i y = 0$. Consequently, $\lambda \in \sigma_p(A)$ and $\sigma_p(\tilde{A}) \subseteq \sigma_p(A)$.

If, on the other hand, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \sigma_p(A)$, then there is a non-trivial $x \in H$ such that $(\tilde{A}_i - \lambda_i)x = 0$ and $(\tilde{A}_i^* - \lambda_i^*)x = 0$ for all $1 \leq i \leq n$. Since $A_i^* |A_i|^{1/2} |\tilde{A}_i|^{1/2} x = |A_i|^{1/2} |\tilde{A}_i|^{1/2} \tilde{A}_i^* x$, $A_i^* |A_i|^{1/2} |\tilde{A}_i|^{1/2} x = \lambda_i |\tilde{A}_i|^{1/2} |A_i|^{1/2} x$, for all $1 \leq i \leq n$. Defining $(0 \neq )y$ by $y = \prod_{i=1}^{n'} |A_i|^{1/2} |\tilde{A}_i|^{1/2} x$, where $\prod_{i=1}^{n'}$ has meaning similar to that above, we have $A_i^* y = \tilde{A}_i y$, for all $i = 1, 2, \ldots, n$ such that $\lambda_i \neq 0$. Since $\lambda_i \in \sigma_p(\tilde{A}_i)$ implies $\lambda_i \in \sigma_p(A_i) = \sigma_{np}(A_i)$ (see Lemma 0), $A_i y = \lambda_i y$ for all $i = 1, 2, \ldots, n$ such that $\lambda_i \neq 0$. Now if $\tilde{A}_i x = 0$, then $0 \in \sigma_p(\tilde{A}_i) = \sigma_p(A_i)$ and $\tilde{A}_i x = 0 \Rightarrow |\tilde{A}_i|^{1/2} V_i^* |\tilde{A}_i|^{1/2} x = 0$

$\Rightarrow \tilde{A}_i |\tilde{A}_i|^{1/2} x = 0 \Rightarrow \tilde{A}_i x = 0$.

$|\tilde{A}_i|^{1/2} x = 0 \Rightarrow \tilde{A}_i x = 0$.

$A_i^* |A_i|^{1/2} x = 0 \Rightarrow A_i |A_i|^{1/2} x = 0$.

$|A_i|^{1/2} x = 0 \Rightarrow A_i x = 0$.

(Line 2 follows since $0 \in \sigma_p(A_i)$. Line 4 follows because $0 \in \sigma_p(A_i) \subseteq \sigma_p(A_i)$.) Consequently, $A_i y = 0$ for such an $i$. Hence $\sigma_p(\tilde{A}) \subseteq \sigma_p(A_i)$. Since $\sigma_p(A_i) = \sigma_{np}(A_i)$ and $\sigma_p(\tilde{A}) = \sigma_{np}(\tilde{A})$, for all $1 \leq i \leq n$, this completes the proof.
Lemma 3. \( \sigma_\pi(\mathcal{A}) = \sigma_{n\pi}(\mathcal{A}) = \sigma_{n\pi}(\overline{\mathcal{A}}) = \sigma_\pi(\overline{\mathcal{A}}). \)

Proof. Letting \( A^0 = (A^0_1, A^0_2, \ldots, A^0_n) \) denote the Berberian extension of \( \mathcal{A} \) (see Theorem B), it follows from Lemma 2 that

\[
\sigma_\pi(\mathcal{A}) = \sigma_\pi(\mathcal{A}^0) = \sigma_{n\pi}(\mathcal{A}^0) = \sigma_{n\pi}(\overline{\mathcal{A}}^0) = \sigma_\pi(\overline{\mathcal{A}}).
\]

We are now in a position to prove the equality of the (Taylor) spectra of \( \mathcal{A} \) and \( \overline{\mathcal{A}} \).

Theorem 1. \( \sigma(\mathcal{A}) = \sigma(\overline{\mathcal{A}}) \).

Proof. Let \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \sigma(\mathcal{A}) \). Then there exists a partition

\[
\{i_1, \ldots, i_m\} \cup \{j_1, \ldots, j_s\} \text{ of } \{1, 2, \ldots, n\}
\]

and a sequence \( \{x_k\} \) of unit vectors in \( H \) such that

\[
(A_{i_k} - \lambda_{i_k})x_k \to 0 \quad \text{and} \quad (A^*_{j_k} - \overline{\lambda}_{j_k})x_k \to 0 \quad \text{as} \quad k \to \infty,
\]

for all \( 1 \leq r \leq m \) and \( 1 \leq \ell \leq s \). (See [7, Corollary 3.3].) Let \( \mathcal{A}^0 \) denote the Berberian extension \( (A^0_1, \ldots, A^0_m, A^0_{i_1}, \ldots, A^0_{i_m}) \) of \( \mathcal{A} \), and let \( \mathcal{B} = (A^0_1, \ldots, A^0_m, A^0_{i_1}, \ldots, A^0_{i_m}) \). Then

\[
(\lambda_{i_1}, \ldots, \lambda_{i_m}, \overline{\lambda}_{j_1}, \ldots, \overline{\lambda}_{j_s}) \in \sigma_p(\mathcal{B}).
\]

Since \( \sigma_p(A^0_{i_k}) = \sigma_p(\overline{A}^0_{j_k}) = \sigma_{n\pi}(\overline{A}^0_{j_k}) \), for all \( 1 \leq r \leq m \), and since

\[
\overline{A}^0_{j_k} |A^*_j|^{1/2} V^*_j |A_j|^{1/2} U^*_j = |\overline{A}^0_{j_k}|^{1/2} V^*_j |A_j|^{1/2} U^*_j A^*_j,
\]

it follows (from an argument similar to that used in the proof of Lemma 2) that \( \sigma_p(\mathcal{B}) \subseteq \sigma_p(\overline{\mathcal{A}}) \) and

\[
\lambda \in \sigma_p(\overline{\mathcal{A}}) \subseteq \sigma(\overline{\mathcal{A}}) \subseteq \sigma(\overline{\mathcal{A}}^*).
\]

Hence \( \lambda \in \sigma(\overline{\mathcal{A}}) \), and \( \sigma(\mathcal{A}) \subseteq \sigma(\overline{\mathcal{A}}) \).

Conversely, if \( \lambda \in \sigma(\overline{\mathcal{A}}) \) then (from an argument similar to that above) \( \overline{\lambda} \in \sigma_p(\overline{\mathcal{A}}) \). This implies that \( \lambda \in \sigma_p(\mathcal{A}^*) \subseteq \sigma(\mathcal{A}^*) \), \( \lambda \in \sigma(\mathcal{A}) \) and \( \sigma(\mathcal{A}) \subseteq \sigma(\mathcal{A}) \). Hence

\( \sigma(\mathcal{A}) = \sigma(\overline{\mathcal{A}}) \), and the proof is complete.

The joint spectral radius \( r(\mathcal{T}) \) and the joint operator norm \( \|\mathcal{T}\| \) of an \( n \)-tuple \( \mathcal{T} = (T_1, T_2, \ldots, T_n) \) are defined by

\[
\|\mathcal{T}\| = \sup\left\{ \left( \sum_{i=1}^n \|T_i x\|^2 \right)^{1/2} : x \in H, \|x\| = 1 \right\},
\]

and

\[
r(\mathcal{T}) = \left( \sum_{i=1}^n |\lambda_i|^2 \right)^{1/2} : \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \sigma(\mathcal{T}) \right\}.
\]

See [6]. The operators \( \mathcal{A} \) and \( \overline{\mathcal{A}} \) being jointly normaloid (see [3, Theorem 9] and [6, Theorem 3.4]), \( r(\mathcal{A}) = \|\mathcal{A}\| \) and \( r(\overline{\mathcal{A}}) = \|\overline{\mathcal{A}}\| \). Theorem 1 thus implies the following result.
Corollary 1. \|A\| = \|\hat{A}\| = \|\tilde{A}\|.

That \|A\| = \|\hat{A}\| for a single operator \(A \in HU(p)\) has been proved by M. Fujii et al. in [10].

Given a semi-normal (i.e., hyponormal or co-hyponormal) operator \(T = X + iY\), a well known result of Putnam [12] states that if a real number \(r \in \sigma(X)\) (or \(r + is \in \sigma(T)\), for some real numbers \(r\) and \(s\)), then there exists a real number \(s\) such that \(r + is \in \sigma(T)\) (resp., \(r \in \sigma(X)\) and \(s \in \sigma(Y)\)). This result extends to doubly commuting \(n\)-tuples of hyponormal operators [4]. Does a similar result hold (for \(A \in HU(p)\) and) doubly commuting \(n\)-tuples in \(HU(p)\)? The technique of this paper (seemingly) does not lend to a proof of this. We do however have the following analogue for \(HU(p)\) operators of a result on \(n\)-tuples of doubly commuting hyponormal operators with spectrum in \(\mathbb{R}^n\). (See [4, Corollary].)

Corollary 2. If \(\sigma(A) \subseteq \mathbb{R}^n\), then \(A_i\) is self-adjoint, for all \(1 \leq i \leq n\).

Proof. Since \(\sigma(A) = \sigma(A) \subseteq \mathbb{R}^n\), \(A_i\) is self-adjoint, for all \(1 \leq i \leq n\), by [4]. Recall that \(A_i\) is normal if and only if \(\hat{A}_i\) is normal [9, Corollary 2]; hence \(A_i\) is self-adjoint, for all \(1 \leq i \leq n\).

Following Chô [2], we define the joint Weyl spectrum \(\sigma_w(T)\) of a commuting \(n\)-tuple \(T\) by

\[ \sigma_w(T) = \bigcap \{\sigma(T + \mathcal{K}) ; \mathcal{K}\text{ is an }n\text{-tuple of compact operators and }\langle T + \mathcal{K}\rangle\text{ is a commuting }n\text{-tuple}\}. \]

Let \(\sigma_{\text{iso}}(T)\) denote the set of isolated points of \(\sigma(T)\) which are joint eigen-values of finite multiplicity of \(T\). It is clear from Theorem 1 that, if \(\lambda\) is an isolated point of \(\sigma(A)\), then \(\lambda\) is an isolated point of \(\sigma(A)\). The operator \(A\) being a doubly commutative \(n\)-tuple of hyponormal operators, an isolated point \(\lambda\) of \(\sigma(A)\) is a point of \(\sigma_p(A)\). Hence by Lemma 2 we have the following result.

Corollary 3. If \(\lambda\) is an isolated point of \(\sigma(A)\), then \(\lambda \in \sigma_p(A)\).

Recall that if \(A\) is \(p\)-hyponormal, then \(\sigma_w(A) = \sigma(A) - \sigma_{\text{iso}}(A)\) by [9] and if \(T\) is a doubly commuting \(n\)-tuple of hyponormal operators, then \(\sigma_w(T) \subseteq \sigma(T) - \sigma_{\text{iso}}(T)\) by [2].

Theorem 2. \(\sigma_w(A) \subseteq \sigma(A) - \sigma_{\text{iso}}(A)\).

Proof. Suppose \((\lambda_1, \lambda_2, \ldots, \lambda_n) \in \sigma_{\text{iso}}(A)\), and let \(N = \ker \left( \sum_{i=1}^{n} (A_i - \lambda_i)^*(A_i - \lambda_i) \right)\).

Since \(\lambda \in \sigma_p(A)\) if and only if \(0 \in \sigma_p(\sum_{i=1}^{n} (A_i - \lambda)^*(A_i - \lambda_i))\), \(N\) is finite dimensional. By Lemma 2, \(\sigma_p(A) = \sigma_{np}(A)\); hence \(N\) reduces \(A_i\) to \(N\) \(= (A_1 | N, A_2 | N, \ldots, A_n | N)\) is normal and \(A_i = A_i | N^\perp = (A_1 | N^\perp, A_2 | N^\perp, \ldots, A_n | N^\perp)\) is a doubly commuting \(n\)-tuple of \(HU(p)\) operators. Let \(P\) be the orthogonal projection of \(H\) onto \(N\); \(P\) is then a compact operator which satisfies \([A_i, P] = [A_i^*, P] = 0\) for all \(i = 1, 2, \ldots, n\). The operator

\[ A + P = \left( A_1 + \frac{1}{\sqrt{n}} P, A_2 + \frac{1}{\sqrt{n}} P, \ldots, A_n + \frac{1}{\sqrt{n}} P \right) \]
is a doubly commuting n-tuple. Let

$$R = (\mathcal{A} + \mathcal{P}) | N = \left( \left( A_1 + \frac{1}{\sqrt{n}} P \right) | N, \left( A_2 + \frac{1}{\sqrt{n}} P \right) | N, \ldots, \left( A_n + \frac{1}{\sqrt{n}} P \right) | N \right),$$

$$\mathcal{F} = (\mathcal{A} + \mathcal{P}) | N^\perp.$$ 

\(R\) and \(\mathcal{F}\) are then doubly commuting n-tuples such that \(\sigma(\mathcal{A} + \mathcal{P}) = \sigma(R) \cup \sigma(\mathcal{F}).\)

Suppose that \(\lambda \in \sigma(\mathcal{A} + \mathcal{P}).\) Then \(\lambda \notin \sigma(R)\) and so \(\lambda\) must be an isolated point of \(\sigma(\mathcal{F}).\) There exists a partition \(\{i_1, \ldots, i_m\} \cup \{j_1, \ldots, j_s\}\) of \(\{1, 2, \ldots, n\}\) and a sequence \(\{x_k\}\) of unit vectors in \(N^\perp\) such that

$$\left( A_{i_k} - \lambda_{i_k} + \frac{1}{\sqrt{n}} P \right)x_k \to 0 \quad \text{and} \quad \left( A_{j_k} - \lambda_{j_k} + \frac{1}{\sqrt{n}} P \right)x_k \to 0 \quad \text{as} \quad k \to \infty.$$ 

But then \(\lambda \in \sigma(\mathcal{A}_i)\) and hence (by Corollary 3) \(\lambda \in \sigma_p(\mathcal{A}_i).\) Thus there exists an \(x \in N^\perp\) such that \((A_i - \lambda_i)x = 0,\) for all \(i = 1, 2, \ldots, n.\) Since this is a contradiction, we must have \(\lambda \notin \sigma_w(\mathcal{F}).\)

**Remarks.** (i) the Taylor–Weyl spectrum of \(\mathcal{F}, \sigma_{TW}(\mathcal{F}),\) is defined to be the set of \(\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)\) such that \((\mathcal{F} - \lambda)\) is not Taylor–Weyl (where \(\mathcal{F} - \lambda\) is said to be Taylor–Weyl if \(\mathcal{F} - \lambda\) is Fredholm and index \((\mathcal{F} - \lambda) = 0).\) Theorem 2 implies that \(\sigma(\mathcal{A}) \cap \sigma_{TW}(\mathcal{A}) \subseteq \sigma_{TW}(\mathcal{F}).\) The inclusion \(\sigma(\mathcal{A}) \cap \sigma_{TW}(\mathcal{A}) \subseteq \sigma_{TW}(\mathcal{F})\) does not hold (even for hyponormal \(\mathcal{A}\)).

(ii) Given a \(p\)-hyponormal operator \(A, \sigma_{\omega}(A) = \sigma_{\omega}(\hat{A})\) by [9]. Does \(\sigma_{\omega}(\mathcal{A}) = \sigma_{\omega}(\hat{A})?\)

The Harte spectrum \(\sigma_{\epsilon}(\mathcal{F})\) of the commutative n-tuple \(\mathcal{F}\) is defined to be \(\sigma_{\epsilon}(\mathcal{F}) = \sigma'(\mathcal{F}) \cup \sigma'^*(\mathcal{F}),\) where \(\sigma'(\mathcal{F})\) (respectively, \(\sigma'^*(\mathcal{F})\)) is the set of \(\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)\) such that \((\mathcal{F} - \lambda_{\alpha})_{1 \leq \alpha \leq n}\) generates a proper left (resp., right) ideal in \(B(H).\) The (Harte) essential spectrum \(\sigma_{\epsilon}(\mathcal{F})\) is defined by \(\sigma_{\epsilon}(\mathcal{F}) = \sigma(a),\) where \(a = (a_1, a_2, \ldots, a_n) = \pi(\mathcal{F})\) and \(\pi\) is the canonical homomorphism of \(B(H)\) onto the Calkin algebra \(B(H)/K(H); K(H)\) is the algebra of compact operators on \(H.\) For a single linear operator, the (Harte) essential spectrum coincides with the essential spectrum; the following extends the conclusion \(\sigma_{\epsilon}(A) = \sigma_{\epsilon}(\hat{A})\) of Lemma 0 to \(\sigma_{\epsilon}(\mathcal{A}).\)

**Theorem 3.** \(\sigma_{\epsilon}(\mathcal{A}) = \sigma_{\epsilon}(\hat{A}).\)

**Proof.** Suppose \(\lambda \in \sigma_{\epsilon}(\mathcal{A}).\) Then, \(\mathcal{A}\) being a hyponormal n-tuple, there exists a sequence \(\{x_k\}\) of unit vectors converging weakly to \(0\) in \(H\) such that

$$\| (\hat{\lambda}_i - \lambda_i) x_k \| \to 0 \quad \text{as} \quad k \to \infty, \quad \text{for all} \quad 1 \leq i \leq n,$$

by [8, Theorem 2.6]. Let \(\{y_k\}\) be the sequence defined by

$$y_k = \left( \prod_{i=1}^{n} |A_i|^{1/2} |\hat{A}_i|^{1/2} x_k \right) / \left( \prod_{i=1}^{n} |A_i|^{1/2} |\hat{A}_i|^{1/2} \right),$$

where “""" on the product \(\prod_{i=1}^{n} \) denotes that only those \(|A_i|s\) and \(|\hat{A}_i|s\) appear in the product.
for which \( \lambda_i \neq 0 \). (Notice that if \( \|A_i^{1/2}x_k\| \) or \( \|A_i^{1/2}|A_i|^{1/2}x_k\| \to 0 \) as \( k \to \infty \), for some \( i \) with \( 1 \leq i \leq n \), then \( \|A_i^{1/2}x_k\| \) and \( \|A_i^{1/2}x_k\| \to 0 \) as \( k \to \infty \).) Since \((x_k, h) \to 0 \) as \( k \to \infty \) for all \( h \in H \), \((y_k, h) \to 0 \) as \( k \to \infty \) and

\[
\|(A_j - \lambda_j)^*y_k\| = \left\| \frac{\prod_{i=1}^n |A_i|^{1/2} |A_i|^{1/2}x_k}{\prod_{i=1}^n |A_i|^{1/2} |A_i|^{1/2}x_k} \right\| \to 0 \quad \text{as} \quad k \to \infty,
\]

for all \( 1 \leq j \leq n \). Thus \( \lambda \in \sigma_c(\mathcal{A}) \) and \( \sigma_c(\mathcal{A}) \subseteq \sigma_c(\mathcal{A}) \).

Consider now \( \lambda \in \sigma_c(\mathcal{A}) = \sigma'_c(\mathcal{A}) \cup \sigma_s(\mathcal{A}) \). Suppose that \( \lambda \in \sigma'_c(\mathcal{A}) \); then there exists a sequence \( \{x_k\} \) of unit vectors converging weakly to \( 0 \) in \( H \) such that \( \|(A_i - \lambda_i)x_k\| \to 0 \) as \( k \to \infty \), for all \( 1 \leq i \leq n \). Defining the sequence \( \{y_k\} \) by

\[
y_k = \frac{\left( \prod_{i=1}^n |A_i|^{1/2} |A_i|^{1/2}x_k \right)}{\left( \prod_{i=1}^n |A_i|^{1/2} |A_i|^{1/2}x_k \right)},
\]

(where \( \prod_{i=1}^n \) has a meaning similar to that above), an argument similar to that above shows that \( \{y_k\} \) is a sequence of unit vectors converging weakly to \( 0 \) in \( H \) such that

\[
\|(A_j - \lambda_j)y_k\| \to 0 \quad \text{as} \quad k \to \infty, \quad \text{for all} \quad 1 \leq i \leq n.
\]

Hence \( \lambda \in \sigma'_c(\mathcal{A}) \). A similar argument shows that if \( \lambda \in \sigma'_s(\mathcal{A}) \) then \( \lambda \in \sigma'_c(\mathcal{A}) \). Thus \( \sigma_c(\mathcal{A}) \subseteq \sigma_c(\mathcal{A}) \), and the proof is complete.

**Corollary 4.** \( \sigma_c(\mathcal{A}) = \sigma'_c(\mathcal{A}) \).

**Proof.** The argument of the proof of Theorem 3 implies that

\[
\sigma'_c(\mathcal{A}) \subseteq \sigma_c(\mathcal{A}) = \sigma'_c(\mathcal{A}) \subseteq \sigma'_c(\mathcal{A}).
\]

**Corollary 5.** \( \sigma_H(\mathcal{A}) = \sigma_c(\mathcal{A}) \cup \sigma_p(\mathcal{A})^* \).

**Proof.** Let \( \sigma_\pi(\mathcal{A}) = \{\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \} \) there exists a sequence \( \{x_k\} \) of unit vectors in \( H \) such that \( \|(A_i - \lambda_i)x_k\| \to 0 \) as \( k \to \infty \), for all \( i = 1, 2, \ldots, n \) denote the joint approximate defect spectrum of \( \mathcal{A} \). Then

\[
\sigma_H(\mathcal{A}) = \sigma'(\mathcal{A}) \cup \sigma^*(\mathcal{A}) = \sigma_\pi(\mathcal{A}) \cup \sigma_\delta(\mathcal{A}).
\]

By Lemma 3, \( \sigma_\pi(\mathcal{A}) = \sigma_\pi(\mathcal{A}) \); applying an argument similar to that used in the proof of Lemma 2 to \( A_i^{|1/2|} \) it is seen that \( \sigma_\pi(\mathcal{A}) = \sigma_\pi(\mathcal{A}) \). We have

\[
\sigma_H(\mathcal{A}) = \sigma_\pi(\mathcal{A}) \cup \sigma_\delta(\mathcal{A}) = \sigma_\pi(\mathcal{A}) \cup \sigma_\delta(\mathcal{A}) = \sigma_\delta(\mathcal{A}) = \sigma_H(\mathcal{A}),
\]
since $A$ is a hyponormal $n$-tuple. Also, since

$$\sigma_H(A) = \sigma_\varepsilon(A) = \sigma_\varepsilon(A) \cup \sigma_p(A)^* = \sigma_\varepsilon(A) \cup \sigma_p(A^*)^*,$$

the proof is complete.

The $n$-tuple $(A_1, A_2, \ldots, A_n)$ is said to be essentially doubly commuting (resp., essentially $\mathcal{HU}(p)$) if the $n$-tuple $(a_1, a_2, \ldots, a_n)$, where $a_i = \pi(A_i)$ for all $1 \leq i \leq n$, and $\pi : B(H) \to B(H) \setminus K(H)$, is doubly commuting (resp., $\mathcal{HU}(p)$). We close this note with the following result.

**Theorem 4.** Suppose $(A_1, A_2, \ldots, A_n)$ is an $n$-tuple of essentially doubly commuting essentially $\mathcal{HU}(p)$ operators. Then $A_1, A_2, \ldots, A_n$ have a common reducing subspace “modulo the compact operators”.

**Proof.** The hypotheses imply that $a_i \in \mathcal{HU}(p)$ for all $1 \leq i \leq n$ and that the $a_is$ are doubly commuting. Since $\sigma'_\varepsilon(A) \cap \sigma'_\varepsilon(A)$ is not empty (this is consequence of the definition of essential spectrum—see [8, Lemma 4.2]), there exists $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \sigma'_\varepsilon(A) \cap \sigma'_\varepsilon(A)$ and a non-zero projection $q$ in (the Calkin algebra) $B(H)/K(H)$ such that

$$a_i q = \lambda_i q \quad (1 \leq i \leq n).$$

Since $\sigma_\varepsilon(a_i) = \sigma_{np}(a_i)$, this implies that $a_i^* q = \lambda_i q \quad (1 \leq i \leq n)$. Consequently $a_i q = (\lambda_i q)^* = (a_i^* q)^* = q a_i$ (1 \leq i \leq n), or, letting $\pi(Q) = q$, $(A_i Q - QA_i)$ is a compact operator, for all $1 \leq i \leq n$. This completes the proof.

**REFERENCES**


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