COUNTING WORD-TREES

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Introduction. In his recent study of free inverse semigroups, Munn [2] introduced and used extensively the concept of a *word-tree*. In this note the number of such trees is found.

DEFINITION. A word-tree on an alphabet A is a finite tree, with at least two points, that satisfies the following conditions.

(WT1) Each line is oriented and is labelled by an element of A.

(WT2) T has no subgraph of the form $o \xrightarrow{a} o \xleftarrow{a} o$ or $o \xleftarrow{a} o \xrightarrow{a} o$ $(a \in A)$.

Lines will be described by an ordered pair of adjacent points and are said to be *similar* if their orientations and labellings are the same.

A line $\alpha\beta$, where β is an endpoint of a word-tree T, is called an *endline* of T.

Isomorphism is defined in the obvious way (preserving orientation and labelling of lines).

A fundamental fact proved in Munn's paper [2, Theorem 2.2] is that a word-tree has no nontrivial automorphism.

Notations. $d(\alpha)$ = degree of point α.

k = cardinal of A, taken to be finite.

$n(p)$ = number of word-trees on A with p points.

$n(p, e)$ = number of word-trees on A with p points and e endpoints.

$$(n)_j = \begin{cases} n(n-1)\dots(n-j+1) & (j = 1, 2, \dots), \\ 1 & (j = 0). \end{cases}$$

Note that $n(p, e) = 0$ if $e < 2$ or $p < e$ and that, for $p > 2$, $n(p, p) = 0$.

Construction. Given a word-tree T with $p-1$ points and e endpoints, we can construct a larger one by adding a new line at any point. The number of dissimilar lines that may be added at α is $2k-d(\alpha)$, since each old line terminating in α imposes one restriction on the new line and, by (WT2), these restrictions are different. The new word-tree has e endpoints if the new line was added at an old endpoint and $e+1$ endpoints otherwise.

Let T_1 and T_2 be the word-trees obtained by adding endlines $\alpha_1\beta_1$ and $\alpha_2\beta_2$ respectively to T, where α_1 and α_2 are distinct points of T. Then, since T has no nontrivial automorphisms, there can be no isomorphism $\phi : T_1 \to T_2$ such that $\beta_1\phi = \beta_2$. Clearly the same result holds if $\alpha_1 = \alpha_2$ but the endlines $\alpha_1\beta_1$ and $\alpha_2\beta_2$ are dissimilar. It follows that there is a $1-1$ correspondence between the possible constructions on word-trees with $p-1$ points and the word-trees on p points with a distinguished endpoint.

Calculation. From the tree T the construction gives $(2k-1)e$ word-trees with e endpoints and $\sum_{\alpha}(2k-d(\alpha))$ with $e+1$ endpoints, where the sum is over all points α of T that are not endpoints of T. But $\sum_{\alpha} d(\alpha)+e=2$ (number of lines in T) $=2(p-2)$ (see, e.g., [**1**, Theorem 4.11]), so that the number of word-trees with $e+1$ endpoints is

$$2k(p-1-e)-2(p-2)+e.$$

Among all possible constructions, a particular tree with p points and e points will occur e times. Thus

$$n(p, e)=\frac{1}{e}\{(2k-1)en(p-1, e)+(2k(p-1-e+1)-2(p-2)+e-1)n(p-1, e-1)\}$$

$$=(2k-1)n(p-1, e)+\left\{\frac{2kp-2p+3}{e}-(2k-1)\right\}n(p-1, e-1), \quad \text{for} \quad p \geqq 3. \tag{1}$$

This recurrence formula is used $p-3$ times to express $n(p)$, where $p \geqq 3$, in terms of $n(3, 2)$, which is $k(2k-1)$. In fact, for $0 \leqq j \leqq p-3$,

$$n(p)=(2kp-2p+j+2)_j \sum_{e=2}^{p-j-1} n(p-j, e)/(e+j)_j. \tag{2}$$

This is proved by induction on j. Application of (1) in (2) gives

$$n(p)=(2kp-2p+j+2)_j \sum_{e=2}^{p-j-1}\left[(2k-1)n(p-j-1, e)+\right.$$

$$\left.+\left\{\frac{2k(p-j)-2p+2j+3}{e}-(2k-1)\right\}n(p-j-1, e-1)\right]\Big/(e+j)_j.$$

In the sum, the coefficient of $n(p-j-1, e)$, when $2 \leqq e \leqq p-j-2$, is

$$\frac{2k-1}{(e+j)_j}+\frac{2k(p-j)-2p+2j+3}{(e+1)(e+1+j)_j}-\frac{2k-1}{(e+1+j)_j}=(2kp-2p+j+3)/(e+j+1)_{j+1}.$$

The other substitute terms are 0. This verifies (2).

In particular, for $j=p-3$, we obtain

$$n(p)=(2kp-2p+p-1)_{p-3} \sum_{e=2}^{2} n(3, e)/(e+p-3)_{p-3}$$

$$=(2kp-p-1)_{p-3}\, k(2k-1)\Big/\frac{(p-1)!}{2}$$

$$=\frac{2k(2k-1)}{(p-1)(p-2)}\binom{2kp-p-1}{p-3}.$$

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REFERENCES

1. F. Harary, *Graph theory* (Reading, Mass., 1969).

2. W. D. Munn, Free inverse semigroups, *Proc. London Math. Soc.*; to appear.

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