ON THE SPECTRUM OF $n$-TUPLES OF $p$-HYPONORMAL OPERATORS

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1. Introduction. Let $B(H)$ denote the algebra of operators (i.e., bounded linear transformations) on the Hilbert space $H$. A $\in B(H)$ is said to be $p$-hyponormal ($0 < p \leq 1$), if $(AA^*)^p \leq (A^*A)^p$. (Of course, a 1-hyponormal operator is hyponormal.) The $p$-hyponormal property is monotonic decreasing in $p$ and a $p$-hyponormal operator is $q$-hyponormal operator for all $0 < q \leq p$. Let $A$ have the polar decomposition $A = U|A|$, where $U$ is a partial isometry and $|A|$ denotes the (unique) positive square root of $A^*A$. If $A$ has equal defect and nullity, then the partial isometry $U$ may be taken to be unitary. Let $\mathcal{HU}(p)$ denote the class of $p$-hyponormal operators for which $U$ in $A = U|A|$ is unitary. $\mathcal{HU}(1/2)$ operators were introduced by Xia and $\mathcal{HU}(p)$ operators for a general $0 < p < 1$ were first considered by Aluthge (see [1, 14]); $\mathcal{HU}(p)$ operators have since been considered by a number of authors (see [3, 4, 5, 9, 10] and the references cited in these papers). Generally speaking, $\mathcal{HU}(p)$ operators have spectral properties similar to those of hyponormal operators. Indeed, let $A \in \mathcal{HU}(p)$, $(0 < p < 1/2)$, have the polar decomposition $A = U|A|$, and define the $\mathcal{HU}(p + 1/2)$ operator $\hat{A}$ by $\hat{A} = |A|^{1/2} U|A|^{1/2}$. Let $\hat{A} = V|\hat{A}|$ with $V$ unitary and $\hat{A}$ be the hyponormal operator defined by $\hat{A} = |\hat{A}|^{1/2} V|\hat{A}|^{1/2}$. Then we have the following result.

Lemma 0. $\sigma_s(A) = \sigma_s(\hat{A})$, where $\sigma_s$ denotes either of the following: point spectrum, approximate point spectrum, eigenvalues of finite multiplicity, spectrum, Weyl spectrum, and essential spectrum.

Recall that an $n$-tuple $\mathcal{A} = (A_1, A_2, \ldots, A_n)$ of operators is said to be doubly commuting if $A_iA_j - A_jA_i = 0$ and $A_i^*A_j - A_jA_i^* = 0$, for all $1 \leq i \neq j \leq n$. Doubly commuting $n$-tuples $\mathcal{A}$ of operators in $\mathcal{HU}(p)$ have been considered by Munee Cho in [3], where it is shown that a weak Putnam theorem holds for $\mathcal{A}$ and that $\mathcal{A}$ is jointly normaloid. In this note we study the relationship between the spectral properties of $\mathcal{A}$ and $\mathcal{A} = (\hat{A}_1, \hat{A}_2, \ldots, \hat{A}_n)$, and prove that $\sigma_s(\mathcal{A}) = \sigma_s(\mathcal{A})$, where $\sigma_s$ is either the joint point spectrum or the joint approximate point spectrum or the joint (Taylor) spectrum. This then leads us to:

(a) $\|\mathcal{A}\| = \|\mathcal{A}\|$
(b) if $\sigma(\mathcal{A}) \in \mathbb{R}^n$, then $A_i$ is self-adjoint, for all $1 \leq i \leq n$.

We show that the (Cho-Takaguchi) joint Weyl spectrum of $\mathcal{A}$ is contained in the (Taylor) spectrum $\sigma(\mathcal{A})$ of $\mathcal{A}$ minus the set of isolated points of $\sigma(\mathcal{A})$ which are joint eigenvalues of finite multiplicity, and that $\mathcal{A}$ and $\mathcal{A}$ have the same (Harte) essential spectrum. We conclude this note with a result (in the spirit of Dash [8, Corollary 4.6]) on the joint eigenvalues of $\mathcal{A}$ in the Calkin algebra.

We assume henceforth, without loss of generality, that $0 < p < 1/2$. Most of the notation that we use in this note is standard (and usually explained at the first instance of

occurrence). The following theorem, the \(n\)-tuple version of the Berberian extension theorem, will play an important role in the sequel.

**Theorem B.** If \( \mathcal{A} = (A_1, A_2, \ldots, A_n) \) is an \(n\)-tuple of commuting operators on \(H\), then there exists a Hilbert space \(H^0 \supset H\) and an isometric *-isomorphism \(A_i \rightarrow A_i^0\), \((1 \leq i \leq n)\), preserving order such that \(\sigma_p(A_i) = \sigma_p(A_i^0) = \sigma_p(A_i^0)\) and \(\sigma_a(\mathcal{A}) = \sigma_a(A_1, A_2, \ldots, A_n) = \sigma_p(A_1^0, A_2^0, \ldots, A_n^0) = \sigma_p(\mathcal{A}^0).\) (Here \(\sigma_p(\mathcal{A})\) and \(\sigma_a(\mathcal{A})\) denote, respectively, the joint spectrum and the joint approximate point spectrum (defined below) of \(\mathcal{A}\).)

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**2. Results.** Throughout the following \(\mathcal{A} = (A_1, A_2, \ldots, A_2)\) will denote a doubly commuting (i.e., \(A_iA_j - A_jA_i = 0\) and \(A_iA_j^* - A_j^*A_i = 0\), for all \(1 \leq i \neq j \leq n\)) \(n\)-tuple of \(\mathcal{H}U(p)\) operators \(A_i\), \((1 \leq i \leq n)\). Given \(A_i = U_i |A_i|\), define \(\hat{A}_i\), by \(\hat{A}_i = |A_i|^{1/2} U_i |A_i|^{1/2};\) also, letting \(\hat{A}_i\) have the polar decomposition \(\hat{A}_i = V_i |\hat{A}_i|\), define \(\hat{A}_i\) by

\[
\hat{A}_i = |\hat{A}_i|^{1/2} V_i |\hat{A}_i|^{1/2} \quad (1 \leq i \leq n).
\]

The \(n\)-tuples \(\mathcal{A}\) and \(\mathcal{A}\) are then defined by \(\mathcal{A} = (\hat{A}_1, \hat{A}_2, \ldots, \hat{A}_n)\) and \(\mathcal{A}_1 = (\hat{A}_1, \hat{A}_2, \ldots, \hat{A}_n)\).

**Lemma 1.** \(\mathcal{A}\) is doubly commuting \(\Rightarrow \mathcal{A}\) is doubly commuting \(\Rightarrow \mathcal{A}\) is doubly commuting. Also, \(\mathcal{A}\) is doubly commuting \(\Rightarrow [A_i, |A_i|] = 0 = [\hat{A}_i, |\hat{A}_i|] = 0\), for \(1 \leq i \neq n\), where \([A, B]\) denotes the commutator \(AB - BA\) of \(A\) and \(B\).

**Proof.** Given \(A_i = U_i |A_i|\) and \(\hat{A}_i = V_i |\hat{A}_i|\), the doubly commuting hypothesis on \(\mathcal{A}\) implies that

\[
[U_i, U_j] = [|A_i|, |A_j|] = [A_i, U_j] = 0,
\]

for all \(1 \leq i \neq j \leq n\). (See [11, Theorems 2 and 4].) Consequently, \(\mathcal{A}\) is doubly commuting and so

\[
[V_i, V_j] = [|\hat{A}_i|, |\hat{A}_j|] = [\hat{A}_i, V_j] = 0,
\]

for all \(1 \leq i \neq j \leq n\). This implies that \(\mathcal{A}\) is doubly commuting. The argument above also implies that \([A_i, \hat{A}_j] = [A_i, \hat{A}^*_j] = [\hat{A}_i, \hat{A}_j] = [\hat{A}_i, \hat{A}^*_j] = 0\), for all \(1 \leq i \neq j \leq n\). Hence, also, \([A_i, |\hat{A}_j|] = [\hat{A}_i, |\hat{A}_j|] = 0\), for all \(1 \leq i \neq j \leq n\).

In the following we shall denote the Taylor joint spectrum of \(\mathcal{A}\) by \(\sigma(\mathcal{A})\). (See [13] for the definition of Taylor spectrum of a commuting \(n\)-tuple of operators.) We say that \(\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)\), \((\lambda_i \in \mathbb{C}\) for all \(1 \leq i \leq n)\), is in the joint approximate point spectrum \(\sigma_a(\mathcal{A})\) of \(\mathcal{A}\) if there exists a sequence \(\{x_k\}\) of unit vectors in \(H\) such that

\[
\| (A_i - \lambda_i) x_k \| \rightarrow 0 \quad \text{as} \quad k \rightarrow \infty,
\]

for all \(1 \leq i \leq n\); \(\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)\), \(\lambda_i \in \mathbb{C}\) for all \(1 \leq i \leq n\), is in the joint point spectrum \(\sigma_p(\mathcal{A})\) of \(\mathcal{A}\) if there exists a non-trivial vector \(x \in H\) such that

\[
(A_i - \lambda_i) x = 0, \quad \text{for all} \quad 1 \leq i \leq n.
\]
We say that $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ is in the normal point spectrum $\sigma_{np}(\mathcal{A})$ of $\mathcal{A}$ if there exists a non-trivial vector $x \in H$ such that $(A_i - \lambda_i)x = 0 \iff (A_i - \lambda_i)^*x = 0$, for all $1 \leq i \leq n$.

**Lemma 2.** $\sigma_p(\mathcal{A}) = \sigma_{np}(\mathcal{A}) = \sigma_{np}(\mathcal{A}) = \sigma_p(\mathcal{A})$.

**Proof.** Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \sigma_p(\mathcal{A})$ and let $x \in H$ be such that $x \neq 0$ and $(A_i - \lambda_i)x = 0$, for all $1 \leq i \leq n$. It is easily seen that $\tilde{A}_i |\tilde{A}_i|^{1/2} |A_i|^{1/2} = |\tilde{A}_i|^{1/2} |A_i|^{1/2} A_i$; hence

$$
\tilde{A}_i |\tilde{A}_i|^{1/2} |A_i|^{1/2} x = \lambda_i |\tilde{A}_i|^{1/2} |A_i|^{1/2} x,
$$

for all $1 \leq i \leq n$. Let

$$
y = \prod_{i=1}^n |\tilde{A}_i|^{1/2} |A_i|^{1/2} x,
$$

where "\" denotes that only those $|A_i|s$, (and so also $|\tilde{A}_i|s$), appear in the product for which $\lambda_i$ in $A_i x = \lambda_i x$ does not equal 0. Then $y$ is non-trivial, and

$$
\tilde{A}_i y = \lambda_i y, \text{ for all } i = 1, 2, \ldots, n \text{ for which } \lambda_i \neq 0.
$$

If $\lambda_i = 0$, i.e. $A_i x = 0$, then $|A_i|^{1/2} x = 0$. This implies that $\tilde{A}_i x = 0$. Since this in turn implies that $|\tilde{A}_i|^{1/2} x = 0$, we conclude that $\tilde{A}_i x = 0$. Since $|A_i, \tilde{A}_i| = 0$ for all $1 \leq i \neq j \leq n$, we have that $\tilde{A}_i, y = 0$. Consequently, $\lambda \in \sigma_{\mathcal{A}}(\mathcal{A})$ and $\sigma_{\mathcal{A}}(\mathcal{A}) \subseteq \sigma_{\mathcal{A}}(\mathcal{A})$.

If, on the other hand, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \sigma_p(\mathcal{A})$, then there is a non-trivial $x \in H$ such that $(\tilde{A}_i - \lambda_i)x = 0$ and $(\tilde{A}_i^* - \lambda_i)x = 0$ for all $1 \leq i \leq n$. Since $A_i^* |A_i|^{1/2} |\tilde{A}_i|^{1/2} x = A_i |A_i|^{1/2} |\tilde{A}_i|^{1/2} x$ for all $1 \leq i \leq n$. Defining $(0 \neq) y$ by

$$
y = \prod_{i=1}^n |A_i|^{1/2} |\tilde{A}_i|^{1/2} x,
$$

where "\" has meaning similar to that above, we have $A_i^* y = \tilde{A}_i y$, for all $i = 1, 2, \ldots, n$ such that $\lambda_i \neq 0$. Since $\lambda_i \in \sigma_p(\tilde{A}_i)$ implies $\lambda_i \in \sigma_{\mathcal{A}}(A_i) = \sigma_{np}(A_i)$ (see Lemma 0), $A_i y = \lambda_i y$ for all $i = 1, 2, \ldots, n$ such that $\lambda_i \neq 0$. Now if $\tilde{A}_i x = 0$, then $0 \in \sigma_{\mathcal{A}}(\tilde{A}_i) = \sigma_{\mathcal{A}}(A_i)$ and

$$
\tilde{A}_i x = 0 \Rightarrow |\tilde{A}_i|^{1/2} V_i |\tilde{A}_i|^{1/2} x = 0
$$

$$
\Rightarrow \tilde{A}_i |\tilde{A}_i|^{1/2} x = 0 \Rightarrow \tilde{A}_i x = 0
$$

$$
\Rightarrow |\tilde{A}_i|^{1/2} x = 0 \Rightarrow \tilde{A}_i x = 0
$$

$$
\Rightarrow A_i^* |A_i|^{1/2} x = 0 \Rightarrow A_i |A_i|^{1/2} x = 0
$$

$$
\Rightarrow |A_i|^{1/2} x = 0 \Rightarrow A_i x = 0
$$

(Line 2 follows since $0 \in \sigma_{\mathcal{A}}(A_i)$. Line 4 follows because $0 \in \sigma_p(A_i) = \sigma_{np}(A_i)$.) Consequently, $A_i y = 0$ for such an $i$. Hence $\sigma_{\mathcal{A}}(\mathcal{A}) \subseteq \sigma_{\mathcal{A}}(\mathcal{A})$. Since $\sigma_{\mathcal{A}}(A_i) = \sigma_{np}(A_i)$ and $\sigma_{\mathcal{A}}(A_i) = \sigma_{np}(\tilde{A}_i)$, for all $1 \leq i \leq n$, this completes the proof.
Lemma 3. \( \sigma_{\pi}(\mathcal{A}) = \sigma_{\pi}(\mathcal{A}) = \sigma_{\pi}(\mathcal{A}) \).

Proof. Letting \( A^0 = (A_1^0, A_2^0, \ldots, A_n^0) \) denote the Berberian extension of \( \mathcal{A} \) (see Theorem B), it follows from Lemma 2 that
\[
\sigma_{\pi}(\mathcal{A}) = \sigma_{\pi}(\mathcal{A}^0) = \sigma_{\pi}(\mathcal{A}^0) = \sigma_{\pi}(\mathcal{A}) = \sigma_{\pi}(\mathcal{A}).
\]
We are now in a position to prove the equality of the (Taylor) spectra of \( \mathcal{A} \) and \( \mathcal{A} \).

Theorem 1. \( \sigma(\mathcal{A}) = \sigma(\mathcal{A}) \).

Proof. Let \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \sigma(\mathcal{A}) \). Then there exists a partition
\[ \{i_1, \ldots, i_m\} \cup \{j_1, \ldots, j_s\} \text{ of } \{1, 2, \ldots, n\} \]
and a sequence \( \{x_k\} \) of unit vectors in \( H \) such that
\[
(A_{i_k} - \lambda_{i_k})x_k \to 0 \quad \text{and} \quad (A_{j_k}^* - \lambda_{j_k})x_k \to 0 \quad \text{as} \quad k \to \infty,
\]
for all \( 1 \leq r \leq m \) and \( 1 \leq s \leq s \). (See [7, Corollary 3.3].) Let \( \mathcal{A}^0 \) denote the Berberian extension \( (A_1^0, \ldots, A_m^0, A_{i_1}^0, \ldots, A_{j_s}^0) \) of \( \mathcal{A} \), and let \( \mathcal{B} = (A_1^0, \ldots, A_m^0, A_{i_1}^0, \ldots, A_{j_s}^0) \). Then
\[ (\lambda_{i_1}, \ldots, \lambda_{i_m}, \lambda_{j_1}, \ldots, \lambda_{j_s}) \in \sigma_p(\mathcal{B}). \]
Since \( \sigma_p(A_r^0) = \sigma_p(A_r^0) = \sigma_p(A_r^0) \), for all \( 1 \leq r \leq m \), and since
\[
\lambda_{i_k}^* |A_{i_k}^0|^{1/2} V_{i_k}^* |A_{i_k}^0|^{1/2} U_{i_k}^* = |A_{i_k}^0|^{1/2} V_{i_k}^* |A_{i_k}^0|^{1/2} U_{i_k}^* A_{i_k}^0, \]
it follows (from an argument similar to that used in the proof of Lemma 2) that \( \sigma_p(\mathcal{B}) \subseteq \sigma_p(\mathcal{B}) \) and
\[ \lambda \in \sigma_p(\mathcal{B}) \subseteq \sigma_p(\mathcal{B}). \]
Hence \( \lambda \in \sigma(\mathcal{A}) \), and \( \sigma(\mathcal{A}) \subseteq \sigma(\mathcal{A}) \).

Conversely, if \( \lambda \in \sigma(\mathcal{A}) \), then (from an argument similar to that above) \( \tilde{\lambda} \in \sigma_p(\mathcal{A}) \). This implies that \( \tilde{\lambda} \in \sigma_p(\mathcal{A}^0) \subseteq \sigma(\mathcal{A}) \), \( \lambda \in \sigma(\mathcal{A}) \) and \( \sigma(\mathcal{A}) \subseteq \sigma(\mathcal{A}) \). Hence \( \sigma(\mathcal{A}) = \sigma(\mathcal{A}) \), and the proof is complete.

The joint spectral radius \( r(\mathcal{T}) \) and the joint operator norm \( \|\mathcal{T}\| \) of an \( n \)-tuple \( \mathcal{T} = (T_1, T_2, \ldots, T_n) \) are defined by
\[
r(\mathcal{T}) = \sup \left\{ |\lambda| = \left( \sum_{i=1}^{n} |\lambda_i|^2 \right)^{1/2} : \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \sigma(\mathcal{T}) \right\}
\]
and
\[
\|\mathcal{T}\| = \sup \left\{ \left( \sum_{i=1}^{n} \|T_ix\|^2 \right)^{1/2} : x \in H, \|x\| = 1 \right\}.
\]
See [6]. The operators \( \mathcal{A} \) and \( \tilde{\mathcal{A}} \) being jointly normaloid (see [3, Theorem 9] and [6, Theorem 3.4]), \( r(\mathcal{A}) = \|\mathcal{A}\| \) and \( r(\tilde{\mathcal{A}}) = \|	ilde{\mathcal{A}}\| \). Theorem 1 thus implies the following result.
COROLLARY 1. \( \| A \| = \| \hat{A} \| = \| \tilde{A} \|. \)

That \( \| A \| = \| \hat{A} \| \) for a single operator \( A \in \mathcal{KU}(p) \) has been proved by M. Fujii et al. in [10].

Given a semi-normal (i.e., hyponormal or co-hyponormal) operator \( T = X + iY \), a well known result of Putnam [12] states that if a real number \( r \in \sigma(X) \) (or \( r + is \in \sigma(T) \), for some real numbers \( r \) and \( s \)), then there exists a real number \( s \) such that \( r + is \in \sigma(T) \) (resp., \( r \in \sigma(X) \) and \( s \in \sigma(Y) \)). This result extends to doubly commuting \( n \)-tuples of hyponormal operators [4]. Does a similar result hold (for \( A \in \mathcal{KU}(p) \) and) doubly commuting \( n \)-tuples in \( \mathcal{KU}(p) \)? The technique of this paper (seemingly) does not lend to a proof of this. We do however have the following analogue for \( \mathcal{KU}(p) \) operators of a result on \( n \)-tuples of doubly commuting hyponormal operators with spectrum in \( \mathbb{R}^n \). (See [4, Corollary].)

COROLLARY 2. If \( \sigma(\mathcal{A}) \subseteq \mathbb{R}^n \), then \( A_i \) is self-adjoint, for all \( 1 \leq i \leq n \).

Proof. Since \( \sigma(\mathcal{A}) = \sigma(\mathcal{A}) \subseteq \mathbb{R}^n \), \( \hat{A}_i \) is self-adjoint, for all \( 1 \leq i \leq n \), by [4]. Recall that \( A_i \) is normal if and only if \( \hat{A}_i \) is normal [9, Corollary 2]; hence \( A_i \) is self-adjoint, for all \( 1 \leq i \leq n \).

Following Chô [2], we define the joint Weyl spectrum \( \sigma_\omega(\mathcal{T}) \) of a commuting \( n \)-tuple \( \mathcal{T} \) by

\[
\sigma_\omega(\mathcal{T}) = \cap \{ \sigma(\mathcal{T} + \mathcal{K}); \mathcal{K} \text{ is an } n \text{-tuple of compact operators and } (\mathcal{T} + \mathcal{K}) \text{ is a commuting } n \text{-tuple}. \}
\]

Let \( \sigma_{\omega_0}(\mathcal{T}) \) denote the set of isolated points of \( \sigma(\mathcal{T}) \) which are joint eigen-values of finite multiplicity of \( \mathcal{T} \). It is clear from Theorem 1 that, if \( \lambda \) is an isolated point of \( \sigma(\mathcal{A}) \), then \( \lambda \) is an isolated point of \( \sigma(\hat{\mathcal{A}}) \). The operator \( \hat{\mathcal{A}} \) being a doubly commutitive \( n \)-tuple of hyponormal operators, an isolated point \( \lambda \) of \( \sigma(\hat{\mathcal{A}}) \) is a point of \( \sigma_p(\mathcal{A}) \). Hence by Lemma 2 we have the following result.

COROLLARY 3. If \( \lambda \) is an isolated point of \( \sigma(\mathcal{A}) \), then \( \lambda \in \sigma_p(\mathcal{A}) \).

Recall that if \( A \) is \( p \)-hyponormal, then \( \sigma_\omega(A) = \sigma(A) - \sigma_{\omega_0}(A) \) by [9] and if \( \mathcal{T} \) is a doubly commuting \( n \)-tuple of hyponormal operators, then \( \sigma_\omega(\mathcal{T}) \subseteq \sigma(\mathcal{T}) - \sigma_{\omega_0}(\mathcal{T}) \) by [2].

THEOREM 2. \( \sigma_\omega(\mathcal{A}) \subseteq \sigma(\mathcal{A}) - \sigma_{\omega_0}(\mathcal{A}) \).

Proof. Suppose \( (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \sigma_{\omega_0}(\mathcal{A}) \), and let \( N = \ker \left\{ \sum_{i=1}^{n} (A_i - \lambda_i)^*(A_i - \lambda_i) \right\} \). Since \( \lambda \in \sigma_p(\mathcal{A}) \) if and only if \( 0 \in \sigma_p \left( \sum_{i=1}^{n} (A_i - \lambda)^*(A_i - \lambda_i) \right) \), \( N \) is finite dimensional. By Lemma 2, \( \sigma_p(\mathcal{A}) = \sigma_p(\hat{\mathcal{A}}) \); hence \( N \) reduces \( \mathcal{A}_1 \), \( \mathcal{A}_0 = \mathcal{A} \mid N = (A_1 \mid N, A_2 \mid N, \ldots, A_n \mid N) \) is normal and \( \mathcal{A}_1 = \mathcal{A} \mid N^{\perp} = (A_1 \mid N^{\perp}, A_2 \mid N^{\perp}, \ldots, A_n \mid N^{\perp}) \) is a doubly commuting \( n \)-tuple of \( \mathcal{KU}(p) \) operators. Let \( P \) be the orthogonal projection of \( \mathcal{H} \) onto \( N \). \( P \) is then a compact operator which satisfies \( [A_i, P] = [A_i^*, P] = 0 \), for all \( i = 1, 2, \ldots, n \). The operator

\[
\mathcal{A} + P = \left( A_1 + \frac{1}{\sqrt{n}} P, A_2 + \frac{1}{\sqrt{n}} P, \ldots, A_n + \frac{1}{\sqrt{n}} P \right)
\]
is a doubly commuting n-tuple. Let
\[ \mathcal{R} = (\mathcal{A} + \mathcal{P}) \mid N = \left( \left( A_1 + \frac{1}{\sqrt{n}} P \right) \mid N, \left( A_2 + \frac{1}{\sqrt{n}} P \right) \mid N, \ldots, \left( A_n + \frac{1}{\sqrt{n}} P \right) \mid N \right), \]
\[ \mathcal{S} = (\mathcal{A} + \mathcal{P}) \mid N^\perp. \]
\( \mathcal{R} \) and \( \mathcal{S} \) are then doubly commuting n-tuples such that \( \sigma(\mathcal{A} + \mathcal{P}) = \sigma(\mathcal{R}) \cup \sigma(\mathcal{S}). \)

Suppose that \( \lambda \in \sigma(\mathcal{A} + \mathcal{P}). \) Then \( \lambda \notin \sigma(\mathcal{R}) \) and so \( \lambda \) must be an isolated point of \( \sigma(\mathcal{S}). \) There exists a partition \( \{i_1, \ldots, i_m\} \cup \{j_1, \ldots, j_s\} \) of \( \{1, 2, \ldots, n\} \) and a sequence \( \{x_k\} \) of unit vectors in \( N^\perp \) such that
\[ a(s_{i_k} + \phi_i) = a(t_{j_k}) \] and \( \sigma(t_{j_k}) = \sigma(s_{i_k}). \) But then \( \lambda \notin \sigma(\mathcal{S}_{i_k}) \) and hence (by Corollary 3) \( \lambda \notin \sigma_p(\mathcal{A}_i). \) Thus there exists an \( x \in N^\perp \) such that \( (A_i - \lambda_i)x = 0, \) for all \( i = 1, 2, \ldots, n. \) Since this is a contradiction, we must have \( \lambda \notin \sigma_m(\mathcal{A}). \)

**Remarks.** (i) the Taylor–Weyl spectrum of \( \mathcal{T}, \sigma_{TW}(\mathcal{T}), \) is defined to be the set of \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \) such that \( (\mathcal{T} - \lambda) \) is not Taylor–Weyl (where \( \mathcal{T} - \lambda \) is said to be Taylor–Weyl if \( \mathcal{T} - \lambda \) is Fredholm and \( \lambda \) is Fredholm index \( (\mathcal{T} - \lambda) = 0). \) Theorem 2 implies that \( \sigma(\mathcal{A}) \setminus \sigma_{TW}(\mathcal{A}) \subseteq \sigma_{00}(\mathcal{A}). \) The inclusion \( \sigma(\mathcal{A}) \setminus \sigma_{TW}(\mathcal{A}) \subseteq \sigma_{00}(\mathcal{A}) \) does not hold (even for hyponormal \( \mathcal{A} \)).

(ii) Given a p-hyponormal operator \( A, \sigma_\omega(A) = \sigma_\omega(\mathcal{A}) \) by [9]. Does \( \sigma_\omega(\mathcal{A}) = \sigma_\omega(\mathcal{A})? \)

The Harte spectrum \( \sigma_\mathcal{H}(\mathcal{T}) \) of the commutative n-tuple \( \mathcal{T} \) is defined to be \( \sigma_\mu(\mathcal{T}) = \sigma(\mathcal{T}) \cup \sigma'(\mathcal{T}), \) where \( \sigma'(\mathcal{T}) \) (respectively, \( \sigma'(\mathcal{T}) \)) is the set of \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \) such that \( \{T_i - \lambda_i\}_{i=1}^n \) generates a proper left (resp., right) ideal in \( B(H). \) The (Harte) essential spectrum \( \sigma_\mathcal{E}(\mathcal{T}) \) is defined by \( \sigma_\mathcal{E}(\mathcal{T}) = \sigma(a), \) where \( a = (a_1, a_2, \ldots, a_n) = \pi(\mathcal{A}) \) and \( \pi \) is the canonical homomorphism of \( B(H) \) onto the Calkin algebra \( B(H)/K(H); K(H) \) is the algebra of compact operators on \( H. \) For a single linear operator, the (Harte) essential spectrum coincides with the essential spectrum; the following extends the conclusion \( \sigma_\mathcal{E}(A) = \sigma_\mathcal{E}(\mathcal{A}) \) of Lemma 0 to \( \sigma_\mathcal{E}(\mathcal{A}). \)

**Theorem 3.** \( \sigma_\mathcal{E}(\mathcal{A}) = \sigma_\mathcal{E}(\mathcal{A}). \)

**Proof.** Suppose \( \lambda \in \sigma_\mathcal{E}(\mathcal{A}). \) Then, \( \mathcal{A} \) being a hyponormal n-tuple, there exists a sequence \( \{x_k\} \) of unit vectors converging weakly to 0 in \( H \) such that
\[ \| (A_i - \lambda_i)x_k \| \to 0 \quad \text{as} \quad k \to \infty, \quad \text{for all} \quad 1 \leq i \leq n, \]
by [8, Theorem 2.6]. Let \( \{y_k\} \) be the sequence defined by
\[ y_k = \left( \prod_{i=1}^n |A_i|^{1/2} |\tilde{A}_i|^{1/2}x_k \right) / \left\| \prod_{i=1}^n |A_i|^{1/2} |\tilde{A}_i|^{1/2}x_k \right\|, \]
where "***" on the product \( \prod_{i=1}^n \) denotes that only those \( |A_i| \)'s and \( |\tilde{A}_i| \)'s appear in the product.
for which \( \lambda_i \neq 0 \). (Notice that if \( \|A_i\|^{1/2}x_k \) or \( \|\hat{A}_i\|^{1/2}x_k \) \( \to 0 \) as \( k \to \infty \), for some \( i \) with \( 1 \leq i \leq n \), then \( \|\hat{A}_i\|^{1/2}x_k \) and \( \|\hat{A}_i\|^{1/2}x_k \) \( \to 0 \) as \( k \to \infty \).) Since \( (x_k, h) \to 0 \) as \( k \to \infty \) for all \( h \in H \), \( (y_k, h) \to 0 \) as \( k \to \infty \) and

\[
\|(A_j - \lambda_j)^*y_k\| = \left\| \frac{\prod_{i=1}^{n} |A_i|^{1/2} \hat{A}_i^{1/2}}{\prod_{i=1}^{n} |A_i|^{1/2} \hat{A}_i^{1/2} x_k} (A_j - \lambda_j)^*x_k \right\| \to 0 \quad \text{as} \quad k \to \infty,
\]

for all \( 1 \leq j \leq n \). Thus \( \lambda \in \sigma_c(\mathcal{A}) \) and \( \sigma_c(\mathcal{A}) \subseteq \sigma_r(\mathcal{A}) \).

Consider now \( \lambda \in \sigma_c(\mathcal{A}) = \sigma_r^+(\mathcal{A}) \cup \sigma_r^{-}(\mathcal{A}) \). Suppose that \( \lambda \in \sigma_r^+(\mathcal{A}) \); then there exists a sequence \( \{x_k\} \) of unit vectors converging weakly to 0 in \( H \) such that \( \|(A_i - \lambda_i)x_i\| \to 0 \) as \( k \to \infty \), for all \( 1 \leq i \leq n \). Defining the sequence \( \{y_k\} \) by

\[
y_k = \frac{\left( \prod_{i=1}^{n} |A_i|^{1/2} \hat{A}_i^{1/2} x_k \right)}{\left( \prod_{i=1}^{n} |A_i|^{1/2} \hat{A}_i^{1/2} x_k \right)},
\]

(where \( \prod_{i=1}^{n} \) has a meaning similar to that above), an argument similar to that above shows that \( \{y_k\} \) is a sequence of unit vectors converging weakly to 0 in \( H \) such that

\[
\|(A_j - \lambda_j)y_k\| \to 0 \quad \text{as} \quad k \to \infty, \quad \text{for all} \quad 1 \leq i \leq n.
\]

Hence \( \lambda \in \sigma_r^+(\mathcal{A}) \). A similar argument shows that if \( \lambda \in \sigma_r^-(\mathcal{A}) \) then \( \lambda \in \sigma_r^-(\mathcal{A}) \). Thus \( \sigma_r(\mathcal{A}) \subseteq \sigma_r(\mathcal{A}) \), and the proof is complete.

**Corollary 4.** \( \sigma_c(\mathcal{A}) = \sigma_r(\mathcal{A}) \).

**Proof.** The argument of the proof of Theorem 3 implies that

\[
\sigma_c(\mathcal{A}) \subseteq \sigma_r(\mathcal{A}) = \sigma_c(\mathcal{A}) = \sigma_r(\mathcal{A}) \subseteq \sigma_r(\mathcal{A}).
\]

**Corollary 5.** \( \sigma_{1*}(\mathcal{A}) = \sigma_c(\mathcal{A}) \cup \sigma_r(\mathcal{A})^* \).

**Proof.** Let \( \sigma_{\mathcal{A}}(\mathcal{A}) = \{\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) : \text{there exists a sequence} \ \{x_k\} \ \text{of unit vectors in} \ H \ \text{such that} \ \|(A_i - \lambda_i)x_k\| \to 0 \ \text{as} \ k \to \infty, \ \text{for all} \ i = 1, 2, \ldots, n \} \) denote the joint approximate defect spectrum of \( \mathcal{A} \). Then

\[
\sigma_{1*}(\mathcal{A}) = \sigma^+(\mathcal{A}) \cup \sigma^-(\mathcal{A}) = \sigma_r(\mathcal{A}) \cup \sigma_b(\mathcal{A}).
\]

By Lemma 3, \( \sigma_{\mathcal{A}}(\mathcal{A}) = \sigma^{\mathcal{A}}(\mathcal{A}) \); applying an argument similar to that used in the proof of Lemma 2 to \( A_i^{1*} \) it is seen that \( \sigma_b(\mathcal{A}) = \sigma_b(\mathcal{A}) \). We have

\[
\sigma_{1*}(\mathcal{A}) = \sigma_r(\mathcal{A}) \cup \sigma_b(\mathcal{A}) = \sigma_r(\mathcal{A}) \cup \sigma_b(\mathcal{A}) = \sigma_r(\mathcal{A}) = \sigma_{1*}(\mathcal{A}),
\]
since $\mathcal{A}$ is a hyponormal $n$-tuple. Also, since
\[ \sigma_H(\mathcal{A}) = \sigma_E(\mathcal{A}) = \sigma_E(\mathcal{A}) \cup \sigma_H(\mathcal{A}^*) = \sigma_E(\mathcal{A}) \cup \sigma_H(\mathcal{A}^*)^*, \]
the proof is complete.

The $n$-tuple $(A_1, A_2, \ldots, A_n)$ is said to be essentially doubly commuting (resp., essentially $\mathcal{HU}(p)$) if the $n$-tuple $(a_1, a_2, \ldots, a_n)$, where $a_i = \pi(A_i)$ for all $1 \leq i \leq n$, and $\pi : B(H) \to B(H) \setminus \mathcal{K}(H)$, is doubly commuting (resp., $\mathcal{HU}(p)$). We close this note with the following result.

**Theorem 4.** Suppose $(A_1, A_2, \ldots, A_n)$ is an $n$-tuple of essentially doubly commuting essentially $\mathcal{HU}(p)$ operators. Then $A_1, A_2, \ldots, A_n$ have a common reducing subspace “modulo the compact operators”.

**Proof.** The hypotheses imply that $a_i \in \mathcal{HU}(p)$ for all $1 \leq i \leq n$ and that the $a_i$s are doubly commuting. Since $\sigma_E(\mathcal{A}) \cap \sigma_E(\mathcal{A})$ is not empty (this is consequence of the definition of essential spectrum—see [8, Lemma 4.2]), there exists $\lambda = (\lambda_1, \ldots, \lambda_n) \in \sigma_E(\mathcal{A}) \cap \sigma_E(\mathcal{A})$ and a non-zero projection $q$ in (the Calkin algebra) $B(H)/\mathcal{K}(H)$ such that
\[ a_i q = \lambda_i q \quad (1 \leq i \leq n). \]
Since $\sigma_E(a_i) = \sigma_{mp}(a_i)$, this implies that $a_i^* q = \lambda_i^* q$ $(1 \leq i \leq n)$. Consequently $a_i q = (\lambda_i q)^* = (a_i^* q)^* = qa_i$ $(1 \leq i \leq n)$, or, letting $\pi(Q) = q$, $(A_i Q - QA_i)$ is a compact operator, for all $1 \leq i \leq n$. This completes the proof.

**REFERENCES**


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