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ANOTHER CLASS OF GRACEFUL TREES

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Abstract

We discuss the problem of constructing large graceful trees from smaller ones and provide a partial answer in the case of the product tree $S_m\{\mathcal{G}\}$ by way of a sample of sufficient conditions on $\mathcal{G}$. Interlaced trees play an important role as building blocks in our constructions, although the resulting valuations are not always interlaced.

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1. Discussion

A *valuation* θ of a tree T on n vertices, $n > 1$, is a bijection between the vertex set of T and the set $N_n = \{1, 2, \ldots, n\}$. Denoting an edge with endpoints u and v by (u, v), the edge (u, v) of T carries under θ the *weight* $\omega(u, v) = |\theta(u) - \theta(v)|$. A valuation θ of T is, in the usage of Golumb (1972), *graceful* if distinct edges carry distinct weights, so that ω is a bijection between the edge set of T and the set N_{n-1}. T is *graceful* if it has a graceful valuation. Note that if θ is a graceful valuation of T then so is θ^+ where

(1) $$\theta^+(v) = n + 1 - \theta(v).$$

Are all trees graceful? Kotzig (see Rosa (1967); Bermond (1979), p. 24) conjectured that they are, but this conjecture remains open. A more specific question was recently given prominence by Cahit (1976): are all complete binary trees graceful? This question, it turned out, had already been answered affirmatively by Stanton and Zarnke (1973) in the course of a more general investigation. Nevertheless, Cahit's note stimulated fresh activity as reported by Guy

(1977). A conspectus of results is provided by Koh, Rogers and Tan (1979) (hereinafter their paper is referred to as KRT) who, in addition to rediscovering some already known constructions, extended the range of results with some new ones. In particular, these authors showed that trees (*interlaced trees*) having a type of graceful valuation known as an *interlaced valuation* were useful as building blocks in these constructions. (A more comprehensive survey of work on graceful graphs has recently been given by Bermond (1979).)

An *interlaced valuation* of a tree T is defined as follows. The *parity set* $\mathcal{P}(v)$ of a vertex v in T is the set of vertices u (including v) for which the number of edges in the shortest path in T between u and v is even. The *base* of T under a valuation θ is the vertex b with $\theta(b) = 1$. A valuation θ of T is a *parity valuation* if it induces, by restriction, a bijection between $\mathcal{P}(b)$ and N_p where b is the base of T under θ and p, the *size* of T under θ, is the cardinality of $\mathcal{P}(b)$. Finally, an *interlaced valuation* is a parity valuation which is also graceful and an *interlaced tree* is a tree with an interlaced valuation. Again notice that if θ is an interlaced valuation of T then so is θ^+ given by (1) as is θ' where

$$\theta'(v) = \begin{cases} p + 1 - \theta(v) & \text{if } \theta(v) \leqslant p, \\ n + p + 1 - \theta(v) & \text{if } p < \theta(v). \end{cases} \tag{2}$$

Several of the valuations of complete binary trees illustrated by Cahit (1976) are interlaced as was noted by Rogers (1978). There is indeed always at least one interlaced tree on n vertices of size p with $1 \leqslant p \leqslant [\frac{1}{2}(n + 1)]$ (where $[x]$ denotes the integer part of x) and so also, in view of (1), of size $n - p$. The *caterpillars* (trees which on deletion of endpoints and adjacent edges leave chains) provide easy examples, but it would be interesting to know how many interlaced trees there are for each n and p. Of special utility in the constructions which we describe later are interlaced trees on n vertices with size $[\frac{1}{2}(n + 1)]$: we call trees with this property *fair trees* and their associated valuations *fair valuations* (see Figure 1).

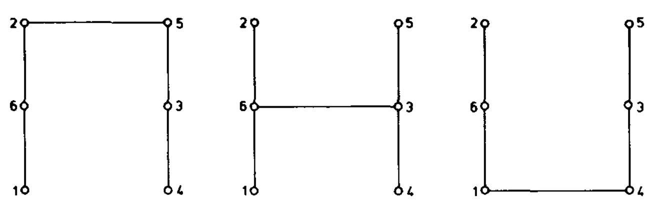

FIGURE 1. Fair trees on six vertices

We introduce this notion of fair trees as a technical device which enables us to construct graceful trees without the 'symmetry' conditions previously imposed (see KRT and Bermond (1979), p. 27). This, in itself, is a new and interesting

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departure which, in conjunction with the staggering technique introduced in Section 2, may suggest further, similar results. But it also gives an additional, large class of graceful trees since fair trees are themselves plentiful.

In the first place, caterpillars again readily provide a collection of examples: for instance, the path (or chain) P_n on $n + 1$ vertices, $n \geqslant 1$, has a fair valuation under which one endpoint is the base. Further, fair trees may be constructed recursively as follows. Let T be a fair tree with fair valuation θ and let (a_1, a_2) be an edge whose deletion decomposes T into two disjoint components T_1 and T_2. Suppose that, for $i = 1$ or 2, b_i is a vertex of T_i such that b_1 and b_2 are an odd distance apart in T and

$$|\theta(b_1) - \theta(b_2)| = \omega(a_1, a_2) = |\theta(a_1) - \theta(a_2)|.$$

If T^* is the tree obtained from T by deleting the edge (a_1, a_2) and adding an edge joining b_1 and b_2, then θ is a fair valuation of T^*. Thus, T^* is again a fair tree. Some examples are shown in Figure 1. (These are examples of what Ruiz Cornejo (1979) calls 'path-like' trees: more generally, 'path-like' trees are otained in this recursive way from the fair valuation of the paths mentioned above.) As a step towards the seemingly difficult problem of characterising interlaced trees it would be interesting to characterise fair trees, although it would probably be difficult to determine the proportion of these among all (respectively, graceful) trees on n vertices.

Although, in full generality, Kotzig's conjecture appears difficult, it is fruitful to ask: when can a large graceful tree be assembled from smaller graceful trees? As an aid in examining this problem, we now introduce the further notion of *product trees*. Thus, let H be a labelled tree on $m + 1$ vertices and let $\mathcal{G} = \mathcal{G}_m$ be a set of rooted trees G_i, $0 \leqslant i \leqslant m$, disjoint from each other and from H. Then we denote by $H\{\mathcal{G}\}$ (respectively, $H(\mathcal{G})$) the tree obtained by identifying the root of G_i with the ith vertex of H for each i, $1 \leqslant i \leqslant m$ (respectively, $0 \leqslant i \leqslant m$). Also $\langle \mathcal{G} \rangle$ is the tree obtained by identifying the roots of the G_i, $0 \leqslant i \leqslant m$. The *apex* of $H\{\mathcal{G}\}$ or $H(\mathcal{G})$ is the vertex of H labelled 0 while that of $\langle \mathcal{G} \rangle$ is the identified roots of the G_i.

There is considerable flexibility in this notation since any tree may be viewed as such a product tree in several ways. In these terms, our problem is to discover conditions on H and $\mathcal{G}$ which ensure that the resulting product tree $H\{\mathcal{G}\}$ or $H(\mathcal{G})$ or $\langle \mathcal{G} \rangle$ is graceful. Some sufficient conditions of this sort are presented in KRT. The purpose of this note is to observe that more can be said in the case where H is the *star* tree S_m on $m + 1$ vertices, $m \geqslant 1$, that is the tree on vertices v and v_i, $1 \leqslant i \leqslant m$, where v_i is adjacent to v and is labelled i while v is labelled 0. In this case, we have the following theorem, Theorem A, which provides a sample of sufficient conditions on $\mathcal{G}$ for $S_m^* = S_m(\mathcal{G})$ to be graceful.

THEOREM A. *Let* G_i, $1 \leqslant i \leqslant m$, *be a tree on* n_i *vertices with graceful valuation* θ_i *with* G_i *rooted at the base under* θ_i. *Then some sufficient conditions for* S_m^* *to be graceful are that for each* i, $1 \leqslant i \leqslant m$:

(A1) G_i *is an isomorphic copy of a tree* G *on* n *vertices with graceful valuation* θ *and* θ_i *is the valuation of* G_i *induced by the isomorphism*;

(A2) $n_i = n$ (*independent of* i) *and* θ_i *is an interlaced valuation under which* G_i *has size* $p_i = p$ (*independent of* i);

(A3) θ_i *is a fair valuation and* $n_i = k[(i + r - 1)/r] + 1$ *where* k *and* r *are fixed positive integers with either* (3a) $k \geqslant 1$ *and* $r = 1$; *or* (3b) $k = 1$, $r = 2$ *and* $m \not\equiv 1$ (*modulo* 4); *or* (3c) $k \geqslant 2$ *and* $r + 1 \geqslant m$;

(A4) θ_i *is a fair valuation and, for some fixed integer* $k \geqslant 2$, $n_1 = k$; $n_2 = 2[\frac{1}{2}k] + 1$ *or* $2[\frac{1}{2}k] + 2$; $n_{i+1} = 2n_i$ *or* $2n_i - 1$, $2 \leqslant i < m - 1$; *and* $n_m = 2n_{m-1} - j$, $0 \leqslant j \leqslant 2$.

Moreover, in all these cases, S_m^* *has a graceful valuation under which the apex is the base.*

Note that if $\mathcal{J}$ is the set of graphs J_i, $1 \leqslant i \leqslant m$, obtained from the G_i by adding a pendent edge adjacent to the root of G_i and rerooting at the new vertex, then $\langle \mathcal{J} \rangle = S_m\{\mathcal{G}\}$. So this theorem, like Theorems 5 and 6 in KRT, may also be regarded as giving sufficient conditions under which an amalgamated product is graceful. Indeed conditions (A1) and (A2) are straightforward cases of the conditions in those theorems and we do not mention them further here. However, condition (A4) comes from application of Theorem 8 of KRT which we state and discuss, as Theorem C, in Section 3, together with a similar result, Theorem B, not previously described, from which the sufficiency of condition (A3c) may, for example, be deduced. Of greater interest are conditions (A3a) and (A3b) in the proof for which we employ a new type of graceful valuation, called a *staggered valuation* (see Section 2). The case $k = 1 = r$ of (A3) has also been established by Pastel and Raynaud (1978) using a different technique.

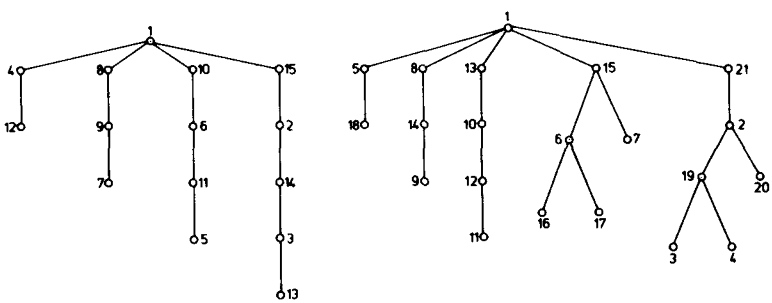

FIGURE 2. Examples of staggered valuations: condition (A3a)

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Although we prove, in Sections 3 and 4, some more general results than those stated in Theorem A, we single out (A3) and (A4) as giving specific classes of graphs not previously shown to be graceful (see Figures 2 and 4). We include the statements of the known 'symmetrical' cases (A1) and (A2) for comparison with these new 'asymmetrical' ones in which the G_i need not have the same number of vertices (compare Bermond (1979), p. 27).

There are several other minor results on the gracefulness of S_m^*. Thus if π is a permutation of N_m, $\pi\mathcal{G}$ is the set of trees $G_{\pi i}$, $0 \leqslant i \leqslant m$, and $S_m\{\mathcal{G}\}$ is graceful, then so is $S_m\{\pi\mathcal{G}\}$. Also, if $S_m\{\mathcal{G}\}$ has a graceful valuation under which the apex is base, then $\mathcal{G}$ may be augumented by any number of trees on single vertices and the result still holds. So the conclusions of Theorem A also follow if $k = 1$ in (A3c) or (A4). However, it is not true, in general, that the apex of S_m^* is the base of a graceful valuation: the tree in Figure 3 with apex a as shown provides a counterexample.

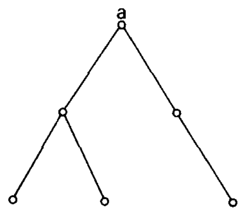

FIGURE 3

In order to break Kotzig's conjecture into more manageable pieces, it may be worthwhile investigating the gracefulness of these product trees more systematically. For example, we conjecture that (A3) is sufficient without the given restrictions on k and r and Theorem C appears to offer some help. This note represents only a small contribution to this project. Golomb (1972) defines the notion of gracefulness for general graphs and we may likewise ask: when are product graphs graceful in this extended sense?

2. Staggered valuations

Let G_i, $1 \leqslant i \leqslant m$, be a tree on n_i vertices, rooted at its base under an interlaced valuation θ_i. Let p_i be the size of G_i under θ_i and write $p_i^+ = n_i - p_i$ so that p_i^+ is the size of G_i under θ_i^+ (see (1)). Further, let $v_{i,j}$, $1 \leqslant j \leqslant n_i$, be the vertex of G_i for which $\theta_i(v_{i,j}) = j$. We also write

$$n = \sum_{i=1}^{m} n_i; \qquad p = \sum_{i=1}^{m} p_i; \qquad p^+ = \sum_{i=1}^{m} p_i^+ = n - p.$$

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S_m^* is then a tree on the $n+1$ vertices v and $v_{i,j}$, $1 \leqslant j \leqslant n_i$; $1 \leqslant i \leqslant m$, with the identifications $v_i = v_{i,1}$, $1 \leqslant i \leqslant m$.

We define a valuation θ of S_m^* by staggering the vertex labels in consecutive runs in the G_i according to the underlying valuations θ_i. Since these are interlaced, this arrangement produces runs of consecutive weights on the edges of the G_i. Moreover, because of the way θ is defined, there is a gap of a single missing weight between numerically adjacent runs. Thus, to ensure that θ is graceful, we need to arrange that these missing weights appear on the edges $(v, v_{i,1})$, $1 \leqslant i \leqslant m$.

In order to define θ explicitly, we introduce the following notation:

$$e(r) = 1, \quad r \text{ even}; \quad = 0, \quad r \text{ odd};$$

$$\bar{r} = m - 2\left[\tfrac{1}{2}(r+1)\right];$$

and

$$n(r) = \sum_{i=1}^{r} n_i; \qquad p(r) = \sum_{i=1}^{r} p_i; \qquad p^+(r) = \sum_{i=1}^{r} p_i^+ = n(r) - p(r),$$

where, by convention, empty or impossible sums are zero (so, for example, $n(0) = p(0) = p^+(0) = 0$). Then we define θ on the vertex set of S_m^* by $\theta(v) = 1$ (so the apex is the base) and, for $0 \leqslant r \leqslant m$,

$$\theta(v_{m-r,j}) = \begin{cases} 1 + n + p(\bar{r}) - p - p^+(r) - (-1)^r j + e(r), & 1 \leqslant j \leqslant p_i; \\ 1 + p^+(\bar{r}) - p^+ - p(r) - (-1)^r j + (n_{m-r} + 1)e(r), & p_i < j \leqslant n_i. \end{cases}$$

Some examples are illustrated in Figures 2 and 4.

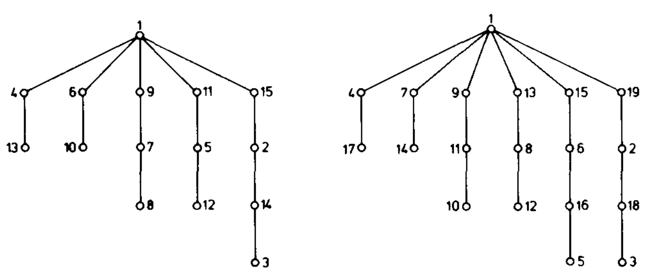

FIGURE 4. Examples of staggered valuations: condition (A3b).

The weights under θ not appearing on the edges of any G_i in S_m^* are, in descending order,

$$n\left(m - \left[\tfrac{1}{2}(r+1)\right]\right) - n\left(\left[\tfrac{1}{2}r\right]\right), \qquad 0 \leqslant r < m.$$

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So θ is graceful if these are equal to the weights under θ on the edges $(v, v_{m-r,1})$, $0 \leqslant r \leqslant m$, that is:

$$\theta(v_{m-r,1}) - 1 = n\left(m - \left[\tfrac{1}{2}(r+1)\right]\right) - n\left(\left[\tfrac{1}{2}r\right]\right), \qquad 0 \leqslant r < m. \tag{3}$$

When condition (3) is satisfied the weights under θ occur, in descending order, on (v, v_m), on the edges of G_m, on (v, v_{m-1}), on the edges of G_1, on (v, v_{m-2}), on the edges of G_{m-1}, on (v, v_{m-3}), on the edges of G_2, and so on. Now (3) gives conditions on the n_i and p_i for θ to be graceful. It is easy to check that if (A3a) or (A3b) holds then these conditions are indeed satisfied (see, for example, Figure (2)), although they are also satisfied in many other cases.

3. Two graceful constructions

Theorems 7 and 8 of KRT are, on the surface, apparently simple results, but they may be used to show that several families of trees are graceful, even if it would be complicated to determine which trees arise in this way. In this section, we establish a kindred result, Theorem B, as well as giving an application of Theorem 8 of KRT. These results provide some conditions under which two trees may be put together to form a larger graceful tree. The condition in Theorem B is similar to those in Theorems 1 and 2 of KRT.

Thus, let T_i, $i = 1, 2$, be disjoint trees on t_i vertices having graceful valuations φ_i with bases b_i. If φ_i is interlaced, then the size of T_i under φ_i is q_i, $i = 1, 2$. Also, $\mathcal{A}(b_i)$ is the set of vertices in T_i adjacent to b_i, $i = 1, 2$.

THEOREM B. *Suppose that φ_2 is an interlaced valuation and that*

$$\{\varphi_1(v) - 1 : v \in \mathcal{A}(b_1)\} = \{t_1 + q_2 - \varphi_1(u) : u \in \mathcal{A}(b_1)\}. \tag{4}$$

Let T be the tree obtained from T_1 and T_2 by making the identification $b_1 = b_2 = b$. Then T has a graceful valuation φ under which the base is b. Moreover, if φ_1 is interlaced, then so is φ.

PROOF. Define φ on the vertex set of T by

$$\varphi(v) = \begin{cases} \varphi_2(v), & \varphi_2(v) \leqslant q_2;\ v \text{ in } T_2; \\ \varphi_2(v) + t_1 - 1, & \varphi_2(v) > q_2;\ v \text{ in } T_2; \\ \varphi_1^+(v) + q_2, & v \text{ in } T_1. \end{cases}$$

Then φ is a valuation of T. The edges of T_2 carry under φ the weights t_1 to $t_1 + t_2 - 1$ while those of T_1 not adjacent to $b = b_1$ carry under φ of the same weights as they carried under φ_1 and so also under φ_1^+ (see (1)). Condition (4)

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ensures that the sets of weights on the edges of T_1 adjacent to $b = b_1$ under φ and φ_1 are the same (as sets). Hence φ is a graceful valuation of T with base b. The parity set of b in T is the union of the parity sets of the b_i in the set T_i. So if φ_1 is also interlaced, then, in view of the way φ is defined, φ is interlaced as well.

Now, by condition (A2) of Theorem A, if G_i is a tree on $k[(i + r - 1)/r] + 1$ vertices, rooted at its base under a fair valuation θ_i, then $S_m\{\mathcal{G}_m\}$, $1 \leqslant i \leqslant m \leqslant r$, is graceful with the apex as base and so (A3c) is sufficient when $m \leqslant r$. In particular, with $m = r$, take $T_1 = S_m\{\mathcal{G}_m\}$ with φ_1 this graceful valuation under which the apex is the base. Also take T_2 to be a chain on $k[(m + r)/r] + 2$ vertices with φ_2 a fair valuation of T_2 having one endpoint as base. Then, by Theorem B, the resulting tree T is graceful and this establishes the sufficiency of (A3c) with $m = r + 1$. Another illustration of Theorem B is shown in Figure 5.

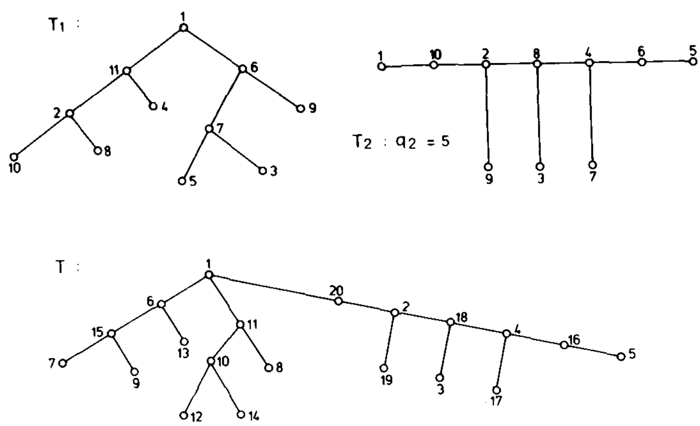

FIGURE 5. An example of Theorem B.

The following is a restatement of Theorem 8 of KRT.

THEOREM C. *Suppose that both φ_1 and φ_2 are interlaced. Suppose further that there are vertices u_i in T_i, $i = 1, 2$, such that either*

(i) $\varphi_1(u_1) - \varphi_2(u_2) = q_1 < \varphi_1(u_1)$

or

(ii) $t_2 + \varphi_1(u_1) - \varphi_2(u_2) = q_1 \geqslant \varphi_1(u_1)$.

Let T be the tree obtained from T_1 and T_2 by joining them at u_1 and u_2 by a new edge. Then T has a graceful valuation φ under which b_1 is the base. Further if, in the two cases (i) $\varphi_2(u_2) \leqslant q_2$ *or* (ii) $\varphi_2(u_2) > q_2$, *then φ is interlaced.*

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The proof is again a matter of defining a graceful valuation φ on T and, since it is given in KRT, we omit it here. However, the following corollary is worth noting.

COROLLARY. *Suppose that $S_m\{\mathcal{G}_m\}$, a tree on n vertices, has an interlaced valuation ψ under which the apex is the base and that the size of $S_m\{\mathcal{G}_m\}$ is p. Suppose further that G_{m+1} has an interlaced valuation χ under which G_{m+1} has size $n - p$. Then $S_{m+1}\{\mathcal{G}_{m+1}\}$ also has an interlaced valuation with apex as base.*

PROOF. The Corollary follows from Theorem C, case (i), on taking $T_1 = S_m\{\mathcal{G}_m\}$ and $T_2 = G_{m+1}$; $\varphi_1 = \psi^+$ (see (1)) and $\varphi_2 = \chi'$ (see (2)); $u_1 = b_1$ and $u_2 = b_2$. (Note that $\varphi_1(u_1) = n$ and $\varphi_2(u_2) = n - p$ so that $\varphi_1(u_1) - \varphi_2(u_2) = p = q_1$.) Thus φ^+ is an interlaced valuation of $T = S_{m+1}\{\mathcal{G}_{m+1}\}$ under which the apex is the base.

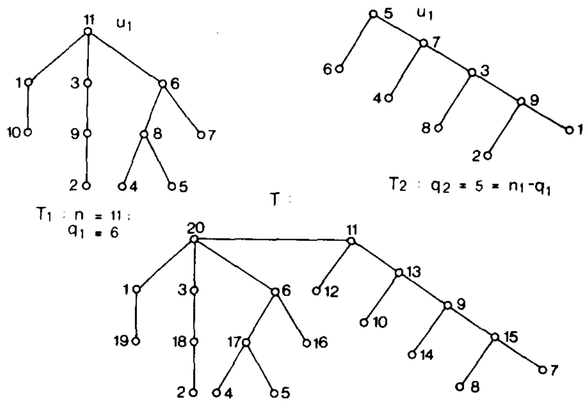

FIGURE 6. An example of Theorem C, case (i) (Corollary).

Condition (A4) then comes, by induction, from the Corollary, starting with $S_1\{\mathcal{G}_1\}$ as a fair tree on $k + 1$ vertices in which the base has valence one and is taken as the apex. An instance of (A4) is shown in Figure 6. Again, many other conditions for the gracefulness of S_m^* may be obtained in this way. Similarly, for the path P_m on $m + 1$ vertices, we may establish conditions on $\mathcal{G}$ for the gracefulness of $P_m(\mathcal{G})$ (compare Figures (9–13) of KRT).

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