BULL. AUSTRAL. MATH. SOC. MOS 26A82, 42A08
VOL. 6 (1972), 11-18.

Error bounds in the approximation of functions

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Let $f(x) \in \text{Lip}\alpha$, $0 < \alpha < 1$, in the range $(-\pi, \pi)$, and periodic with period 2π, outside this range. Also let

$$(*) \qquad f(x) \sim \frac{1}{2} a_0 + \sum_{\nu=1}^{\infty} \left(a_\nu \cos\nu x + b_\nu \sin\nu x\right) \equiv \sum_{\nu=0}^{\infty} A_\nu(x) .$$

We define the norm as

$$\|f\|_p = \left\{\int_0^{2\pi} |f(x)|^p dx\right\}^{1/p} , \quad p \geq 1 ,$$

and let the degree of approximation be given by

$$E_n^*(f) = \min_{T_n} \|f-T_n\|_p$$

where $T_n(x)$ is some n-th trigonometric polynomial.

We define a generating sequence $\{p_n\}$ such that it is non-negative, non-increasing and

$$(**) \qquad P(n) = p_0 + p_1 + \ldots + p_n \to \infty \quad \text{as} \quad n \to \infty .$$

Approximation of functions belonging to the class $\text{Lip}\alpha$ by the (c, δ), $0 < \delta \leq 1$, mean of its Fourier series is due to Chapman and Riesz. The following is the main result of our paper:

THEOREM. *If* $f(x)$ *is periodic and belongs to the class* $\text{Lip}(\alpha, p)$, $0 < \alpha \leq 1$, *and if the sequence* $\{p_n\}$ *is defined*

Received 2 August 1971.

https://doi.org/10.1017/S0004972700044208 Published online by Cambridge University Press

as in (**), *and if*

$$\left(\int_1^n \frac{(P(y))^q}{y^{q\alpha+2-q}}\,dy\right)^{1/q} = O\left(P(n)/n^{\alpha+1/q-1}\right) ,$$

then

$$E_n^*(f) = \min_{N_n} \|f-N_n\|_p = O\left(1/n^{\alpha}\right) ,$$

where $N_n(x)$ *is the* $(N,\, p_n)$ *mean of the Fourier series* (*) *and, in particular,* $T_n(x) = N_n(x)$.

1.

We define the norm as

(1.1) $$\|f\|_p = \left\{\int_0^{2\pi} |f(x)|^p dx\right\}^{1/p} , \quad p \geq 1 ,$$

and let the degree of approximation be given by (see [7])

(1.2) $$E_n^*(f) = \min_{T_n} \|f-T_n\|_p .$$

Here $T_n(x)$ is some n-th trigonometric polynomial.

Let $f(x) \in \mathrm{Lip}\alpha$, $0 < \alpha < 1$, in the interval $(-\pi,\, \pi)$, and periodic with period 2π outside this range. Also let

(1.3) $$f(x) \sim \frac{1}{2}a_0 + \sum_{\nu=1}^{\infty} \left(a_\nu \cos \nu x + b_\nu \sin \nu x\right)$$

$$= \sum_{\nu=0}^{\infty} A_\nu(x) .$$

We write

(1.4) $$\phi(x,\, t) = \frac{f(x+t)+f(x-t)}{2} - f(x) .$$

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2.

The following theorems are known:

THEOREM A. *If the periodic function* $f(x)$ *belongs to the class* $\mathrm{Lip}\alpha$ *, for* $0 < \alpha < 1$ *, then the* (c, δ) *mean of its Fourier series for* $0 < \alpha < \delta \le 1$ *, gives*

$$\max_{0\le x\le 2\pi} \left|f(x)-\sigma_n^{\delta}(x)\right| = O\left(1/n^{\alpha}\right) , \tag{2.1}$$

and for $0 < \alpha \le \delta \le 1$ *satisfies*

$$\max_{0\le x\le 2\pi} \left|f(x)-\sigma_n^{\delta}(x)\right| = O\left(\log n/n^{\alpha}\right) , \tag{2.2}$$

where $\sigma_n^{\delta}(x)$ *is the* (c, δ) *mean of the partial sum of* (1.3).

THEOREM B. *If the periodic function* $f(x)$ *belongs to the class* $\mathrm{Lip}\alpha$ *, for* $0 < \alpha \le 1$ *, then the* $(c, 1)$ *mean of its Fourier series is given by*

$$\frac{1}{n}\sum_{k=1}^{n} \left|f(x)-S_k(x)\right| = O\left(\log n/n^{\alpha}\right) , \tag{2.3}$$

where S_k *is the partial sum of* (1.3).

It is known [3] that for $\alpha = 1$ the order of (2.3) is not $O(1/n)$.

Theorem A was proved by Chapman and Riesz (see [1]) independently. Theorem B is a simplified form of the result due to Alexits and Leindler [3]. Later Alexits and Králik [2] changed the summation in (2.3) from $k = n$ to $k = 2n - 1$, along with some other improvements.

Let $\{p_n\}$ be a non-negative, non-increasing generating sequence for the (N, p_n) method such that

$$P_n \equiv P(n) = p_0 + p_1 + \dots + p_n \to \infty \text{ as } n \to \infty . \tag{2.4}$$

Some of the related recent work on the Nörlund method (N, p_n) is due to Izumi and Izumi [4] and Sahney [6].

The object of this paper is to prove the following theorems:

THEOREM 1. *If* $f(x)$ *is periodic and belongs to the class* $\mathrm{Lip}(\alpha, p)$

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for $0 < \alpha < 1$, *such that* $0 < \alpha < \delta \le 1$, *then*

(2.5)
$$E_n^*(f) = \min_{T_n} \left\| f - \sigma_n^{(\delta)} \right\|_p = O\left(1/n^{\alpha - 1/p}\right)$$

where $\sigma_n^{(\delta)}(x)$ *is the* (c, δ) *mean of* (1.3). *Here, in particular,* $T_n(x) = \sigma_n^{\delta}(x)$.

THEOREM 2. *If* $f(x)$ *is periodic and belongs to the class* $\mathrm{Lip}(\alpha, p)$ *for* $0 < \alpha \le 1$, *and if the sequence* $\{p_n\}$ *is as defined in* (2.4) *with the other requirements therein and if*

(2.6)
$$\left(\int_1^n \frac{(P(y))^q}{y^{q\alpha+2-q}} \, dy \right)^{1/q} = O\left(\frac{P(n)}{n^{\alpha+1/q-1}} \right) ,$$

then

(2.7)
$$E_n^*(f) = \min_{T_n} \| f - N_n \|_p = O\left(\frac{1}{n^{\alpha}} \right) ,$$

where $N_n(x)$ *is the* (N, p_n) *mean of* (1.3), *and, in particular,* $T_n(x) = N_n(x)$.

3.

Proof of Theorem 1. Following Zygmund [7] we can write

(3.1)
$$f(x) - \sigma_n^{(\delta)}(x) = \frac{1}{\pi A_n^{\delta}} \int_0^{\pi} \phi(t) \sum_{k=0}^{n} \frac{A_k^{\delta-1} \sin kt}{t} \, dt + o(1)$$
$$= \frac{1}{\pi A_n^{\delta}} \left[\int_0^{\pi/n} + \int_{\pi/n}^{\pi} \right] \phi(t) \sum_{k=0}^{n} \frac{A_k^{\delta-1} \sin kt}{t} \, dt + o(1)$$
$$= I_1 + I_2 + o(1) \text{ , say.}$$

By Hölder's inequality and the fact that $\phi(x) \in \mathrm{Lip}(\alpha, p)$, we get

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$$(3.2)\qquad I_1 = \frac{1}{\pi A_n^{\delta}} \int_0^{\pi/n} \frac{\phi(t)}{t^{\alpha}} \sum_{k=0}^{n} \frac{A_k^{\delta-1}\sin kt}{t^{1-\alpha}}\,dt$$

$$= O\left(\frac{1}{n^{\delta}}\right)\left(\int_0^{\pi/n} \left|\frac{\phi(t)}{t^{\alpha}}\right|^p dt\right)^{1/p}\left(\int_0^{\pi/n} \left|\sum_{k=0}^{n} \frac{A_k^{\delta-1}\sin kt}{t^{1-\alpha}}\right|^q dt\right)^{1/q}$$

$$= O\left(\frac{1}{n^{\delta}}\right)\cdot o(1)\left\{\int_0^{\pi/n} \left(O\left(\frac{n^{\delta}\cdot nt}{t^{1-\alpha}}\right)\right)^q dt\right\}^{1/q}$$

$$= o(n)\left(\int_0^{\pi/n} t^{\alpha q}dt\right)^{1/q}$$

$$= o(n)\left(\left[t^{1+\alpha q}\right]_0^{\pi/n}\right)^{1/q}$$

$$= o\left(\frac{1}{n^{\alpha-(1-1/q)}}\right)$$

$$= o\left(\frac{1}{n^{\alpha-1/p}}\right) ,$$

where $1/p + 1/q = 1$ such that $1 \le p \le \infty$.

Also, similarly, as above

$$(3.3)\qquad I_2 = O\left(\frac{1}{n^{\delta}}\right)\left(\int_{\pi/n}^{\pi} \left|\frac{\phi(t)}{t^{\alpha}}\right|^p dt\right)^{1/p}\left(\int_{\pi/n}^{\pi} \left|\sum_{k=0}^{n} \frac{A_k^{\delta-1}\sin kt}{t^{1-\alpha}}\right|^q dt\right)^{1/q}$$

$$= O\left(\frac{1}{n^{\delta}}\right)o(1)\left(\int_{\pi/n}^{\pi} \left(\frac{1}{t}\right)^{\delta q} \frac{dt}{t^{(1-\alpha)q}}\right)^{1/q}$$

$$= o\left(\frac{1}{n^{\delta}}\right)\left(\int_{\pi/n}^{\pi} \frac{dt}{t^{q+\delta q-\alpha q}}\right)^{1/q}$$

$$= o\left(\frac{1}{n^{\delta}}\right)\left(\left[t^{1-q+\alpha q-\delta q}\right]_{\pi/n}^{\pi}\right)^{1/q}$$

$$= o\left(\frac{1}{n^{\delta}}\right) + o\left(\frac{1}{n^{\delta}}\right)\left(\frac{1}{n^{1/q-1+\alpha-\delta}}\right)$$

$$= o\left(\frac{1}{n^{\delta}}\right) + o\left(\frac{1}{n^{\alpha-1/p}}\right) .$$

Hence

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(3.4) $$I = O\left(\frac{1}{n^{\alpha-1/p}}\right) + O\left(\frac{1}{n^{\delta}}\right) + O\left(\frac{1}{n^{\alpha-1/p}}\right) .$$

Therefore

(3.5) $$E_n^*(f) = \min_{\sigma_n^{(\delta)}} \left\|f-\sigma_n^{\delta}\right\|_p = O\left(1/n^{\alpha-1/p}\right) .$$

4.

Proof of Theorem 2. Following Zygmund [7] we write

(4.1) $$f(x) - N_n(x) = \frac{2}{\pi P(n)} \int_0^{\pi} \frac{\phi(t)}{t} \sum_{k=0}^{n} p_k \sin(n-k)t\,dt + o(1)$$
$$= \frac{2}{\pi P(n)} \left(\int_0^{\pi/n} + \int_{\pi/n}^{\pi}\right) \frac{\phi(t)}{t} \sum_{k=0}^{n} p_k \sin(n-k)t\,dt + o(1)$$
$$= J_1 + J_2 + o(1) \text{ , say.}$$

Applying Hölder's inequality and then by the fact that $\phi(x) \in \mathrm{Lip}(\alpha, p)$, we have

(4.2) $$J_1 = O\left(\frac{1}{P(n)}\right) \left(\int_0^{\pi/n} \left|\frac{\phi(t)}{t^{\alpha}}\right|^p dt\right)^{1/p} \left(\int_0^{\pi/n} \left|\frac{\sum_{k=0}^{n} p_k \sin(n-k)t}{t^{1-\alpha}}\right|^q dt\right)^{1/q}$$
$$= O\left(\frac{1}{P(n)}\right) \cdot O(1) \left\{\int_0^{\pi/n} \left(\frac{O(P(n)\cdot nt)}{t^{1-\alpha}}\right)^q dt\right\}^{1/q}$$
$$= O(n) \left(\left[t^{\alpha q+1}\right]_0^{\pi/n}\right)^{1/q}$$
$$= O(n) \cdot O\left(1/n^{\alpha+1/q}\right)$$
$$= O\left(1/n^{\alpha+(1/q-1)}\right)$$
$$= O\left(1/n^{\alpha-1/p}\right) .$$

Similarly

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$$(4.3)\quad J_2 = O\left(\frac{1}{P(n)}\right)\left(\int_{\pi/n}^{\pi}\left|\frac{\phi(t)}{t^{\alpha}}\right|^p dt\right)^{1/p}\left(\int_{\pi/n}^{\pi}\left|\frac{\sum_{k=0}^{n} p_k \sin(n-k)t}{t^{1-\alpha}}\right|^q dt\right)^{1/q}$$

$$= O\left(\frac{1}{P(n)}\right)\cdot O(1)\left(\int_{\pi/n}^{\pi}\left|\frac{P(1/t)}{t^{1-\alpha}}\right|^q dt\right)^{1/q}$$

$$= O\left(\frac{1}{P(n)}\right)\left(\int_1^n \frac{(P(y))^q}{y^{(\alpha-1)q}}\cdot\frac{dy}{y^2}\right)^{1/q}$$

$$= O\left(\frac{1}{P(n)}\right)\left(\int_1^n \frac{(P(y))^q}{y^{\alpha q-q+2}}\, dy\right)^{1/q}$$

$$= O\left(\frac{1}{P(n)}\right)\cdot O\left(\frac{P(n)}{n^{\alpha+1/q-1}}\right)$$

$$= O\left(1/n^{\alpha-1/p}\right) .$$

Hence

$$(4.4)\quad E_n^*(f) = \min_{N_n} \|f-N_n\|_p = O\left(\frac{1}{n^{\alpha-1/p}}\right) .$$

This completes the proof of Theorem 2.

5.

As $p \to \infty$ (and therefore $q = 1$), Theorem 1 is equivalent to Theorem A, for $0 < \alpha < \delta \le 1$. Also Theorem 2, for $p_n = \binom{n+\delta-1}{\delta-1}$ is equivalent to Theorem 1.

References.

[1] G. Alexits, *Convergence problems of orthogonal series* (Pergamon Press, New York, Oxford, London, Paris, 1961).

[2] G. Alexits und D. Králik, "Über die Approximation mit starken de la Vallée-Poussinschen Mitteln", *Acta Math. Acad. Sci. Hungar.* 16 (1965), 43-49.

https://doi.org/10.1017/S0004972700044208 Published online by Cambridge University Press

[3] G. Alexits und L. Leindler, "Über die Approximation im starken Sinne", *Acta Math. Acad. Sci. Hungar.* 16 (1965), 27-32.

[4] Masako Izumi and Shin-ichi Izumi, "On the Nörlund summability of Fourier series", *Proc. Japan Acad.* 45 (1969), 773-778.

[5] G.G. Lorentz, *Approximation of functions* (Holt, Rinehart & Winston, New York; Chicago, Illinois; Toronto, Ontario; 1966).

[6] Badri N. Sahney, "On the Nørlund summability of Fourier series", *Canad. J. Math.* 22 (1970), 86-91.

[7] A. Zygmund, *Trigonometric series*, Vol. 1, 2nd ed. (Cambridge University Press, Cambridge, 1959).

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https://doi.org/10.1017/S0004972700044208 Published online by Cambridge University Press