CHARACTER MODULES, DIMENSION AND PURITY

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1. Introduction. In this paper we use the Bourbaki conventions for rings and modules: all rings are associative but not necessarily commutative and have an identity element; all modules are unital.

For any \( A \)-module \( M \), let \( M^* = \text{Hom}_Z(M, \mathbb{Q}/\mathbb{Z}) \) denote the character module of \( M \), where \( \mathbb{Q} \) denotes the rationals and \( \mathbb{Z} \) the integers. Then \( * \) is an exact contravariant zero-reflecting (i.e. \( M = 0 \) if and only if \( M^* = 0 \)) functor from the category of left \( A \)-modules to the category of right \( A \)-modules. Details may be found in Lambek [6], whose results in [5] we extend.

We introduce the absolute pure dimension (apd), relate it to the weak and injective dimensions, denoted \( \text{wd} \) and \( \text{injd} \) respectively, in terms of character modules, and deduce some results on the corresponding (left) global dimensions, denoted \( \text{lapgl}, \text{wgl}, \text{and lgl} \) respectively. We conclude with some remarks on absolute purity. Basic facts on purity and regularity can be found in the author’s paper [4]. Facts on dimensions used can be found in Cartan–Eilenberg [2].

2. The dimension theorems. For any module \( M \), we define \( \text{apd} M \) to be the least integer \( n \) such that \( \text{Ext}^{n+1}(N, M) = 0 \) for all finitely presented modules \( N \). If no such integer exists, \( \text{apd} M \) is defined to be infinite. Clearly \( \text{apd} M \leq \text{injd} M \) for all \( M \), and equality holds if \( A \) is noetherian, since \( \text{injd} M \leq n \) if and only if \( \text{Ext}^{n+1}(A/I, M) = 0 \) for all ideals \( I \) of \( A \). Later we shall show the converse that, if \( \text{apd} M = \text{injd} M \) for all \( M \), then \( A \) must be noetherian.

**Theorem 2.1.** For all \( M \), we have \( \text{apd} M^* = \text{wd} M = \text{injd} M^* \).

**Proof.** Since \( \mathbb{Q}/\mathbb{Z} \) is \( \mathbb{Z} \)-injective, we have \( \text{Ext}^n(N, M^*) = (\text{Tor}^n(N, M))^* \) for all \( N, M \) and all \( n \geq 0 \). (See Cartan–Eilenberg [2], p. 120.) Hence

\[
\text{apd} M^* \leq n \iff \text{Ext}^{n+1}(N, M^*) = 0 \quad \text{for all finitely presented } N
\]

\[
\iff \text{Tor}^{n+1}(N, M) = 0 \quad \text{for all finitely presented } N
\]

\[
\iff \text{wd} M \leq n
\]

\[
\iff \text{Tor}^{n+1}(N, M) = 0 \quad \text{for all } N
\]

\[
\iff \text{Ext}^{n+1}(N, M^*) = 0 \quad \text{for all } N
\]

\[
\iff \text{injd} M^* \leq n.
\]

**Corollary.**

\[ \text{wgl} A \leq \text{lapgl} A \leq \text{lgl} A. \]

We recall that a ring \( A \) is *left coherent* if and only if every finitely presented left \( A \)-module
is coherent, which means that all finitely generated submodules are finitely presented. For details see Bourbaki [1, pp. 62–63].

For the following result we shall use the fact that over a coherent ring every finitely presented module has a resolution by finitely generated free modules.

**Theorem 2.2.** For left coherent rings, \( \text{wd} M^* = \text{apd} M \), for all left \( A \)-modules \( M \).

**Proof.** Since \( A \) is left coherent, we have, by Cartan–Eilenberg [2, pp. 120–121],

\[
\text{Tor}^n(M^*, N) = (\text{Ext}^n(N, M))^* \quad \text{for all } M, \text{ all finitely presented } N \text{ and all } n \geq 0; \text{ and}
\]

\[
\text{apd} M \leq n \iff \text{Ext}^{n+1}(N, M) = 0 \quad \text{for all finitely presented } N,
\]

\[
\iff \text{Tor}^{n+1}(M^*, N) = 0 \quad \text{for all finitely presented } N,
\]

\[
\iff \text{wd} M^* \leq n.
\]

Hence \( \text{wd} M^* = \text{apd} M \).

**Corollary 1.** For left coherent rings, \( \text{lapgl} A = \text{wgl} A \).

**Corollary 2.** For coherent rings, \( \text{apd} M \) is the least integer \( n \) such that \( \text{Ext}^{n+1}(A/I, M) = 0 \) for all finitely generated ideals \( I \) of \( A \).

**Proof.** Use Theorems 2.2 and 2.3.

**Theorem 2.3.** For any left \( A \)-module \( M \) we have: \( \text{wd} M \leq n \) if and only if \( \text{Tor}^{n+1}(A/I, M) = 0 \) for all finitely generated right ideals \( I \) of \( A \).

**Proof.** By definition \( \text{wd} M \leq n \) if and only if \( \text{Tor}^{n+1}(N, M) = 0 \) for all right modules \( N \). Since \( \text{Tor}^n \) commutes with direct limits for all \( n \) and since every module is the direct limit of its finitely generated submodules, we can immediately reduce the problem to the case where \( N \) is finitely generated. By induction on the number of generators of \( N \) we can reduce to the cyclic module case: \( A/I, I \) a right ideal of \( A \). Since \( I \) is the direct limit of finitely generated right ideals, we get the desired result.

**Corollary 1.** \( \text{wgl} A \leq 1 + \sup \text{wd} I \) with the sup taken over all finitely generated left (or right) ideals of \( A \). If \( \text{wgl} A \geq 1 \) we have equality. Hence \( \text{wgl} A \leq 1 \) if and only if all finitely generated left (or right) ideals are flat.

**Proof.** By the theorem, \( \text{wgl} A = \sup \text{wd} A/I \) with the sup taken over all finitely generated left (or right) ideals. But \( \text{wd} (A/I) = 1 + \text{wd} I \) unless \( \text{wd} A/I = 0 \). If \( \text{wgl} A \geq 1 \), then \( A \) is not regular; hence some \( I \) is not pure in \( A \) and \( A/I \) is not flat.

**Corollary 2.** A ring \( A \) is left semihereditary if and only if it is left coherent and \( \text{wgl} A \leq 1 \).

**Proof.** Any finitely generated module is projective if and only if it is finitely presented and flat. Apply this result to any finitely left ideal of \( A \).

**3. Purity.** A short exact sequence of left \( A \)-modules \( E : 0 \to E_1 \to E_2 \to E_3 \to 0 \) is pure exact in the sense of Cohn [3] if and only if, for all (or equivalently for all finitely presented)
right $A$-modules $M$, we have
\[ M \otimes E : 0 \to M \otimes E_1 \to M \otimes E_2 \to M \otimes E_3 \to 0 \]
exact. We have shown in [4] that this is equivalent to $\text{Hom}(N, E_2) \to \text{Hom}(N, E_3)$ being epic for all finitely presented $N$.

A module is called absolutely pure if and only if it is pure in every overmodule; actually it suffices to test in the injective hull.

**Theorem 3.1.** A module is absolutely pure if and only if $\text{apd} M = 0$.

**Proof.** Let $I$ denote the injective hull of $M$. For every module $N$ we have an exact sequence:
\[ \text{Hom}(N, I) \to \text{Hom}(N, I/\text{apd} M) \to \text{Ext}(N, M) \to \text{Ext}(N, I) = 0. \]
Hence $\alpha_N$ is epic if and only if $\text{Ext}(N, M) = 0$.

Therefore $M$ is pure in $I$ if and only if $\text{Ext}(N, M) = 0$ for all finitely presented $N$, i.e., if and only if $\text{apd} M = 0$.

**Corollary.** $A$ is left noetherian if and only if $\text{apd} M = \text{injd} M$ for all left $A$-modules $M$.

**Proof.** We have discussed the "only if" part. The "if" part now follows from Maddox [7]. Note that the dimension zero case suffices.

**References**

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