HIGHER-ORDER OPTIMALITY CONDITIONS FOR A MINIMAX
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Higher-order necessary and sufficient optimality conditions for a nonsmooth minimax problem with infinitely many constraints of inequality type are established under suitable basic assumptions and regularity conditions.

1. INTRODUCTION

Let $C$ be a nonempty subset of a normed space $X$, and let $Q$ and $B$ be nonempty sets. Let $f_\alpha (\alpha \in Q)$ and $g_\beta (\beta \in B)$ be real-valued functions on $X$. We consider the following minimax problem:

\[(P) \quad \min \{F(x) \mid x \in C, G(x) \leq 0\},\]

where $F(x) := \sup_{\alpha \in Q} f_\alpha(x)$ and $G(x) := \sup_{\beta \in B} g_\beta(x)$.

Our aim here is to develop higher-order optimality conditions for (P) by using suitable approximations to the functions involved. Thus our results will be formulated in terms of approximating functions $\phi^{(k)}(\cdot)$ and $\psi^{(k)}(\cdot)$. These may be thought of as substitutes for the $k$-th order directional derivatives of $f_\alpha$ and $g_\beta$, but are considerably more general. For instance, $\phi^{(k)}(\cdot)$ and $\psi^{(k)}(\cdot)$ do not need to be positively homogeneous of degree $k$. The lack of homogeneity forces us to use a particular kind of regularity condition (namely condition 2.3 below). Our approach extends the technique we have used in [6] to derive first-order conditions. Some optimality conditions from [8] are included as special cases in our results.

2. HIGHER-ORDER NECESSARY OPTIMALITY CONDITIONS

In the following we fix a reference point $x_0 \in C$ which is feasible for problem (P). We assume that $F(x_0)$ is finite. Let

\[Q_0 := \{\alpha \in Q \mid f_\alpha(x_0) = F(x_0)\}, \quad B_0 := \{\beta \in B \mid g_\beta(x_0) = G(x_0)\}.\]
We assume that $Q$, $B$ are compact topological spaces, and that the mappings $\alpha \mapsto f_\alpha(x_0)$ and $\beta \mapsto g_\beta(x_0)$ are upper semicontinuous. Then $Q_0$ and $B_0$ are nonempty and compact. We recall [2, p.55] that the contingent cone to $C$ at $x_0$ is the set

$$K_C(x_0) := \{d \in X \mid \exists \{d_n\} \subseteq X, \{t_n\} \subseteq \mathbb{R} : d_n \to d, t_n \downarrow 0, x_0 + t_n d_n \in C\}.$$ 

To derive necessary optimality conditions for (P), we introduce functions which play the roles of higher-order generalised directional derivatives of $f_\alpha$ and $g_\beta$. So, for each $\alpha \in Q$, $\beta \in B$, let $\varphi^{(i)}_\alpha$, $i \in I := \{1, \ldots, k\}$, and $\psi^{(j)}_\beta$, $j \in J := \{1, \ldots, p\}$, be real-valued functions on $X$ satisfying the following:

**Assumption 2.1.**

(a) $\varphi^{(i)}_\alpha(0) = 0$, $\psi^{(j)}_\beta(0) = 0$ for all $\alpha \in Q$, $\beta \in B$, $i \in I$, $j \in J$.

(b) The mappings $\alpha \mapsto \varphi^{(i)}_\alpha(d)$ and $\beta \mapsto \psi^{(i)}_\beta(d)$ are continuous for all $d \in K_C(x_0)$, $i \in I$, $j \in J$.

(c) If $d_n \to d$ as $n \to \infty$, then, for each $i \in I$,

$$\liminf_{n \to \infty} \left[ \varphi^{(i)}_\alpha(d_n) - \varphi^{(i)}_\alpha(d) \right] \leq 0 \quad \text{uniformly in } \alpha,$$

and, for each $j \in J$,

$$\liminf_{n \to \infty} \left[ \psi^{(j)}_\beta(d_n) - \psi^{(j)}_\beta(d) \right] \leq 0 \quad \text{uniformly in } \beta.$$

(d) The mapping $d \mapsto \max_{\alpha \in Q_0} \varphi^{(k)}_\alpha(d)$ is upper semicontinuous.

Let us introduce relations between $f_\alpha$ and $\varphi^{(k)}_\alpha$, $g_\beta$ and $\psi^{(p)}_\beta$.

**Basic Assumption 2.2.** For all $d \in K_C(x_0)$ and sequences $d_n \to d$, $t_n \downarrow 0$ satisfying $x_0 + t_n d_n \in C$,

$$\varphi^{(k)}_\alpha(d) \geq \liminf_{n \to \infty} \frac{1}{t_n} \left[ f_\alpha(x_0 + t_n d_n) - f_\alpha(x_0) - \sum_{i=1}^{k-1} t_n^i \varphi^{(i)}_\alpha(d_n) \right] \quad \text{uniformly in } \alpha,$$

and

$$\psi^{(p)}_\beta(d) \geq \liminf_{n \to \infty} \frac{1}{t_n} \left[ g_\beta(x_0 + t_n d_n) - g_\beta(x_0) - \sum_{j=1}^{p-1} t_n^j \psi^{(j)}_\beta(d_n) \right] \quad \text{uniformly in } \beta.$$ 

To proceed further let us introduce the sets

$$M(x_0) := \left\{ d \in K_C(x_0) \mid \max_{\alpha \in Q_0} \varphi^{(i)}_\alpha(d) \leq 0 \quad \forall i \in I \setminus \{k\}, \max_{\beta \in B_0} \psi^{(j)}_\beta(d) \leq 0 \quad \forall j \in J \right\},$$

$$\widetilde{M}(x_0) := \left\{ d \in K_C(x_0) \mid \max_{\alpha \in Q_0} \varphi^{(i)}_\alpha(d) < 0 \quad \forall i \in I \setminus \{k\}, \max_{\beta \in B_0} \psi^{(j)}_\beta(d) < 0 \quad \forall j \in J \right\}.$$
Moreover, for $V \subseteq Q$ and $W \subseteq B$ let us define
\[
C(V, W) := \left\{ d \in K_C(x_0) \mid \varphi^{(i)}_\alpha(d) < 0 \ \forall \alpha \in V, i \in I; \ \psi^{(j)}_\beta(d) < 0 \ \forall \beta \in W, j \in J \right\}.
\]

Let us introduce a regularity condition of the type used in [3].

**REGULARITY CONDITION 2.3.**

(i) For any closed sets $V$ and $W$ satisfying \( Q_0 \subseteq V \subseteq Q \) and \( B_0 \subseteq W \subseteq B \) it holds that $C(V, W) \neq \emptyset$ implies $0 \in \text{cl}C(V, W)$.

(ii) $M(x_0) \subseteq \text{cl} \hat{M}(x_0)$.

Note that 2.3(i) holds, if the functions $\varphi^{(i)}$, $\psi^{(j)}$ are positively homogeneous.

We are now in a position to formulate a general necessary optimality condition of order $k$ for (P), which is the main result of the paper.

**THEOREM 2.4.** Let $x_0$ be a local minimiser for (P). Assume that assumption 2.1, the basic assumption 2.2, and the regularity condition 2.3 hold. Then

\[
\max_{\alpha \in Q_0} \varphi^{(k)}_\alpha(d) \geq 0 \ \forall \ d \in M(x_0).
\]

**PROOF:** Suppose that (1) is not true. By 2.3.(ii), there exists $\overline{d} \in \text{cl} \hat{M}(x_0)$ such that
\[
\max_{\alpha \in Q_0} \varphi^{(k)}_\alpha(\overline{d}) < 0.
\]

By 2.1.(d), $d \mapsto \max_{\alpha \in Q_0} \varphi^{(k)}_\alpha(d)$ is upper semicontinuous. So we can assume that $\overline{d} \in \hat{M}(x_0)$, that is, $\overline{d} \in K_C(x_0)$ and
\[
\max_{\alpha \in Q_0} \varphi^{(i)}_\alpha(d) < 0 \ \forall i \in I, \ \max_{\beta \in B_0} \psi^{(j)}_\beta(d) < 0 \ \forall j \in J.
\]

We choose $\delta > 0$ such that $\varphi^{(i)}_\alpha(d) \leq -2\delta$ for all $\alpha \in Q_0$, $i \in I$, and $\psi^{(j)}_\beta(d) \leq -2\delta$ for all $\beta \in B_0$, $j \in J$. We define
\[
U_1 := \{ \alpha \in Q \mid \varphi^{(i)}_\alpha(d) < -\delta \ \forall i \in I \}, \quad U_2 := \{ \beta \in B \mid \psi^{(j)}_\beta(d) < -\delta \ \forall j \in J \}.
\]

Then $Q_0 \subseteq U_1$, $B_0 \subseteq U_2$. By 2.1.(b), $U_1$ and $U_2$ are open, and
\[
\varphi^{(i)}_\alpha(d) \leq -\delta \ \forall \alpha \in \text{cl}U_1, i \in I, \quad \psi^{(j)}_\beta(d) \leq -\delta \ \forall \beta \in \text{cl}U_2, j \in J.
\]

So $\overline{d} \in C(\text{cl}U_1, \text{cl}U_2)$ and, by 2.3.(i), $0 \in \text{cl}C(\text{cl}U_1, \text{cl}U_2)$. Hence there exists a sequence $\{h_n\} \subseteq C(\text{cl}U_1, \text{cl}U_2)$ converging to 0.
We assert that there exist $d \in K_c(x_0)$ and $\epsilon > 0$ satisfying

$$
\varphi^{(i)}_\alpha(d) + \epsilon \leq F(x_0) - f_\alpha(x_0) \quad \forall i \in I,
$$

(2)

$$
\psi^{(j)}_\beta(d) + \epsilon \leq G(x_0) - g_\beta(x_0) \quad \forall j \in J
$$

for every $\alpha \in Q$, $\beta \in B$. To prove this, observe that $f_\alpha(x_0) < F(x_0)$ for all $\alpha \in Q \setminus U_1$ and $g_\beta(x_0) < G(x_0)$ for all $\beta \in B \setminus U_2$. Since $Q \setminus U_1$, $B \setminus U_2$ are compact and $\alpha \mapsto f_\alpha(x_0)$, $\beta \mapsto g_\beta(x_0)$ are upper semicontinuous, there exists $\epsilon_1 > 0$ such that

$$
f_\alpha(x_0) \leq F(x_0) - 2\epsilon_1 \quad \forall \alpha \in Q \setminus U_1, \quad g_\beta(x_0) \leq G(x_0) - 2\epsilon_1 \quad \forall \beta \in B \setminus U_2.
$$

Since $h_n \to 0$, by 2.1.(a) and (c) there exists $m \in \mathbb{N}$ with

$$
\varphi^{(i)}_\alpha(h_m) \leq \epsilon_1 \quad \forall \alpha \in Q, \; i \in I, \quad \psi^{(j)}_\beta(h_m) \leq \epsilon_1 \quad \forall \beta \in B, \; j \in J.
$$

Let $d := h_m$. Then (2) holds for all $\alpha \in Q \setminus U_1$, $\beta \in B \setminus U_2$, and every $\epsilon \leq \epsilon_1$. On the other hand, $d \in C(\text{cl} \; U_1, \text{cl} \; U_2)$. Hence by 2.1.(b) and the compactness of $\text{cl} \; U_1$, $\text{cl} \; U_2$ there exists $\epsilon_2 > 0$ such that

$$
\varphi^{(i)}_\alpha(d) + \epsilon_2 \leq 0 \quad \forall \alpha \in \text{cl} \; U_1, \; i \in I, \quad \psi^{(j)}_\beta(d) + \epsilon_2 \leq 0 \quad \forall \beta \in \text{cl} \; U_2, \; j \in J.
$$

This implies (2) for all $\alpha \in \text{cl} \; U_1$, $\beta \in \text{cl} \; U_2$, and every $\epsilon \leq \epsilon_2$.

Now we use (2) to find a sequence $\{x_n\}$ of feasible points for problem (P) converging to $x_0$ such that $F(x_n) < F(x_0)$ for all $n$, which contradicts the hypothesis that $x_0$ is a local minimiser.

Since $d \in K_c(x_0)$, there exist sequences $\{d_n\} \subseteq X$, $\{t_n\} \subseteq \mathbb{R}$ such that $d_n \to d$, $t_n \downarrow 0$, and $x_n := x_0 + t_n d_n \in C$ for all $n$. Replacing these by appropriate subsequences we can assume that, for every $n$,

$$
\frac{1}{t_n^k} \left[ f_\alpha(x_n) - f_\alpha(x_0) - \sum_{i=1}^{k-1} t_n^i \varphi^{(i)}_\alpha(d_n) \right] \leq \varphi^{(k)}_\alpha(d) + \frac{\epsilon}{2} \quad \forall \alpha \in Q,
$$

by 2.2,

$$
\varphi^{(i)}_\alpha(d_n) \leq \varphi^{(i)}_\alpha(d) + \epsilon \quad \forall \alpha \in Q, \; i \in I,
$$

$$
\psi^{(j)}_\beta(d_n) \leq \psi^{(j)}_\beta(d) + \epsilon \quad \forall \beta \in B, \; j \in J
$$

$$
\frac{1}{t_n^p} \left[ g_\beta(x_n) - g_\beta(x_0) - \sum_{j=1}^{p-1} t_n^j \psi^{(j)}_\beta(d_n) \right] \leq \psi^{(p)}_\beta(d) + \frac{\epsilon}{2} \quad \forall \beta \in B.
$$
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by 2.1.(c), and \( \sum_{i=1}^{k} t^i_n \leq 1, \sum_{j=1}^{p} t^j_n \leq 1 \). Then, for all \( \alpha \in Q, \)

\[
 f_{\alpha}(x_n) \leq t^k_n \left( \varphi^{(k)}_{\alpha}(d) + \frac{\varepsilon}{2} \right) + \sum_{i=1}^{k-1} t^i_n \varphi^{(i)}_{\alpha}(d_n) + f_{\alpha}(x_0) \\
\leq t^k_n \left( \varphi^{(k)}_{\alpha}(d) + \varepsilon \right) + \sum_{i=1}^{k-1} t^i_n \left( \varphi^{(i)}_{\alpha}(d) + \varepsilon \right) + f_{\alpha}(x_0) - t^k_n \frac{\varepsilon}{2} \\
\leq \sum_{i=1}^{k} t^i_n \left( F(x_0) - f_{\alpha}(x_0) \right) + f_{\alpha}(x_0) - t^k_n \frac{\varepsilon}{2} \\
\leq F(x_0) - t^k_n \frac{\varepsilon}{2} ,
\]

and similarly \( g_{\beta}(x_n) \leq G(x_0) - t^k_n \varepsilon/2 \) for all \( \beta \in B \). Hence \( F(x_n) < F(x_0) \) and \( G(x_n) < G(x_0) \leq 0 \).

In what follows, we give an application of Theorem 2.4.

**EXAMPLE 2.5:** Recall that \( \varphi^{(k)}_{\alpha}(x_0; d), \) the upper Dini directional derivative of order \( k \) of \( f_{\alpha} \) at \( x_0 \) in the direction \( d \), is defined recursively as follows (see for example, \([4, 5, 8]\)):

\[
\varphi^{(k)}_{\alpha}(x_0; d) := \limsup_{h \to 0, t \to 0} \frac{1}{t^k} \left[ f_{\alpha}(x_0 + th) - f_{\alpha}(x_0) - \sum_{i=1}^{k-1} t^i \varphi^{(i)}_{\alpha}(x_0; h) \right],
\]

provided that each \( \varphi^{(i)}_{\alpha} \) is real-valued.

Note that the mapping \( d \mapsto \varphi^{(k)}_{\alpha}(x_0; d) \) is upper semicontinuous. By applying Theorem 2.4 to the upper Dini directional derivatives we obtain:

**COROLLARY 2.6.** Let \( x_0 \) be a local minimiser for \((P)\). Assume that for

\[
\varphi^{(i)}_{\alpha}(d) := \frac{1}{t^i} \varphi^{(i)}_{\alpha}(x_0; d) \quad \forall i \in I, \quad \psi^{(j)}_{\beta}(d) := \frac{1}{t^j} \psi^{(j)}_{\beta}(x_0; d) \quad \forall j \in J,
\]

assumptions 2.1.(b)-(d) and 2.3.(ii) hold. Suppose, in addition, that the limits in the definitions of \( \varphi^{(k)}_{\alpha} \) and \( \psi^{(j)}_{\beta} \) at \( x_0 \) are uniformly in \( \alpha \) and \( \beta \), respectively. Then

\[
\max_{\alpha \in Q_0} \varphi^{(k)}_{\alpha}(x_0; d) \geq 0
\]

holds for every \( d \in K_C(x_0) \) with

\[
\max_{\alpha \in Q_0} \varphi^{(i)}_{\alpha}(x_0; d) \leq 0 \quad \forall i \in I \setminus \{k\}, \quad \max_{\beta \in B_0} \psi^{(j)}_{\beta}(x_0; d) \leq 0 \quad \forall j \in J.
\]
PROOF: It is easy to see that assumption 2.1.(a) and the basic assumption 2.2 are satisfied. Moreover, 2.3.(i) holds since \( f^{(i)}_{2} \), \( g^{(j)}_{3} \) are positively homogeneous. So the conclusion follows from Theorem 2.4.

Example 2.7: If \( f_{a} \) is \((k - 1)\) times Fréchet differentiable on \( X \) \((k > 1)\) and the Fréchet derivative of order \( k \) of \( f_{a} \) at \( x_0 \), \( f^{(k)}_{a}(x_0) \), exists, then (see for example, [7])

\[
f^{(k)}_{a}(x_0)d^{k} = f^{(k)}_{a}(x_0; d) \quad \forall d \in X,
\]

where \( d^{k} := (d, \ldots, d) \in X^{k} \). Similarly as in Example 2.5, we get the necessary condition for a local minimiser in this case:

\[
\max_{a \in D} f^{(k)}_{a}(x_0)d^{k} \geq 0
\]

for every \( d \in K_{C}(x_0) \) with

\[
\max_{a \in D} f^{(i)}_{a}(x_0)d^{i} \leq 0 \quad \forall i \in I \setminus \{k\}, \quad \max_{\beta \in B_{0}} g^{(j)}_{\beta}(x_0)d^{j} \leq 0 \quad \forall j \in J.
\]

3. Higher-Order Sufficient Optimality Conditions

In this section we assume that \( X = \mathbb{R}^{m} \).

Definition 3.1: [10] The point \( x_0 \in D \) is said to be a strict local minimiser of order \( k \) for the mathematical program \( \text{min}\{F(x) \mid x \in D\} \) if there exist \( \sigma > 0 \) and a neighbourhood \( U \) of \( x_0 \) such that

\[
F(x) \geq F(x_0) + \sigma \|x - x_0\|^k \quad \forall x \in U \cap D.
\]

Let \( x_0 \) be a feasible point for (P). As in the previous section, we consider real-valued functions \( \varphi^{(i)}_{\alpha}, \alpha \in Q, i \in I, \) and \( \psi^{(j)}_{\beta}, \beta \in B, j \in J, \) on \( \mathbb{R}^{m} \), and we define

\[
M(x_0) := \left\{ d \in K_{C}(x_0) \left| \sup_{\alpha \in Q} \varphi^{(i)}_{\alpha}(d) \leq 0 \forall i \in I \setminus \{k\}, \sup_{\beta \in B_{0}} \psi^{(j)}_{\beta}(d) \leq 0 \forall j \in J \right. \right\}.
\]

Let us introduce relations between \( f_{a} \) and \( \varphi^{(i)}_{\alpha}, g_{\beta} \) and \( \psi^{(j)}_{\beta} \).

Basic Assumption 3.2. For all \( d \in K_{C}(x_0) \) and sequences \( d_{n} \to d, t_{n} \downarrow 0 \) satisfying \( x_0 + t_{n}d_{n} \in C \) we have

\[
\varphi^{(i)}_{\alpha}(d) \leq \limsup_{n \to \infty} \frac{1}{t_{n}} \left[ f_{a}(x_0 + t_{n}d_{n}) - f_{a}(x_0) \right] \quad \forall \alpha \in Q, i \in I,
\]

\[
\psi^{(j)}_{\beta}(d) \leq \limsup_{n \to \infty} \frac{1}{t_{n}} \left[ g_{\beta}(x_0 + t_{n}d_{n}) - g_{\beta}(x_0) \right] \quad \forall \beta \in B, j \in J.
\]

A higher-order sufficient optimality condition for (P) can be stated as follows.
THEOREM 3.3. Let \( G(x_0) = 0 \) and let the basic assumption 3.2 be satisfied. Assume that

\[
\sup_{\alpha \in Q_0} \varphi^{(k)}_\alpha(d) > 0 \quad \forall d \in M(x_0) \setminus \{0\}.
\]

Then \( x_0 \) is a strict local minimiser of order \( k \) for (P).

PROOF: Assume that \( x_0 \) is not a strict local minimiser of order \( k \) for (P). Then there exists a sequence \( \{x_n\} \subseteq C \setminus \{x_0\} \) such that \( G(x_n) \leq 0, \|x_n - x_0\| \leq 1/n \), and \( F(x_n) < F(x_0) + \|x_n - x_0\|^k/n \) for every \( n \). Since \( f_\alpha(x_0) = F(x_0) \) for \( \alpha \in Q_0 \), we obtain

\[
f_\alpha(x_n) \leq f_\alpha(x_0) + \|x_n - x_0\|^k/n \quad \forall \alpha \in Q_0, n \in \mathbb{N}.
\]

Let \( t_n := \|x_n - x_0\| \) and \( d_n := (x_n - x_0)/t_n \). Then \( \|d_n\| = 1 \), so by the compactness of the unit sphere in \( \mathbb{R}^m \) there exists a subsequence \( \{d_{n_\nu}\} \) converging to \( d \) with \( \|d\| = 1 \). Since \( t_{n_\nu} \downarrow 0 \) and \( x_0 + t_{n_\nu}d_{n_\nu} = x_n \in C \), it follows that \( d \in K_C(x_0) \).

Using \( G(x_0) = 0 \) we have \( g_\beta(x_n) - g_\beta(x_0) = g_\beta(x_n) \leq G(x_n) \leq 0 \) for all \( \beta \in B_0, n \in \mathbb{N} \). Thus

\[
\psi^{(j)}_\beta(d) \leq \limsup_{\nu \to \infty} \frac{g_\beta(x_{n_\nu}) - g_\beta(x_0)}{t_{n_\nu}^k} \leq 0 \quad \forall \beta \in B_0, j \in J
\]

by 3.2. By combining (4) and (3) we obtain

\[
\varphi^{(i)}_\alpha(d) \leq \limsup_{\nu \to \infty} \frac{f_\alpha(x_{n_\nu}) - f_\alpha(x_0)}{t_{n_\nu}^i} \leq \limsup_{\nu \to \infty} \frac{\|x_{n_\nu} - x_0\|^k}{n_\nu t_{n_\nu}^i} = \lim_{\nu \to \infty} \frac{t_{n_\nu}^{k-i}}{n_\nu} = 0 \quad \forall \alpha \in Q_0, i \in I.
\]

Hence \( d \in M(x_0) \setminus \{0\} \) and \( \sup_{\alpha \in Q_0} \varphi^{(k)}_\alpha(d) \leq 0 \), which contradicts (3).

Theorem 3.3 includes [8] Corollary 2.1 as a special case.

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