COMPOSITIO MATHEMATICA

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Compositio Math. **153** (2017), 1050–1118.

doi:10.1112/S0010437X17007011

FOUNDATION COMPOSITIO MATHEMATICA

LONDON MATHEMATICAL SOCIETY EST. 1865

https://doi.org/10.1112/S0010437X17007011 Published online by Cambridge University Press

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Abstract

After the work of Kisin, there is a good theory of canonical integral models of Shimura varieties of Hodge type at primes of good reduction. The first part of this paper develops a theory of Hodge type Rapoport–Zink formal schemes, which uniformize certain formal completions of such integral models. In the second part, the general theory is applied to the special case of Shimura varieties associated with groups of spinor similitudes, and the reduced scheme underlying the Rapoport–Zink space is determined explicitly.

Contents

Received 1 November 2015, accepted in final form 1 December 2016, published online 10 April 2017.
2010 Mathematics Subject Classification 11G18, 14G35 (primary).
Keywords: Shimura variety, spinor group, Rapoport–Zink space.
B.H. is partially supported by NSF grants DMS-1201480 and DMS-1501583. G.P. is partially supported by NSF grants DMS-1102208 and DMS13-60733.
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https://doi.org/10.1112/S0010437X17007011 Published online by Cambridge University Press

1. Introduction

This paper contributes to the theory of integral models of Shimura varieties, and to the related theory of Rapoport–Zink formal schemes. We concentrate our attention on Shimura varieties of Hodge type with hyperspecial level subgroup. In this case, canonical smooth integral models of the Shimura varieties were constructed by Kisin (see also work of Vasiu). Using these models, we give a construction of certain Hodge type Rapoport–Zink formal schemes, and describe their field-valued points in terms of certain *refined* affine Deligne–Lusztig sets.

A large portion of the paper concerns what is arguably the most interesting family of Shimura varieties that are of Hodge but not of PEL type: those associated to the spinor similitude groups of quadratic spaces over $\mathbb{Q}$ of signature $(d,2)$. For this family of Shimura varieties, we use our results on Rapoport–Zink spaces to explicitly describe the basic (supersingular) locus in the reduction modulo p of the canonical integral model.

In what follows, we describe our results in more detail. First, we will discuss the construction of Rapoport–Zink formal schemes for general Hodge type Shimura varieties, and then we will explain the description of the supersingular locus in the reduction modulo p of the Shimura varieties for spinor similitude groups.

1.1 Rapoport–Zink spaces for Hodge type Shimura varieties

Let $(G,\mathcal{H})$ be a Hodge type Shimura datum with reflex field $E\subset\mathbb{C}$. Fix a prime $p>2$ and a sufficiently small compact open subgroup

$$U=U_pU^p\subset G(\mathbb{A}_f)$$

with $U_p\subset G(\mathbb{Q}_p)$ hyperspecial. This implies that G extends to a reductive group scheme over $\mathbb{Z}_{(p)}$, denoted the same way, with $U_p=G(\mathbb{Z}_p)$. Denote by $\mathrm{Sh}_U(G,\mathcal{H})$ the corresponding Shimura variety; it is a smooth quasi-projective variety over E with complex points

$$\mathrm{Sh}_U(G,\mathcal{H})(\mathbb{C})=G(\mathbb{Q})\backslash\mathcal{H}\times G(\mathbb{A}_f)/U.$$

1.1.1 For $(G,\mathcal{H})$ to be of Hodge type means that there is an embedding of Shimura data

$$(G,\mathcal{H})\to(\mathrm{GSp}_{2g},\mathcal{H}_{2g}),\tag{1.1.1.1}$$

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where $\mathcal{H}_{2g}$ is the union of the upper and lower Siegel half-spaces of genus g. This embedding may be chosen in a particular way: we can find a self-dual symplectic space (C, ψ) over $\mathbb{Z}_{(p)}$ and a closed immersion

$$G \hookrightarrow \mathrm{GSp}(C, \psi) \tag{1.1.1.2}$$

of reductive groups over $\mathbb{Z}_{(p)}$ whose generic fiber induces (1.1.1.1).

Moreover, G can be realized as the pointwise stabilizer of a finite set of tensors $(s_\alpha) \subset C^\otimes$. Here $C^\otimes$ is the *total tensor algebra*; it is defined as the direct sum of all free $\mathbb{Z}_{(p)}$-modules that can be formed from C using the operations of taking duals, tensor products, symmetric powers, and exterior powers. In particular, if we set

$$D = \mathrm{Hom}(C, \mathbb{Z}_{(p)}) \tag{1.1.1.3}$$

with its contragredient action $(gd)(c) = d(g^{-1}c)$ of G, then $C^\otimes = D^\otimes$ as representations of G.

For a prime $v \mid p$ of E, Kisin [Kis10] has proved that the Shimura variety $\mathrm{Sh}_U(G, \mathcal{H})$ over E admits a canonical smooth integral model

$$\mathscr{S} = \mathscr{S}_U(G, \mathcal{H})$$

over the localization $\mathcal{O}_{E,(v)}$. The integral model is constructed, using (1.1.1.1), as the normalization of the Zariski closure of $\mathrm{Sh}_U(G, \mathcal{H})$ in the integral model of a Siegel moduli variety. In particular, $\mathscr{S}$ carries over it a 'universal' family of abelian varieties with additional structure, obtained as the pull-back of the universal family over the Siegel variety. The universal family on $\mathscr{S}$ depends on the choice of Hodge embedding (1.1.1.1), but the integral model $\mathscr{S}$ does not.

The special fiber of the canonical integral model comes with its *Newton stratification*, whose strata are defined by fixing the isogeny class of the universal p-divisible group with additional structure. Among the Newton strata there is a distinguished closed stratum, called the *basic locus* (see [RR96, Wor13]). For many Hodge type Shimura varieties, the basic locus is the *supersingular locus*: the locus of points at which the universal abelian variety is isogenous to a product of supersingular elliptic curves. This will be the case for the spinor similitude Shimura varieties discussed below in § 1.2.

When $\mathrm{Sh}_U(G, \mathcal{H})$ is a PEL type Shimura variety, the completion of the integral model along the basic locus is described via the *p-adic uniformization theorem* of Rapoport and Zink [RZ96] as a quotient of what is now called a Rapoport–Zink formal scheme.

1.1.2 The first main result of this paper is the construction of Rapoport–Zink formal schemes for general Hodge type Shimura varieties as above. Such a construction also appears in the recent preprints of Kim [Kim13, Kim14]. We have followed Kim in the characterization of our formal schemes as moduli spaces of quasi-isogenies between p-divisible groups endowed with so-called *crystalline Tate tensors*; however, our construction of these spaces is more direct than Kim's, and uses the existence of the integral model $\mathscr{S}$.

Let k be an algebraic closure of the residue field of the place $v \mid p$ fixed above, and let $W = W(k)$ be the ring of Witt vectors of k. For us, a Hodge type Rapoport–Zink formal scheme over W is characterized in terms of the *local Shimura–Hodge datum* $(G_{\mathbb{Z}_p}, b_{x_0}, \mu_{x_0}, C_{\mathbb{Z}_p})$ attached to a point $x_0 \in \mathscr{S}(k)$. The reductive group scheme $G_{\mathbb{Z}_p}$ over $\mathbb{Z}_p$ and the representation

$$G_{\mathbb{Z}_p} \hookrightarrow \mathrm{GL}(C_{\mathbb{Z}_p})$$

were described above, and we must now explain the meaning of b_{x_0} and μ_{x_0}.

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Denote by X_0 the p-divisible group of the fiber of the universal abelian scheme at x_0, and let $\mathbb{D}(X_0)$ be its contravariant Grothendieck–Messing crystal. The evaluation $\mathbb{D}(X_0)(W)$ of the crystal on W is the Dieudonné module of X_0. Kisin shows that this comes equipped with a collection of *crystalline tensors*

$$t_{\alpha,0} \in \mathbb{D}(X_0)(W)^{\otimes},$$

which are Frobenius invariant in $\mathbb{D}(X_0)(W)^{\otimes}[1/p]$. Moreover, there is a W-module isomorphism

$$D \otimes_{\mathbb{Z}_p} W \xrightarrow{\sim} \mathbb{D}(X_0)(W) \tag{1.1.2.1}$$

identifying $s_\alpha \otimes 1$ with $t_{\alpha,0}$. Under any such identification the Frobenius operator on $\mathbb{D}(X_0)(W)$ induces an operator on $D \otimes_{\mathbb{Z}_p} W$ of the form

$$F = b_{x_0} \circ \sigma$$

for some $b_{x_0} \in G(K)$. Here $\sigma \in \mathrm{Aut}(W)$ lifts the absolute Frobenius on k, and $K = W[1/p]$ is the fraction field of W.

Kisin shows that the Hodge filtration on $\mathbb{D}(X_0)(k)$ is split by a G_k-valued cocharacter, which is the reduction of a minuscule cocharacter

$$\mu_{x_0} : \mathbb{G}_{mW} \to G_W$$

satisfying $b_{x_0} \in G(W)\mu_{x_0}^{\sigma}(p)G(W)$.

The $G(W)$-conjugacy class of μ_{x_0} is independent of (1.1.2.1) and it agrees with the conjugacy class of the *inverse* of the Deligne cocharacter $\mu_h : \mathbb{G}_{m\mathbb{C}} \to G_{\mathbb{C}}$ associated to the symmetric domain $\mathcal{H}$. More precisely, μ_{x_0} and μ_h^{-1} become conjugate after we fix an isomorphism $\mathbb{C} \xrightarrow{\sim} \bar{K}$ whose restriction to $E \hookrightarrow \bar{K}$ induces the place v chosen above.

Having fixed x_0 and (1.1.2.1), we abbreviate $b = b_{x_0}$ and $\mu = \mu_{x_0}$. Define an algebraic group J_b over $\mathbb{Q}_p$ with functor of points

$$J_b(R) = \{g \in G(R \otimes_{\mathbb{Q}_p} K) : gb\sigma(g)^{-1} = b\}$$

for any $\mathbb{Q}_p$-algebra R. The element b is *basic* if and only if J_b is an inner form of G.

Theorem A. *There exists a formal scheme RZ_G over $\mathrm{Spf}(W)$ that is formally smooth and locally formally of finite type, admits a left action of $J_b(\mathbb{Q}_p)$, and has the following properties.*

(i) *It is a formal closed subscheme of the usual Rapoport–Zink formal scheme $\mathrm{RZ}(X_0)$ over $\mathrm{Spf}(W)$ representing pairs (X, ρ) of a p-divisible group X and a quasi-isogeny $\rho : X_0 \dashrightarrow X$, as in [RZ96].*

(ii) *There is a bijection*

$$\mathrm{RZ}_G(k) \xrightarrow{\sim} X_{G,b,\mu^\sigma}(k),$$

where $X_{G,b,\mu^\sigma}(k)$ is the affine Deligne–Lusztig set

$$\big\{g \in G(K) : g^{-1}b\sigma(g) \in G(W)\mu^\sigma(p)G(W)\big\}/G(W).$$

(iii) *Assume in addition that b is basic, or, equivalently, that the point x_0 lies in the basic locus. Then there is an isomorphism of formal schemes*

$$\Theta^b : I(\mathbb{Q})\backslash \mathrm{RZ}_G \times G(\mathbb{A}_f^p)/U^p \xrightarrow{\sim} (\widehat{\mathscr{S}_W})_{/\mathscr{S}_b}.$$

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Here $(\widehat{\mathscr{S}_W})_{/\mathscr{S}_b}$ is the completion of the base change $\mathscr{S}_W$ along the basic locus $\mathscr{S}_b$ of the special fiber, and I is a reductive group over $\mathbb{Q}$, which is an inner form of G admitting identifications

$$I(\mathbb{Q}_\ell) = \begin{cases} J_b(\mathbb{Q}_p) & \text{if } \ell = p, \\ G(\mathbb{Q}_\ell) & \text{if } \ell \neq p, \end{cases}$$

and with $I(\mathbb{R})$ compact modulo center.

In fact, $\mathrm{RZ}_G(k)$ can be identified with the set of isomorphism classes of triples $(X, \rho, (t_\alpha))$ in which X is a p-divisible group over k,

$$(t_\alpha) \subset \mathbb{D}(X)(W)^{\otimes}$$

is a collection of Frobenius invariant tensors, and $\rho : X_0 \dashrightarrow X$ is a quasi-isogeny identifying t_α with $t_{\alpha,0}$. Some additional technical properties are required; see Definition 2.3.3. We can give a similar moduli description of the R-valued points of RZ_G for R any formally smooth, formally finitely generated W-algebra, but not for general R. This description uniquely determines RZ_G.

Remark 1.1.3. As noted earlier, Theorem A already appears in the recent preprints of Kim [Kim13, Kim14]. It is only our *construction* of the space RZ_G that is new. It is essential for our construction (but not for Kim's) that the local Shimura–Hodge datum $(G_{\mathbb{Z}_p}, b, \mu, C_{\mathbb{Z}_p})$ arises from a point $x_0 \in \mathscr{S}(k)$ on a global Hodge type Shimura variety as above. Given results on the non-emptiness of Newton strata for Shimura varieties of Hodge type which have been recently announced by Kisin, Madapusi Pera, and Shin, one should be able to show that this happens most of the time; we would like to return to this question on another occasion.

1.1.4 We can also give a concrete description of $\mathrm{RZ}_G(k')$ when k'/k is any finitely generated field extension and the p-divisible group X_0 is formal. This involves the new notion of a *refined* affine Deligne–Lusztig set, which we now explain. Let W' be the Cohen ring of k', let $K' = W'[1/p]$ be its fraction field, and suppose $\sigma : W' \to W'$ is an appropriate lift of Frobenius (see Proposition 2.4.8). The following theorem is then obtained by using Zink's theory of displays and windows.

Theorem B. *There is a bijection*

$$\mathrm{RZ}_G(k') \xrightarrow{\sim} X_{G,b,\mu^\sigma,\sigma}(k'),$$

where the refined affine Deligne–Lusztig set $X_{G,b,\mu^\sigma,\sigma}(k')$ is, by definition, the image of the natural map

$$\{g \in G(K') : g^{-1} b\sigma(g) \in G(W')\mu^\sigma(p)\} \to G(K')/G(W').$$

Our refined affine Deligne–Lusztig set is a subset of the *naive* affine Deligne–Lusztig set

$$\{g \in G(K') : g^{-1} b\sigma(g) \in G(W')\mu^\sigma(p)G(W')\}/G(W'),$$

and equality holds if k' is perfect. The above description of $\mathrm{RZ}_G(k')$ is entirely group-theoretical (i.e. does not involve p-divisible groups), and is thus quite useful.

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Remark 1.1.5. There is a simpler parallel theory when we consider only the points of Rapoport–Zink formal schemes with values in perfect $\mathbb{F}_p$-algebras. Indeed, then one can even use the Witt vector affine Grassmannian as in [Zhu17, BS15] to obtain a more straightforward and general construction of (at least) the reduced locus of Rapoport–Zink schemes, but *only up to perfection*. These constructions allow one to also consider non-minuscule coweights. However, this comes at the cost of passing to the non-finite type perfection which loses a lot of information.

In contrast, in this paper we can consider points with values in non-perfect rings, but we must restrict to minuscule coweights connected to Shimura varieties. Allowing non-perfect rings as in Theorem B is essential for the application to spinor Shimura varieties described below. A different approach towards describing Rapoport–Zink formal schemes as functors on more general (not necessarily perfect) rings directly from the group data is pursued in work in preparation by one of us (G.P.) with O. Bültel.

Remark 1.1.6. A direct construction of an adic analytic space corresponding to the limit of Rapoport–Zink spaces over all p-level subgroups has been given by Scholze and Weinstein [SW13] using Scholze's perfectoid spaces. Recently, there has been further progress in defining related spaces by Scholze using his theory of diamonds. The constructions of Scholze and of Scholze and Weinstein concern the generic fiber, and do not provide a uniformization of the integral model.

1.2 Spinor similitude Shimura varieties

In large part, our motivation for studying Rapoport–Zink spaces for Hodge type Shimura varieties is to apply the general theory to the Shimura varieties associated with spinor similitude groups.

By combining our general results, specialized to the case of GSpin, with the linear algebra of lattices in quadratic spaces as in [HP14], we obtain a very explicit description of the basic locus of the special fiber of the integral model, and of the underlying reduced scheme of the corresponding Rapoport–Zink formal scheme.

1.2.1 Start with an odd prime p and a self-dual quadratic space (V, Q) over $\mathbb{Z}_{(p)}$ of signature $(d, 2)$ with $d \geqslant 1$. The corresponding bilinear form is denoted by

$$[x, y] = Q(x + y) - Q(x) - Q(y). \tag{1.2.1.1}$$

This determines a reductive group scheme $G = \mathrm{GSpin}(V)$ over $\mathbb{Z}_{(p)}$. By a slight abuse of notation, we sometimes use the same letter to denote the generic fiber of G.

Define a hyperspecial subgroup

$$U_p = G(\mathbb{Z}_p) \subset G(\mathbb{Q}_p).$$

By setting $U = U^p U_p$ for any sufficiently small compact open subgroup $U^p \subset G(\mathbb{A}_f^p)$, we obtain a d-dimensional Shimura variety $\mathrm{Sh}_U(G, \mathcal{H})$ over $\mathbb{Q}$. Here $G(\mathbb{R})$ acts on the hermitian domain

$$\mathcal{H} = \{z \in V_{\mathbb{C}} : [z, z] = 0, [z, \bar{z}] < 0\}/\mathbb{C}^\times \tag{1.2.1.2}$$

via the natural surjection $G \to \mathrm{SO}(V)$.

1.2.2 The group G is, by definition, a subgroup of the unit group of the Clifford algebra $C = C(V)$, and hence G acts on C by left multiplication. For an appropriate choice of perfect symplectic form ψ on C, this defines a closed immersion (1.1.1.2) of reductive groups over $\mathbb{Z}_{(p)}$,

Thus we find ourselves in exactly the situation described in § 1.1. Let $\mathscr{S} = \mathscr{S}_U(G, \mathcal{H})$ be the canonical smooth integral model over $\mathbb{Z}_{(p)}$, equipped with the universal abelian scheme determined

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by the symplectic embedding (1.1.1.2). This universal abelian scheme is also known as the *Kuga–Satake abelian scheme*, and the locus of points

$$\mathscr{S}_{\mathrm{ss}} \subset \mathscr{S} \otimes_{\mathbb{Z}_{(p)}} k \tag{1.2.2.1}$$

at which it is supersingular is precisely the basic locus. As before, we set $k = \bar{\mathbb{F}}_p$ and $W = W(k)$.

Fix a supersingular point $x_0 \in \mathscr{S}(k)$, and let $(G_{\mathbb{Z}_p}, b, \mu, C_{\mathbb{Z}_p})$ be the corresponding unramified local Shimura–Hodge datum as in § 1.1. Let $\mathrm{RZ} = \mathrm{RZ}_G$ be the associated formal scheme over W, as in Theorem A. Our main result is an explicit description of the underlying reduced locally finite type k-scheme $\mathrm{RZ}^{\mathrm{red}}$. First, we give a formula for its dimension.

Theorem C. *Let $n = d + 2$ be the dimension of $V_{\mathbb{Q}_p}$. All irreducible components of $\mathrm{RZ}^{\mathrm{red}}$ are isomorphic, and are smooth of dimension*

$$\dim(\mathrm{RZ}^{\mathrm{red}}) = \frac{1}{2}\begin{cases} n-4 & \text{if } n \text{ is even and } \det(V_{\mathbb{Q}_p}) = (-1)^{n/2}, \\ n-3 & \text{if } n \text{ is odd}, \\ n-2 & \text{if } n \text{ is even and } \det(V_{\mathbb{Q}_p}) \neq (-1)^{n/2}, \end{cases}$$

where the equalities involving $\det(V_{\mathbb{Q}_p})$ are understood to be in $\mathbb{Q}_p^\times$ modulo squares. Equivalently,

$$\dim(\mathrm{RZ}^{\mathrm{red}}) = \begin{cases} (d/2) - 1 & \text{if } V_{\mathbb{Q}_p} \text{ is a sum of hyperbolic planes}, \\ \lfloor d/2 \rfloor & \text{otherwise}. \end{cases}$$

1.2.3 In fact, we give essentially a complete description of $\mathrm{RZ}^{\mathrm{red}}$, in the same spirit as the work of Vollaard [Vol10], Vollaard and Wedhorn [VW11], Rapoport *et al.* [RTW14], and the authors [HP14] for some unitary Shimura varieties. To explain its structure requires some more notation.

Consider the quadratic space V_K over $K = W[1/p]$, with its natural action $G_K \to \mathrm{SO}(V_K)$. The operator $\Phi = b \circ \sigma$ makes V_K into a slope 0 isocrystal, and its subspace of Φ-invariant vectors V_K^Φ is a $\mathbb{Q}_p$-quadratic space of the same dimension and determinant as $V_{\mathbb{Q}_p}$, but with different Hasse invariant. In fact, the self-duality of V implies that $V_{\mathbb{Q}_p}$ has Hasse invariant 1, and so V_K^Φ has Hasse invariant -1.

A *vertex lattice* is a $\mathbb{Z}_p$-lattice $\Lambda \subset V_K^\Phi$ satisfying $p\Lambda \subset \Lambda^\vee \subset \Lambda$. The quadratic form pQ on V_K^Φ induces a quadratic form on the $\mathbb{F}_p$-vector space

$$\Omega_0 = \Lambda/\Lambda^\vee.$$

The *type* $t_\Lambda = \dim(\Omega_0)$ of Λ is even, and satisfies $2 \leqslant t_\Lambda \leqslant t_{\max}$, where

$$t_{\max} = \begin{cases} n-2 & \text{if } n \text{ is even and } \det(V_{\mathbb{Q}_p}) = (-1)^{n/2}, \\ n-1 & \text{if } n \text{ is odd}, \\ n & \text{if } n \text{ is even and } \det(V_{\mathbb{Q}_p}) \neq (-1)^{n/2}. \end{cases} \tag{1.2.3.1}$$

One may characterize Ω_0 as the unique quadratic space over $\mathbb{F}_p$ of dimension t_Λ that admits no Lagrangian (= totally isotropic of dimension $t_\Lambda/2$) subspace.

Of course the base change of Ω_0 to k does admit Lagrangian subspaces, and we define a smooth projective k-variety S_Λ with k-points

$$S_\Lambda(k) = \left\{ \text{Lagrangians } \mathscr{L} \subset \Omega_0 \otimes_{\mathbb{F}_p} k : \dim(\mathscr{L} + \Phi(\mathscr{L})) = \frac{t_\Lambda}{2} + 1 \right\}.$$

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Here $\Phi = \mathrm{id} \otimes \sigma$ is the operator on $\Omega_0 \otimes k$ induced by the absolute Frobenius $\sigma(x) = x^p$ on k. The variety $S_\Lambda = S_\Lambda^+ \sqcup S_\Lambda^-$ has two connected components, which are (non-canonically) isomorphic, and smooth of dimension $(t_\Lambda/2) - 1$. As we will explain in § 6.5.4, these can be identified with closures of Deligne–Lusztig varieties for $\mathrm{SO}(\Omega_0)$.

Theorem D. *The Rapoport–Zink formal scheme* $\mathrm{RZ} = \mathrm{RZ}_G$ *admits a decomposition*

$$\mathrm{RZ} = \bigsqcup_{\ell \in \mathbb{Z}} \mathrm{RZ}^{(\ell)}$$

with the following properties.

(i) *Each open and closed formal subscheme* $\mathrm{RZ}^{(\ell)}$ *is connected, and*

$$\mathrm{RZ}^{(\ell)} \xrightarrow{\sim} \mathrm{RZ}^{(\ell+1)}.$$

(ii) *Each connected component* $\mathrm{RZ}^{(\ell)}$ *has a collection of closed formal subschemes* $\mathrm{RZ}_\Lambda^{(\ell)} \subset \mathrm{RZ}^{(\ell)}$ *indexed by the vertex lattices* $\Lambda \subset V_K^\Phi$*, and the underlying reduced schemes satisfy*

$$\mathrm{RZ}_\Lambda^{(\ell),\mathrm{red}} \xrightarrow{\sim} S_\Lambda^{\pm}.$$

Moreover, for any vertex lattices Λ_1 *and* Λ_2*,*

$$\mathrm{RZ}_{\Lambda_1}(k) \cap \mathrm{RZ}_{\Lambda_2}(k) = \begin{cases} \mathrm{RZ}_{\Lambda_1 \cap \Lambda_2}(k) & \textit{if } \Lambda_1 \cap \Lambda_2 \textit{ is a vertex lattice,} \\ \emptyset & \textit{otherwise.} \end{cases}$$

(iii) *The irreducible components of* $\mathrm{RZ}^{(\ell),\mathrm{red}}$ *are precisely the closed subschemes* $\mathrm{RZ}_\Lambda^{(\ell),\mathrm{red}}$ *indexed by the vertex lattices of type* $t_\Lambda = t_{\max}$.

Loosely speaking, the theorem asserts that the irreducible components of $\mathrm{RZ}^{\mathrm{red}}$, their intersections, the intersections of their intersections, etc. are all isomorphic to varieties of the form $S_\Lambda^{\pm}$ for various choices of Λ. The following result is an immediate corollary of this and the uniformization result of Theorem A.

Theorem E. *For* $U^p \subset G(\mathbb{A}_f^p)$ *sufficiently small, every irreducible component of the supersingular locus (1.2.2.1) is isomorphic to a connected component of the smooth projective* k*-variety*

$$\left\{ \text{Lagrangians } \mathscr{L} \subset \Omega_0 \otimes k : \dim(\mathscr{L} + \Phi(\mathscr{L})) = \frac{t_{\max}}{2} + 1 \right\},$$

where Ω_0 *is the unique quadratic space over* $\mathbb{F}_p$ *having dimension* $t_{\max}$*, and admitting no Lagrangian subspace. In particular, all irreducible components of* $\mathscr{S}_{\mathrm{ss}}$ *are smooth and projective of dimension*

$$\dim(\mathscr{S}_{\mathrm{ss}}) = \frac{t_{\max}}{2} - 1 = \begin{cases} (d/2) - 1 & \textit{if } V_{\mathbb{Q}_p} \textit{ is a sum of hyperbolic planes,} \\ \lfloor d/2 \rfloor & \textit{otherwise.} \end{cases}$$

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1.3 Applications and directions of further inquiry

1.3.1 One motivation for wanting such an explicit description of the supersingular locus for GSpin Shimura varieties is because of its relevance to conjectures of Kudla [Kud04] relating intersections of special cycles on orthogonal Shimura varieties to derivatives of Eisenstein series. Indeed, Kudla and Rapoport [KR99, KR00] were able to verify many cases of these conjectures for Shimura varieties attached to the low-rank groups $\mathrm{GSpin}(2,2)$ and $\mathrm{GSpin}(3,2)$, and their arguments depend in an essential way on having concrete descriptions of the supersingular loci.

With the results of § 1.2 now in hand, it should be possible to extend the results of these papers to all Shimura varieties of type $\mathrm{GSpin}(d,2)$. Some results in this direction will appear in the forthcoming Boston College PhD thesis of Cihan Soylu.

1.3.2 Görtz and He [GH15] have studied all basic minuscule affine Deligne–Lusztig varieties for equicharacteristic discrete-valued fields. They give a list of cases where these affine Deligne–Lusztig varieties can be expressed as a union of classical Deligne–Lusztig varieties, and that list contains equicharacteristic analogues of the GSpin Rapoport–Zink spaces considered here. In fact, these spaces are the only (absolutely simple) types in their list with hyperspecial level subgroups which are not of EL or PEL type. The results of Görtz and He in the equicharacteristic case are analogous to our mixed characteristic results.

There are other Hodge type cases for which a similar description should be possible, but for more general parahoric level subgroups. Extending our construction of Rapoport–Zink formal schemes to the general parahoric case, by using, for example, the integral models of Shimura varieties given in [KP15], is an interesting problem. If this is done, then our results should extend to cover all the cases listed in [GH15]. This will probably require generalizing, via Bruhat–Tits theory, the algebra of lattices in quadratic spaces we use in this paper. In another direction, it would also be interesting to understand our results from the point of view of the stratifications introduced by Chen and Viehmann in [CV15].

1.3.3 In the cases considered in [GH15], the affine Deligne–Lusztig varieties are unions of Ekedahl–Oort (EO) strata. Such strata can be defined in the hyperspecial mixed characteristic case following [Zha13] or [Vie14]. In the GSpin case considered here, the EO strata should be indexed by the possible types t_Λ of vertex lattices Λ. In fact, we expect that each EO stratum is the union of all *Bruhat–Tits strata*

$$\mathrm{BT}_\Lambda = \mathrm{RZ}_\Lambda^{\mathrm{red}} \smallsetminus \bigcup_{\Lambda' \subsetneq \Lambda} \mathrm{RZ}_{\Lambda'}^{\mathrm{red}}$$

in the sense of § 6.5, with Λ ranging over all vertex lattices of the corresponding type.

1.4 Organization and contents

In § 2 we first fix notation and recall some general facts about windows and crystals for p-divisible groups, and about local Shimura data. When the local Shimura datum $(G,[b],\{\mu\})$ is of Hodge type, and after fixing a suitable Hodge embedding, we define in § 2.3 a functor $\mathrm{RZ}_G^{\mathrm{nilp}}$ on p-nilpotent algebras. We also consider a functor $\mathrm{RZ}_G^{\mathrm{fsm}}$ defined (only) on formally smooth, formally of finite type p-adic algebras, which is essentially given by a limit of values of $\mathrm{RZ}_G^{\mathrm{nilp}}$. In § 2.4 we describe the field-valued points of these functors via refined affine Deligne–Lusztig sets.

In § 3.1 we switch to the global set-up of Shimura varieties and recall some properties of the canonical integral models constructed by Kisin. Then in § 3.2 we prove the first main result of the paper (Theorem 3.2.1): roughly speaking, we show that when the local Shimura datum is

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obtained from a global one, the functor $\mathrm{RZ}_G^{\mathrm{fsm}}$ is representable by a formal scheme RZ_G. In § 3.3 we prove a uniformization theorem for the formal completion of the integral model of the Shimura variety along its basic locus.

The rest of the paper is devoted to Rapoport–Zink formal schemes and Shimura varieties for spinor similitude groups.

In § 4 we describe the corresponding local Shimura data and define the GSpin Rapoport–Zink formal schemes. We devote § 5 to the algebra of certain type of lattices ('vertex lattices' and 'special lattices') in quadratic spaces. This, together with our previous general results, is used in § 6 to describe the reduced scheme underlying the basic GSpin Rapoport–Zink formal schemes (see especially § 6.4). Finally, in § 7, we apply our local results to the global problem of describing the supersingular loci of Shimura varieties of type GSpin.

1.5 Notation and conventions

Throughout the paper, $k = \bar{\mathbb{F}}_p$, where $p > 2$. The absolute Frobenius on k is denoted by $\sigma(x) = x^p$. We also denote by σ the induced automorphism of the ring of Witt vectors $W = W(k)$ and its fraction field $K = W[1/p]$.

2. Rapoport–Zink spaces of Hodge type

2.1 Preliminaries

In this section we introduce notation for various categories of W-algebras. We also recall some facts about divided power thickenings and crystals of p-divisible groups, and Zink's theory of windows.

2.1.1 As in [RZ96], we will denote by Nilp_W the category of W-schemes S such that p is Zariski locally nilpotent in $\mathcal{O}_S$. Denote by

$$\mathrm{ANilp}_W \subset \mathrm{Nilp}_W^{op}$$

the full subcategory of Noetherian W-algebras in which p is nilpotent. We denote by $\mathrm{ANilp}_W^{\mathrm{f}}$ the category of Noetherian adic W-algebras in which p is nilpotent, and embed

$$\mathrm{ANilp}_W \subset \mathrm{ANilp}_W^{\mathrm{f}}$$

as a full subcategory by endowing any W-algebra in ANilp_W with its p-adic topology.

We say that an adic W-algebra A is *formally finitely generated* if A is Noetherian, and if A/I is a finitely generated W-algebra for some ideal of definition $I \subset A$. Thus $\mathrm{Spf}(A)$ is a formal scheme which is formally of finite type over $\mathrm{Spf}(W)$. If, in addition, p is nilpotent in A, then A is a quotient of $W/(p^n)[\![x_1, \dots, x_r]\!][y_1, \dots, y_s]$ for some n, r, and s.

We will denote by

$$\mathrm{ANilp}_W^{\mathrm{fsm}} \subset \mathrm{ANilp}_W^{\mathrm{f}}$$

the full subcategory whose objects are W-algebras that are formally finitely generated and formally smooth over $W/(p^n)$, for some $n \geqslant 1$.

2.1.2 As in [RZ96, § 2.1], every formal scheme $\mathfrak{X}$ over $\mathrm{Spf}(W)$ defines a functor on Nilp_W. We restrict this functor to ANilp_W, and then extend to $\mathrm{ANilp}_W^{\mathrm{f}}$ as follows: for A in $\mathrm{ANilp}_W^{\mathrm{f}}$ with ideal of definition I, define $\mathfrak{X}(A)$ to be the set

$$\mathfrak{X}(A) = \mathrm{Hom}_{\mathrm{Spf}(W)}(\mathrm{Spf}(A), \mathfrak{X}) = \varprojlim_n \mathfrak{X}(A/I^n).$$

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2.1.3 If R is an object of $\mathrm{ANilp}_W^{\mathrm{fsm}}$, the quotient $\bar{R} = R/pR$ satisfies the condition [deJ95, (1.3.1.1)]. Thus, by [deJ95, Lemma 1.3.3], $\bar{R}$ admits a divided power (PD) thickening

$$\widetilde{R} \to \widetilde{R}/p\widetilde{R} = \bar{R}$$

by a formally smooth, p-adically complete W-algebra $\widetilde{R}$, unique up to non-canonical isomorphism. The absolute Frobenius on $\bar{R}$ lifts to $\widetilde{R}$. The formal smoothness of R over W/p^nW implies that $R \xrightarrow{\sim} \widetilde{R}/p^n\widetilde{R}$, and hence $\widetilde{R}$ also provides a PD thickening $\widetilde{R} \to R$. One can show that $\widetilde{R}$ is a quotient of a W-algebra of the form

$$W[\![x_1, \ldots, x_r]\!]\{y_1, \ldots, y_s\} = \varprojlim_n W/(p^n)[\![x_1, \ldots, x_r]\!][y_1, \ldots, y_s],$$

and hence is Noetherian.

2.1.4 Suppose that R is any k-algebra admitting a p-basis in the sense of [BM90, § 1.1]. An explicit construction of a PD thickening $\widetilde{R} \to R$ is then explained in [BM90]. This applies in particular when $R = k'$ is any field extension of k, in which case $\widetilde{R}$ is isomorphic to the Cohen ring W' of k'.

Recall that the Cohen ring W' is the unique, up to non-canonical isomorphism, discrete valuation ring with k' as a residue field and p as uniformizer. It is flat over the Witt ring W of k. If $(x_i)_i$ is a p-basis of k', then a choice of elements $y_i \in W'$ with $x_i^p \equiv y_i \pmod{pW'}$ determines a lift $\sigma : W' \to W'$ of the absolute Frobenius $k' \to k'$. Set $K' = W'[1/p]$.

2.1.5 Continue with the above notation, and fix a lift $\sigma : W' \to W'$ of the Frobenius of the field k'. The triple (W', pW', k') gives a frame for k' in the sense of [Zin01]. As in [Zin01, Definition 2], a *Dieudonné W'-window* over k' consists of a triple (M, M_1, F), in which

- M is a free finitely generated W'-module;
- $M_1 \subset M$ is a W'-submodule such that $pM \subset M_1 \subset M$;
- $F : M \to M$ is a σ-semi-linear map such that $F(M_1) \subset pM$, and $p^{-1}F(M_1)$ generates M as an W'-module.

These conditions imply $F(M) \subset p^{-1}F(M_1)$, and so are equivalent to the conditions appearing in [Zin01, Definition 2]. If (M, M_1, F) is a Dieudonné W'-window then

$$M_1 = F^{-1}(pM) = \{x \in M[1/p] : F(x) \in pM\}.$$

If M is a free finitely generated W'-module, and $F : M[1/p] \to M[1/p]$ is a σ-semi-linear map such that

- $pM \subset F^{-1}(pM) \subset M$, and
- $F(F^{-1}(M))$ generates M as a W'-module,

then $F(M) \subset M$ and $(M, F^{-1}(pM), F)$ is a Dieudonné W'-window.

A Dieudonné W'-window is called simply a *W'-window* when the additional nilpotence condition of [Zin01, Definition 3] is satisfied.

2.1.6 Let S be a scheme such that p is Zariski locally nilpotent in $\mathcal{O}_S$. Set

$$\bar{S} = S \otimes_{\mathbb{Z}_p} \mathbb{F}_p,$$

and denote by $\sigma : \bar{S} \to \bar{S}$ the absolute Frobenius morphism.

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For a p-divisible group X over S, we will denote by $\mathbb{D}(X)$ its contravariant Dieudonné crystal. It is a crystal of locally free $\mathcal{O}_S/\mathbb{Z}_p$-modules of rank equal to the height $h(X)$ of X. We refer the reader to [Mes72, BBM82, deJ95] for background on the construction and properties of the Dieudonné crystal.

The crystal $\mathbb{D}(X)$ is equipped with the Hodge filtration

$$\mathrm{Fil}^1(X) = \mathrm{Lie}(X)^* \subset \mathbb{D}(X)_S,$$

where $\mathbb{D}(X)_S$ is the pull-back of $\mathbb{D}(X)$ to the Zariski site of S; it is a locally free $\mathcal{O}_S$-module of rank $h(X)$, and the $\mathcal{O}_S$-submodule $\mathrm{Fil}^1(X)$ is locally a direct summand. We also have the Frobenius morphism

$$F : \sigma^*\mathbb{D}(X) \to \mathbb{D}(X), \tag{2.1.6.1}$$

where the pull-back $\sigma^*\mathbb{D}(X)$ is defined as in the above references.

Define crystals

$$\mathbf{1} = \mathbb{D}(\mathbb{Q}_p/\mathbb{Z}_p), \quad \mathbf{1}(-1) = \mathbb{D}(\mu_{p^\infty}),$$

and note that $\mathbf{1}$ is the structure sheaf $\mathcal{O}_S/\mathbb{Z}_p$ with the usual Frobenius structure and $\mathrm{Fil}^1 = (0)$. We often confuse a global section t of a crystal $\mathbb{D}$ with the corresponding morphism of crystals $t : \mathbf{1} \to \mathbb{D}$.

We define $\mathbb{D}(X)^*$ to be the $\mathcal{O}_S/\mathbb{Z}_p$-linear dual with the dual filtration. Note that $\mathbb{D}(X^\vee)^* = \mathbb{D}(X)(-1)$, where $X^\vee$ is the dual p-divisible group. There is a Frobenius structure on $\mathbb{D}(X)^*$ as in (2.1.6.1) but it is defined only 'up to isogeny', i.e. only after we view $\mathbb{D}(X)^*$ as an isocrystal as below.

We define the category of isocrystals over S as follows.

- Objects are crystals $\mathbb{D}$ of locally free $\mathcal{O}_S/\mathbb{Z}_p$-modules. We write $\mathbb{D}[1/p]$ if we view $\mathbb{D}$ as an isocrystal.
- Morphisms $\mathbb{D}[1/p] \to \mathbb{D}'[1/p]$ are given by global sections of the Zariski sheaf $\underline{\mathrm{Hom}}(\mathbb{D}, \mathbb{D}')[1/p]$ over S, where $\mathrm{Hom}(\mathbb{D}, \mathbb{D}')$ is taken in the category of crystals of locally free $\mathcal{O}_S/\mathbb{Z}_p$-modules.

Every quasi-isogeny $\rho : X \dashrightarrow X'$ of p-divisible groups over S, in the sense of [RZ96, Definition 2.8], induces an isomorphism of isocrystals

$$\mathbb{D}(\rho) : \mathbb{D}(X')[1/p] \xrightarrow{\sim} \mathbb{D}(X)[1/p]. \tag{2.1.6.2}$$

The *total tensor algebra* $\mathbb{D}(X)^\otimes$ is defined as the direct sum of all the crystals of locally free $\mathcal{O}_S/\mathbb{Z}_p$-modules which can be formed from $\mathbb{D}(X)$ using the operations of taking duals, tensor products, symmetric powers and exterior powers. It is a crystal of locally free $\mathcal{O}_S/\mathbb{Z}_p$-modules over S. The Hodge filtration on $\mathbb{D}(X)_S$ induces a natural filtration $\mathrm{Fil}^\bullet(\mathbb{D}(X)_S^\otimes)$ on $\mathbb{D}(X)_S^\otimes$, and the Frobenius morphism (2.1.6.1) induces an isomorphism of isocrystals

$$F : \sigma^*\mathbb{D}(X)^\otimes[1/p] \xrightarrow{\sim} \mathbb{D}(X)^\otimes[1/p].$$

For any quasi-isogeny $\rho : X \dashrightarrow X'$ of p-divisible groups over S, the isomorphism (2.1.6.2) extends to

$$\mathbb{D}(\rho) : \mathbb{D}(X')^\otimes[1/p] \xrightarrow{\sim} \mathbb{D}(X)^\otimes[1/p].$$

A similar discussion applies to formal schemes S over $\mathrm{Spf}(\mathbb{Z}_p)$, as in [deJ95, ch. 2].

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2.1.7 Suppose X is a formal p-divisible group over a field k' of characteristic p. Again letting W' be the Cohen ring of k', the evaluation

$$\mathbb{D}(X)(W') = \varprojlim_n \mathbb{D}(X)(W'/p^n W')$$

of the crystal $\mathbb{D}(X)$ on W' has a natural structure of a W'-window over k'. (Combine the proof of [Zin01, Theorem 1.6] with [Zin02, Theorem 6].)

By [Zin01, Theorem 4], the functor $X \mapsto \mathbb{D}(X)(W')$ gives an anti-equivalence of categories between formal p-divisible groups over k' and W'-windows over k'. More precisely, the equivalence of [Zin01] uses the covariant Dieudonné crystal, and we compose the functor defined there with Cartier duality.

If k' is perfect, classical Dieudonné theory (or [Zin01, Theorem 3.2]) gives in the same way an anti-equivalence between (all) p-divisible groups over k' and Dieudonné modules over $W' = W(k')$.

2.2 Local Shimura data

For the remainder of § 2, G is a connected reductive group scheme over $\mathbb{Z}_p$. The generic fiber of G is therefore a connected reductive group over $\mathbb{Q}_p$, and is unramified in the sense that it is quasi-split and split over an unramified extension of $\mathbb{Q}_p$. Conversely, every unramified connected reductive group over $\mathbb{Q}_p$ is isomorphic to the generic fiber of such a G.

2.2.1 Let $([b], \{\mu\})$ be a pair consisting of:

- a $G(\bar{K})$-conjugacy class $\{\mu\}$ of cocharacters $\mu : \mathbb{G}_{m\bar{K}} \to G_{\bar{K}}$;
- a σ-conjugacy class $[b]$ of elements $b \in G(K)$.

Here b and b' are σ-conjugate if there is $g \in G(K)$ with $b' = gb\sigma(g)^{-1}$.

We let $E \subset \bar{K}$ be the field of definition of the conjugacy class $\{\mu\}$. This is the *local reflex field*. Denote by $\mathcal{O}_E$ its valuation ring and by k_E its (finite) residue field. In fact, under our assumption on G, the field $E \subset \bar{K}$ is contained in K and there is a cocharacter $\mu : \mathbb{G}_{mE} \to G_E$ in the conjugacy class $\{\mu\}$ that is defined over E; see [Kot84, Lemma (1.1.3)]. In fact, we can find a representative μ that extends to an integral cocharacter

$$\mu : \mathbb{G}_{m\mathcal{O}_E} \to G_{\mathcal{O}_E}, \tag{2.2.1.1}$$

and the $G(\mathcal{O}_E)$-conjugacy class of such an μ is well defined. In what follows, we usually assume that μ is such a representative.

To the conjugacy class $\{\mu\}$ we associate the homogeneous space

$$M_{G,\mu} = G_{\mathcal{O}_E}/P_\mu$$

over $\mathcal{O}_E$, in which $P_\mu \subset G_{\mathcal{O}_E}$ is the parabolic subgroup defined by μ. More precisely, P_μ is the parabolic subgroup such that $P_\mu \times_{\mathcal{O}_E} W$ contains exactly the root groups U_a of the split group G_W, for all roots a with $a \cdot \mu \geqslant 0$. The group $P_\mu \times_{\mathcal{O}_E} W$ stabilizes the filtration defined by μ in any representation of G_W.

We write $\mu^\sigma = \sigma(\mu)$ for the Frobenius conjugate of (2.2.1.1).

Definition 2.2.2 (Cf. [RV14, Definition 5.1]). A *local unramified Shimura datum* is a triple $(G, [b], \{\mu\})$, in which G is a connected reductive group over $\mathbb{Z}_p$, the pair $([b], \{\mu\})$ is as above, and we assume:

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(i) $\{\mu\}$ is minuscule;

(ii) for some (equivalently, any) integral representative (2.2.1.1) of $\{\mu\}$, the σ-conjugacy class $[b]$ has a representative

$$b \in G(W)\mu^{\sigma}(p)G(W). \tag{2.2.2.1}$$

By [RR96, Theorem 4.2], assumptions (i) and (ii) imply that $[b]$ lies in the set $B(G_{\mathbb{Q}_p}, \{\mu\})$ of neutral acceptable elements for $\{\mu\}$; see [RV14, Definition 2.3]. In particular, $(G_{\mathbb{Q}_p}, [b], \{\mu\})$ is a local Shimura datum in the sense of [RV14, Definition 5.1].

DEFINITION 2.2.3. The local unramified Shimura datum $(G, [b], \{\mu\})$ is of *Hodge type* if there exists a closed group scheme embedding $\iota : G \hookrightarrow \mathrm{GL}(C)$, for a free $\mathbb{Z}_p$-module C of finite rank, with the following properties: the central torus $\mathbb{G}_m \subset \mathrm{GL}(C)$ is contained in G, and, after a choice of basis $C_{\mathcal{O}_E} \xrightarrow{\sim} \mathcal{O}_E^n$, the composite cocharacter

$$\iota \circ \mu : \mathbb{G}_{m\mathcal{O}_E} \to \mathrm{GL}_{n,\mathcal{O}_E}$$

is the inverse of the minuscule cocharacter[1]

$$a \mapsto \mathrm{diag}(a^{(r)}, 1^{(n-r)})$$

for some $1 \leqslant r < n$.

DEFINITION 2.2.4. Let $(G, [b], \{\mu\})$ be a local unramified Shimura datum of Hodge type. A *local Hodge embedding datum* for $(G, [b], \{\mu\})$ consists of:

– a group scheme embedding $\iota : G \hookrightarrow \mathrm{GL}(C)$ as in Definition 2.2.3;

– the $G(W)$-σ-conjugacy class $\{gb\sigma(g)^{-1} : g \in G(W)\}$ of a representative

$$b \in G(W)\mu^{\sigma}(p)G(W)$$

of $[b]$, where $\mu : \mathbb{G}_{mW} \to G_W$ is chosen to be an integral representative of the $G(\bar{K})$-conjugacy class $\{\mu\}$. Note that such a representative μ is unique up to $G(W)$-conjugacy.

The quadruple (G, b, μ, C), where μ is given up to $G(W)$-conjugation, and b up to $G(W)$-σ-conjugation, is a *local unramified Shimura–Hodge datum.*

By definition, there is a surjection $(G, b, \mu, C) \mapsto (G, [b], \{\mu\})$ from the set of local unramified Shimura–Hodge data to the set of local unramified Shimura data of Hodge type.

Fix a local unramified Shimura–Hodge datum (G, b, μ, C), and set $D = \mathrm{Hom}_{\mathbb{Z}_p}(C, \mathbb{Z}_p)$ with the contragredient action of G.

LEMMA 2.2.5. *Up to isomorphism, there is a unique p-divisible group*

$$X_0 = X_0(G, b, \mu, C)$$

over k whose contravariant Dieudonné module is $\mathbb{D}(X_0)(W) = D_W$ with Frobenius $F = b \circ \sigma$. Moreover, the Hodge filtration

$$VD_k \subset D_k = \mathbb{D}(X_0)(k)$$

is induced by a conjugate of the reduction $\mu_k : \mathbb{G}_{mk} \to G_k$.

[1] The notation $a^{(r)}$ means that there are r copies of a.

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Proof. By our assumption on μ in Definition 2.2.3, we have $\mu(p)D_W \subset D_W$. Therefore, by (2.2.2.1), the lattice $D_W \subset D_W[1/p]$ is F-stable. To determine VD_W, write $b = h'\mu^\sigma(p)h$ with $h, h' \in G(W)$, so that

$$\begin{aligned} VD_W = pF^{-1}D_W &= p\sigma^{-1}(b^{-1}D_W) \\ &= \sigma^{-1}(h^{-1}p\mu^\sigma(p)^{-1}h'^{-1}D_W) \\ &= h_1 p\mu(p)^{-1}h_1^{-1}D_W \end{aligned}$$

for $h_1 = \sigma^{-1}(h^{-1}) \in G(W)$. Observe that $p\mu(p)^{-1}D_W \subset D_W$, and in fact the filtration

$$(p\mu(p)^{-1}D_W)/pD_W \subset D_W/pD_W = D_k$$

is induced by $\mu_k : \mathbb{G}_{mk} \to G_k$.

The above calculation shows that $VD_W \subset D_W$, and that the Hodge filtration $VD_k \subset D_k$ is induced by the conjugate $\bar{h}_1\mu_k\bar{h}_1^{-1}$. □

2.2.6 By [Kis10, Proposition (1.3.2)], there is finite list (s_α) of tensors s_α in the total tensor algebra $C^\otimes$ that 'cut out' the group G, in the sense that

$$G(R) = \{g \in \mathrm{GL}(C \otimes_{\mathbb{Z}_p} R) : g \cdot (s_\alpha \otimes 1) = (s_\alpha \otimes 1), \forall \alpha\}$$

for all $\mathbb{Z}_p$-algebras R. Using the canonical isomorphism $C^\otimes = D^\otimes$, the tensors $s_\alpha \in C^\otimes$ determine tensors $s_\alpha \otimes 1 \in D^\otimes \otimes_{\mathbb{Z}_p} W$. Thus, if X_0 is the p-divisible group of Lemma 2.2.5, we obtain tensors

$$t_{\alpha,0} = s_\alpha \otimes 1 \in D^\otimes \otimes_{\mathbb{Z}_p} W = \mathbb{D}(X_0)(W)^\otimes,$$

which are Frobenius invariant when viewed in $\mathbb{D}(X_0)(W)^\otimes[1/p]$. The tensors $t_{\alpha,0}$ uniquely determine morphisms of crystals $t_{\alpha,0} : \mathbf{1} \to \mathbb{D}(X_0)^\otimes$ over $\mathrm{Spec}(k)$, such that each

$$t_{\alpha,0} : \mathbf{1}[1/p] \to \mathbb{D}(X_0)[1/p]^\otimes$$

is Frobenius equivariant.[2] Here, as before, we denote by $\mathbf{1} = \mathbb{D}(\mathbb{Q}_p/\mathbb{Z}_p)$ the crystal determined by the Dieudonné module W with $F = \sigma$. Using Lemma 2.2.5, we easily see that $t_{\alpha,0}(k) \in \mathrm{Fil}^0(\mathbb{D}(X_0)(k)^\otimes)$.

2.2.7 By [Kot85], every σ-conjugacy class in $G(K)$ is decent in the sense of [RZ96, Definition 1.8]. By [RZ96, Proposition 1.12], any $b \in G(K)$ determines a smooth affine group scheme J_b over $\mathbb{Q}_p$ with functor of points

$$J_b(R) = \{g \in G(R \otimes_{\mathbb{Q}_p} K) : gb\sigma(g)^{-1} = b\}$$

for any $\mathbb{Q}_p$-algebra R. Up to isomorphism, J_b depends only on the σ-conjugacy class $[b]$.

2.2.8 Let $\mathbb{T}$ be the pro-torus over $\mathbb{Z}_p$ with character group $\mathbb{Q}$. For any $\mathbb{Z}_p$-algebra R, an R-point $z \in \mathbb{T}(R)$ consists of a tuple

$$z = (z_m \in R^\times)_{m \in \mathbb{Z}_{>0}}$$

such that $z_m = z_{md}^d$ for all positive m and d. The character indexed by the rational number s/t sends $z \mapsto z_t^s$.

[2] Since $\mathbb{D}(X_0)^\otimes$ also involves the dual, the Frobenius is not defined on $\mathbb{D}(X_0)^\otimes$, but only on $\mathbb{D}(X_0)^\otimes[1/p]$; see § 2.1.6.

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Kottwitz [Kot85] attaches to every $b \in G(K)$ a *slope cocharacter*

$$\nu_b : \mathbb{T}_K \to G_K$$

such that for any representation $\psi : G_{\mathbb{Q}_p} \to \mathrm{GL}(M)$ on a $\mathbb{Q}_p$-vector space M, the decomposition

$$M_K = \bigoplus_{s/t \in \mathbb{Q}} M_K^{s/t}$$

of M_K determined by the cocharacter $\psi \circ \nu_b : \mathbb{T}_K \to \mathrm{GL}(M_K)$ agrees with the slope decomposition of the isocrystal $(M_K, \psi(b) \circ \sigma)$. The slope cocharacter depends only on the σ-conjugacy class $[b]$.

An element $b \in G(K)$ is *basic* if its slope cocharacter ν_b factors through the center of G_K. By [Kot85], b is basic if and only if the group J_b is an inner form of G.

2.3 Rapoport–Zink formal schemes and functors

In this subsection we define Rapoport–Zink formal schemes and functors associated to a local unramified Shimura–Hodge datum (G, b, μ, C) as defined in § 2.2. We start by recalling the definition of some 'classical' Rapoport–Zink functors.

2.3.1 Suppose that X_0 is any p-divisible group over k. The *Rapoport–Zink space* $\mathrm{RZ}(X_0)$ of deformations of X_0 up to quasi-isogeny is, as in [RZ96], the formal scheme over $\mathrm{Spf}(W)$ that represents the functor assigning to each scheme S in Nilp_W the set of isomorphism classes of pairs (X, ρ) in which

- X is a p-divisible group over S;
- $\rho : X_0 \times_k \bar{S} \dashrightarrow X \times_S \bar{S}$ is a quasi-isogeny, where $\bar{S} = S \otimes_W k$.

Suppose now that X_0 comes with a principal polarization $\lambda_0 : X_0 \xrightarrow{\sim} X_0^\vee$. The *symplectic Rapoport–Zink space* $\mathrm{RZ}(X_0, \lambda_0)$ is the formal scheme over $\mathrm{Spf}(W)$ that represents the functor that assigns to each S in Nilp_W the set of isomorphism classes of triples (X, λ, ρ) in which

- X is a p-divisible group over S;
- $\lambda : X \xrightarrow{\sim} X^\vee$ is a principal polarization;
- $\rho : X_0 \times_k \bar{S} \dashrightarrow X \times_S \bar{S}$ is a quasi-isogeny that respects polarizations up to a scalar, in the sense that, Zariski locally on $\bar{S}$, we have

$$\rho^\vee \circ \lambda \circ \rho = c^{-1}(\rho) \cdot \lambda_0,$$

  for some $c(\rho) \in \mathbb{Q}_p^\times$.

By [RZ96] the formal schemes $\mathrm{RZ}(X_0)$ and $\mathrm{RZ}(X_0, \lambda_0)$ are formally smooth and locally formally of finite type over W, and forgetting the polarization defines a closed immersion $\mathrm{RZ}(X_0, \lambda_0) \to \mathrm{RZ}(X_0)$.

2.3.2 Suppose that (G, b, μ, C) is a local unramified Shimura–Hodge datum. Choose tensors (s_α) that cut out G as in § 2.2.6. Denote by

$$X_0 = X_0(G, b, \mu, C)$$

the corresponding p-divisible group over k of Lemma 2.2.5, with its Frobenius invariant crystalline tensors $(t_{\alpha,0})$.

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Definition 2.3.3. Consider the functor

$$\mathrm{RZ}_G^{\mathrm{nilp}} = \mathrm{RZ}^{\mathrm{nilp}}_{G,b,\mu,C,(s_\alpha)} : \mathrm{ANilp}_W \to \mathrm{Sets}$$

that assigns to each $R \in \mathrm{ANilp}_W$ the set of isomorphism classes of triples $(X, \rho, (t_\alpha))$ in which

- (X, ρ) consists of a p-divisible group over $\mathrm{Spec}(R)$ and a quasi-isogeny

$$\rho : X_0 \otimes_k \bar{R} \dashrightarrow X \otimes_R \bar{R},$$

  with $\bar{R} = R/pR$, as in the definition of the Rapoport–Zink formal scheme $\mathrm{RZ}(X_0)$;
- the collection (t_α) consists of morphisms of crystals $t_\alpha : \mathbf{1} \to \mathbb{D}(X)^{\otimes}$ over $\mathrm{Spec}(R)$ with $t_\alpha : \mathbf{1}[1/p] \to \mathbb{D}(X)^{\otimes}[1/p]$ Frobenius equivariant,

satisfying the following properties.

(i) For some nilpotent ideal $J \subset R$ with $p \in J$, the pull-back of t_α over $\mathrm{Spec}(R/J)$ is identified with $t_{\alpha,0}$ under the isomorphism of isocrystals

$$\mathbb{D}(\rho) : \mathbb{D}(X_{R/J})^{\otimes}[1/p] \xrightarrow{\sim} \mathbb{D}(X_0 \times_k R/J)^{\otimes}[1/p]$$

induced by the quasi-isogeny ρ.

(ii) The sheaf of G_W-sets over $\mathrm{CRIS}(\mathrm{Spec}(R)/W)$ given by isomorphisms

$$\underline{\mathrm{Isom}}_{t_\alpha, s_\alpha \otimes 1}(\mathbb{D}(X), D \otimes_{\mathbb{Z}_p} R)$$

that respect the tensors as indicated, is a crystal of G_W-torsors, i.e. a crystal fppf locally isomorphic to the crystal defined by G_W.

(iii) There exists an étale cover $\{U_i\}$ of $\mathrm{Spec}(R)$, and for each i an isomorphism

$$\mathbb{D}(X_{U_i})_{U_i} \xrightarrow{\sim} D \otimes_{\mathbb{Z}_p} \mathcal{O}_{U_i}$$

of vector bundles respecting the tensors t_α and $s_\alpha \otimes 1$ as in (ii), such that the Hodge filtration

$$\mathrm{Fil}^1(X_{U_i}) \subset \mathbb{D}(X_{U_i})_{U_i} \xrightarrow{\sim} D \otimes_{\mathbb{Z}_p} \mathcal{O}_{U_i}$$

is induced by a cocharacter that is $G(U_i)$-conjugate to μ.

Two triples $(X, \rho, (t_\alpha))$ and $(X', \rho', (t'_\alpha))$ are identified if there is an isomorphism $X \xrightarrow{\sim} X'$ of p-divisible groups that respects the rest of the data in the obvious manner.

Above, $\mathrm{CRIS}(\mathrm{Spec}(R)/W)$ denotes the big fppf crystalline site of $\mathrm{Spec}(R)$ over $(W, (p), \gamma)$ with γ the natural PD structure, as in [BBM82, 1.1]. Condition (ii) implies that, for any nilpotent PD thickening $R' \to R$ of R, the $\mathrm{Spec}(R')$-scheme of isomorphisms of finite locally free R'-modules

$$T_{R'} = \underline{\mathrm{Isom}}_{R', t_\alpha(R'), s_\alpha \otimes 1}(\mathbb{D}(X)(R'), D \otimes_{\mathbb{Z}_p} R')$$

is a $G_{R'}$-torsor.

Remark 2.3.4. There is a distinguished point

$$x_0 = (X_0, \rho_0, (t_{\alpha,0})) \in \mathrm{RZ}^{\mathrm{nilp}}(k),$$

defined by taking ρ_0 to be the identity quasi-isogeny $X_0 \dashrightarrow X_0$.

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Remark 2.3.5. (a) Suppose $(p) \subset J' \subset J$ with J' also nilpotent. Then a power of the Frobenius of R/J' factors through R/J. Since t_α are Frobenius equivariant we obtain that condition (i) is independent of the ideal J. In particular, we can simply take $J = (p)$.

(b) Conditions (ii) and (iii) together imply the following. The $\mathrm{Spec}(R)$-scheme of R-linear isomorphisms

$$\mathbb{D}(X)(R) \xrightarrow{\sim} D \otimes_{\mathbb{Z}_p} R$$

identifying $t_\alpha(R)$ with $s_\alpha \otimes 1$, and identifying the Hodge filtration $\mathrm{Fil}^1(X) \subset \mathbb{D}(X)(R)$ with standard filtration $F_\mu \otimes_W R \subset D \otimes_{\mathbb{Z}_p} R$ defined by μ, is a $P_\mu \times_W R$-torsor.

(c) For R in $\mathrm{ANilp}_W^{\mathrm{f}}$, the categories of p-divisible groups over $\mathrm{Spf}(R)$ and over $\mathrm{Spec}(R)$ are naturally equivalent, by [deJ95, Lemma 2.4.4]. For R in $\mathrm{ANilp}_W^{\mathrm{fsm}}$, the argument in the proof of [deJ95, Proposition 2.4.8] shows that each morphism of crystals $\hat{t}_\alpha : \mathbf{1} \to \mathbb{D}(X)^{\otimes}$ over $\mathrm{Spf}(R)$ is induced by a unique morphism of crystals $t_\alpha : \mathbf{1} \to \mathbb{D}(X)^{\otimes}$ over $\mathrm{Spec}(R)$.

(d) For $\bar{R}$ of finite type over k, we will see that it is enough to verify (i) over one closed point of each connected component of $\mathrm{Spec}(R)$; see Lemma 3.2.8 and its proof.

2.3.6 Define a functor $\mathrm{RZ}_G^{\mathrm{fsm}}$ on $\mathrm{ANilp}_W^{\mathrm{fsm}}$ by setting

$$\mathrm{RZ}_G^{\mathrm{fsm}}(A) = \varprojlim_n \mathrm{RZ}_G^{\mathrm{nilp}}(A/I^n),$$

where I is an ideal of definition of A.

Assume that I is chosen with $p \in I$. By Remark (c) above and the rigidity of quasi-isogenies [Dri76], we see that elements of $\mathrm{RZ}_G^{\mathrm{fsm}}(A)$ correspond to isomorphism classes of triples $(X, \rho, (t_\alpha))$, in which X is a p-divisible group over $\mathrm{Spec}(A)$,

$$\rho : X_0 \times_k A/I \dashrightarrow X \times_A A/I$$

is a quasi-isogeny, and t_α a morphism of crystals over $\mathrm{Spec}(A)$, such that (i), (ii), and (iii) above are satisfied. The definition is independent of the choice of I.

Since any object A of $\mathrm{ANilp}_W^{\mathrm{fsm}}$ is also an object of ANilp_W, it makes sense to consider $\mathrm{RZ}_G^{\mathrm{nilp}}(A)$. We will rarely do this unless A is discrete, in which case

$$\mathrm{RZ}_G^{\mathrm{nilp}}(A) = \mathrm{RZ}_G^{\mathrm{fsm}}(A).$$

The difference between $\mathrm{RZ}_G^{\mathrm{fsm}}(A)$ and $\mathrm{RZ}_G^{\mathrm{nilp}}(A)$ is that, in the former, we ask that the quasi-isogeny ρ only exists over A/I, with I an ideal of definition of the adic algebra A. For A in $\mathrm{ANilp}_W^{\mathrm{fsm}}$ it will often be the case that $\mathrm{RZ}_G^{\mathrm{nilp}}(A) = \emptyset$, while $\mathrm{RZ}_G^{\mathrm{fsm}}(A) \neq \emptyset$.

2.3.7 The closed immersion $\iota : G \hookrightarrow \mathrm{GL}(C)$ induces an injective homomorphism from the group $J_b(\mathbb{Q}_p)$ into the group $\mathrm{Aut}_{\mathbb{Q}_p}(X_0)$ of quasi-automorphisms of the p-divisible group X_0, i.e. of automorphisms of X_0 up to isogeny. In addition, we can see that the induced action of $J_b(\mathbb{Q}_p)$ on $\mathbb{D}(X_0)(W)^{\otimes}[1/p]$ preserves the tensors $t_{\alpha,0}$. Therefore, the group $J_b(\mathbb{Q}_p)$ acts on the functors $\mathrm{RZ}_G^{\mathrm{nilp}}$ and $\mathrm{RZ}_G^{\mathrm{fsm}}$ on the left by

$$g \cdot (X, \rho, (t_\alpha)) = (X, \rho \circ g^{-1}, (t_\alpha)). \tag{2.3.7.1}$$

2.4 Field-valued points and affine Deligne–Lusztig sets

We now introduce some refined affine Deligne–Lusztig sets, and show that these can be used to parametrize the set $\mathrm{RZ}_G^{\mathrm{nilp}}(k')$ for any finitely generated field extension k'/k.

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2.4.1 For G, b and $\mu : \mathbb{G}_{mW} \to G_W$ as in the beginning of § 2.2.1, the 'classical' affine Deligne–Lusztig set is

$$X_{G,b,\mu}(k) = \{g \in G(K) : g^{-1} b\sigma(g) \in G(W)\mu(p)G(W)\}/G(W).$$

We will define an analogous set for any extension field k'/k.

Let W' be the Cohen ring of k', let $K' = W'[1/p]$ be its fraction field, and let $\sigma : W' \to W'$ be a lift of the absolute Frobenius as in § 2.1.4. Consider the set

$$\{g \in G(K') : g^{-1} b\sigma(g)\mu(p)^{-1} \in G(W')\}, \tag{2.4.1.1}$$

and define

$$Q_\mu(W') = G(W') \cap \mu^{\sigma^{-1}}(p)^{-1} G(W')\mu^{\sigma^{-1}}(p),$$

the intersection taking place in $G(K')$.

The right translation action of $Q_\mu(W')$ on $G(K')$ preserves (2.4.1.1). Indeed, if $q \in Q_\mu(W')$ and g belongs to (2.4.1.1), then $U_g = g^{-1}b\sigma(g)\mu(p)^{-1}$ belongs to $G(W')$, and hence so does

$$(gq)^{-1}b\sigma(gq)\mu(p)^{-1} = q^{-1}g^{-1}b\sigma(g)\sigma(q)\mu(p)^{-1} = q^{-1}U_g\mu(p)\sigma(q)\mu(p)^{-1}.$$

Thus gq belongs to (2.4.1.1).

Definition 2.4.2. The *refined affine Deligne–Lusztig set* is the quotient

$$X_{G,b,\mu,\sigma}(k') = \{g \in G(K') : g^{-1}b\sigma(g)\mu(p)^{-1} \in G(W')\}/Q_\mu(W').$$

Similarly, we have the *naive affine Deligne–Lusztig set*

$$X^{\mathrm{naive}}_{G,b,\mu,\sigma}(k') = \{g \in G(K') : g^{-1}b\sigma(g) \in G(W')\mu(p)G(W')\}/G(W').$$

For simplicity, we will often omit σ from the list of subscripts. However, we do not know if the set $X_{G,b,\mu,\sigma}(k')$ is independent of the choice of the lift of Frobenius σ.

Proposition 2.4.3. *The refined affine Deligne–Lusztig sets have the following properties.*

(i) *Sending $gQ_\mu(W')$ to $gG(W')$ defines an injection*

$$\phi(k') : X_{G,b,\mu}(k') \hookrightarrow X^{\mathrm{naive}}_{G,b,\mu}(k') \subset G(K')/G(W').$$

If k' is perfect, then $\phi(k')$ is a bijection.

(ii) *If b' is σ-conjugate to b, say $b' = h^{-1}b\sigma(h)$ with $h \in G(K)$, then $g \mapsto hg$ defines a bijection*

$$X_{G,b',\mu}(k') \xrightarrow{\sim} X_{G,b,\mu}(k').$$

(iii) *If k' is perfect, then $g \mapsto \sigma^{-1}(b^{-1}g)$ defines a bijection*

$$X_{G,b,\mu^\sigma}(k') \xrightarrow{\sim} X_{G,b,\mu}(k').$$

Proof. We first show (i). The condition for g in the refined notion is stronger than the condition in the naive notion; since $Q_\mu(W') \subset G(W')$ the map is well defined. It remains to show that it is injective. Let $g,\ g' \in G(K')$, and assume

$$U_g := g^{-1}b\sigma(g)\mu(p)^{-1} \in G(W'),$$
$$U_{g'} := g'^{-1}b\sigma(g')\mu(p)^{-1} \in G(W').$$

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Suppose there is an $h \in G(W')$ such that $g' = gh$. Then we obtain

$$U_{g'} = h^{-1}g^{-1}b\sigma(g)\sigma(h)\mu(p)^{-1} = h^{-1}U_g\mu(p)\sigma(h)\mu(p)^{-1}.$$

As $h^{-1}U_g$ and $U_{g'}$ are in $G(W')$, we see that the element

$$\mu(p)\sigma(h)\mu(p)^{-1} = \sigma(\mu^{\sigma^{-1}}(p)h\mu^{\sigma^{-1}}(p)^{-1}) \in G(K)$$

actually lies in $G(W')$. Since $\sigma : W'/p^iW' \to W'/p^iW'$ is injective for all i, and G is affine and flat over $\mathbb{Z}_p$, this implies that

$$\mu^{\sigma^{-1}}(p)h\mu^{\sigma^{-1}}(p)^{-1} \in G(W').$$

It follows that $h \in Q_\mu(W')$. This shows the injectivity of the map $\phi(k')$.

Now suppose that k' is perfect so that σ^{-1} makes sense on $W' = W(k')$. If $g \in G(K')$ is such that $g^{-1}b\sigma(g) = h_1\mu^\sigma(p)h_2$ with $h_i \in G(W')$, then $g' = g\sigma^{-1}(h_2)h_1$ satisfies the refined condition. Hence $\phi(k')$ is surjective.

Part (ii) is routine. To show part (iii), observe that for $h = \sigma^{-1}(b^{-1}g)$, we have

$$h^{-1}b\sigma(h) = \sigma^{-1}(g^{-1}b)bb^{-1}g = \sigma^{-1}(g^{-1})\sigma^{-1}(b)g = \sigma^{-1}(g^{-1}b\sigma(g))$$

and the result follows. □

2.4.4 Suppose $\iota : G \hookrightarrow \mathrm{GL}_n$ is a closed immersion of group schemes over $\mathbb{Z}_p$, and set $\nu = \iota \circ \mu : \mathbb{G}_{mW} \to \mathrm{GL}_{n,W}$. For $\tau = \sigma^{-1} \in \mathrm{Aut}(W)$, we have

$$G(W') = G(K') \cap \mathrm{GL}_n(W')$$

and

$$\mu^\tau(p)^{-1}G(W')\mu^\tau(p) = G(K') \cap \nu^\tau(p)^{-1}\mathrm{GL}_n(W')\nu^\tau(p),$$

the intersections taking place in $\mathrm{GL}_n(K')$. The embedding ι then induces injections

$$G(K')/G(W') \hookrightarrow \mathrm{GL}_n(K')/\mathrm{GL}_n(W')$$

and

$$G(K')/Q_\mu(W') \hookrightarrow \mathrm{GL}_n(K')/Q_\nu^{\mathrm{GL}_n}(W').$$

Moreover, $g \in G(K')$ satisfies $g^{-1}b\sigma(g)\mu(p)^{-1} \in G(W')$ if and only if $\iota(g) \in \mathrm{GL}_n(K')$ satisfies the corresponding condition with (G, b, μ) replaced by $(\mathrm{GL}_n, \iota(b), \nu)$. It follows that ι defines an injection

$$X_{G,b,\mu}(k') \hookrightarrow X_{\mathrm{GL}_n,\iota(b),\nu}(k').$$

2.4.5 We now return to the set-up of § 2.3. Assume that $(G, [b], \{\mu\})$ is an unramified local Shimura datum of Hodge type. Fix a corresponding local Shimura–Hodge datum (G, b, μ, C) and a set of tensors (s_α) that cuts out $G \subset \mathrm{GL}(C)$. We then have the functor $\mathrm{RZ}_G^{\mathrm{nilp}}$ as before.

Fix a point

$$(X, \rho, (t_\alpha)) \in \mathrm{RZ}_G^{\mathrm{nilp}}(k').$$

Consider the value $M = \mathbb{D}(X)(W')$ of the crystal $\mathbb{D}(X)$ on the Cohen ring W' of k', viewed as a PD thickening of k'. We have the tensors $t_\alpha(W') \in M^\otimes$, which are Frobenius invariant in $M^\otimes[1/p]$, and the quasi-isogeny ρ, which induces an isomorphism

$$\mathbb{D}(\rho) : M[1/p] \xrightarrow{\sim} M_0[1/p]$$

such that $\mathbb{D}(\rho)(t_\alpha(W')) = t_{\alpha,0}(W) \otimes 1$, by (i) of Definition 2.3.3.

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Lemma 2.4.6. *Under these assumptions, the scheme*

$$T = \underline{\mathrm{Isom}}_{W', t_\alpha(W'), s_\alpha \otimes 1}(M, D \otimes_{\mathbb{Z}_p} W')$$

is a trivial G'_W-torsor over $\mathrm{Spec}(W')$.

Proof. Note that T is an affine finite type W'-scheme carrying an action of the group scheme G'_W. We will first show that T is a G'_W-torsor over $\mathrm{Spec}(W')$.

For any $m \geqslant 1$, we have the nilpotent PD thickening $W'/p^m W' \to k'$. Therefore, by condition (ii) in the definition of $\mathrm{RZ}_G^{\mathrm{nilp}}$, the base change $T \times_{W'} W'/p^m W'$ is a $G \times_{\mathbb{Z}_p} W'/p^m W'$-torsor. It follows from the local criterion of flatness that T is W'-flat and hence also faithfully flat (since the special fiber is non-empty). Since G acts transitively on the points of T it now follows that T is an (fppf locally trivial) G'_W-torsor over $\mathrm{Spec}(W')$. Since G is smooth, the torsor T splits locally for the étale topology of $\mathrm{Spec}(W')$.

We can easily see that the generic fiber $T \times_{W'} K'$ is a trivial $G_{K'}$-torsor with a section constructed using a composition of $\mathbb{D}(\rho)(W')$ with the identification $\mathbb{D}(X_0)(W') \cong D \otimes_{\mathbb{Z}_p} W'$. By [Nis82, Theorem 5.2] (a very special case of a conjecture of Grothendieck), which applies since G is quasi-split, T is a trivial torsor. □

2.4.7 We now describe $\mathrm{RZ}_G^{\mathrm{nilp}}(k')$ in terms of a refined affine Deligne–Lusztig set. The following may be standard, but we could not find a reference.

Proposition 2.4.8. *Suppose that k'/k is a finitely generated field extension and denote by W' the Cohen ring of k'. There exists a lift of Frobenius $\sigma : W' \to W'$ with the following property. We can write $k' = \varinjlim R$, where R are finitely generated smooth k-algebras, each having a finite p-basis, such that for each R that appears in the limit there is a W-flat formally smooth p-adically complete and separated lift $\widetilde{R}$ of R with $\widetilde{R} \subset W'$ lifting $R \to k'$ which is such that $\sigma(\widetilde{R}) \subset \widetilde{R}$.*

Proof. Suppose that R is a finitely generated smooth k-algebra which is a domain and is such that k' is the fraction field of R. By replacing R by a localization we can assume that the differentials $\Omega_{R/k}$ are a free R-module of rank equal to the Krull dimension of R; let dx_i, $i = 1, \ldots, d$, be an R-basis of $\Omega_{R/k}$. In this situation, the absolute Frobenius $\phi_R : R \to R$ is injective and makes R into a finitely generated R-module. Therefore, by [EGAIV, ch. 0, Proposition (21.1.7)], the tuple (x_i) with $i = 1, \ldots, d$ is a system of p-generators of R over k, i.e. $R = k[R^p, (x_i)]$. In fact, we can easily see that, since dx_i are R-linearly independent, the x_i are p-independent (cf. [Mat80, p. 276]). Therefore, the x_i form a p-basis of R over k.

If we start with a p-basis (x_i) of k', then [Mat80, Theorem 86] implies that (dx_i) are a basis of the k'-vector space $\Omega_{k'/k}$. If $R \subset k'$ is any smooth finite type k'-algebra with $k' = \mathrm{Frac}(R)$ such that $x_i \in R$ and (dx_i) generate $\Omega_{R/k}$, then, by the above, (x_i) also provide a p-basis of R.

Since k is perfect, we can write $k' = \varinjlim R$, where R is as above. Now, as in [BM90, § 1.1], using the p-basis (x_i) we obtain a concrete construction of the Cohen ring

$$W' = \varprojlim_n A_n(k'),$$

and of a W-flat lift

$$\widetilde{R} = \varprojlim_n A_n(R)$$

of R. Here $A_n(R)$ and $A_n(k')$, are certain subrings of the truncated Witt vector rings $W_n(R)$ and $W_n(k')$. By [BM90], sending $x_i \in R$ to $x_i \in k'$ gives ring homomorphisms $i_n : A_n(R) \to A_n(k')$.

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Since $R \to k'$ is injective, $A_n(R) \to A_n(k')$ is injective, and so also $\widetilde{i} : \widetilde{R} \to W'$ is injective. Recall that, by [BM90, Proposition 1.2.6], a lift of Frobenius on $A_n(R)$, respectively $W_n(k')$, is uniquely determined by giving (arbitrary) lifts $y_{i,n} \in A_n(R)$, respectively $A_n(k')$, of all the elements x_i^p. Therefore, we can choose this way lifts of Frobenius on $\widetilde{R}$ and W' that are compatible under $\widetilde{i} : \widetilde{R} \hookrightarrow W'$. □

2.4.9 Let (G, b, μ, C) be a local unramified Shimura–Hodge datum as in Definition 2.2.4, and let (s_α) be tensors in $C^\otimes$ that cut out $G \subset \mathrm{GL}(C)$, as in § 2.2.6.

Let k'/k be a finitely generated field extension, and suppose that the lift $\sigma : W' \to W'$ of the Frobenius is chosen as in Proposition 2.4.8. Suppose also that the Dieudonné module structure on D_W determined by $\iota(b) \in \mathrm{GL}(C_W)$ has no zero slopes (equivalently, the base point p-divisible group X_0 over k defined in Lemma 2.2.5 is formal).

THEOREM 2.4.10. *Under the above assumptions, there are natural bijections*

$$\pi : \varinjlim_R \mathrm{RZ}_G^{\mathrm{nilp}}(R) \xrightarrow{\sim} \mathrm{RZ}_G^{\mathrm{nilp}}(k') \xrightarrow{\sim} X_{G,b,\mu^\sigma,\sigma}(k'),$$

where the limit over R is as in Proposition 2.4.8 above.

Proof. Let X_0 be the p-divisible group of Lemma 2.2.5, and recall that $\mathrm{RZ}(X_0)$ is the (undecorated) Rapoport–Zink formal scheme from § 2.3.1. Notice that $(X, \rho, (t_\alpha)) \mapsto (X, \rho)$ defines an injection $\mathrm{RZ}_G^{\mathrm{nilp}}(k') \hookrightarrow \mathrm{RZ}(X_0)(k')$, as t_α is determined by $t_\alpha(W')$.

Similarly, $\mathrm{RZ}_G^{\mathrm{nilp}}(R)$ injects to $\mathrm{RZ}(X_0)(R)$. Indeed, t_α is determined by $t_\alpha(\widetilde{R})$, and, as $\widetilde{R}$ is torsion-free, $t_\alpha(\widetilde{R})$ is determined by

$$t_\alpha(\widetilde{R})[1/p] = \mathbb{D}(\rho)^{-1}(t_{\alpha,0}[1/p]).$$

Since $\mathrm{RZ}(X_0)$ is formally locally of finite type over W, we have

$$\mathrm{RZ}(X_0)(k') = \varinjlim_R \mathrm{RZ}(X_0)(R),$$

and so $\varinjlim_R \mathrm{RZ}_G^{\mathrm{nilp}}(R) \hookrightarrow \mathrm{RZ}_G^{\mathrm{nilp}}(k')$.

Pick a point $x = (X, \rho, (t_\alpha)) \in \mathrm{RZ}_G^{\mathrm{nilp}}(k')$, and consider the value $M := \mathbb{D}(X)(W')$ of the crystal $\mathbb{D}(X)$ on W', endowed with the tensors $t_\alpha(W') \in M^\otimes$. By Lemma 2.4.6, the scheme

$$T_x = \underline{\mathrm{Isom}}_{W', t_\alpha(W'), s_\alpha \otimes 1}(M, D \otimes_{\mathbb{Z}_p} W')$$

is a trivial G-torsor over $\mathrm{Spec}(W')$.

If $k' = k$ and $x = x_0$ is the base point of Remark 2.3.4, then there is an isomorphism

$$\beta_0 : M_0 := \mathbb{D}(X_0)(W) \xrightarrow{\sim} D \otimes_{\mathbb{Z}_p} W$$

with $\beta_0^\otimes(t_{\alpha,0}(W)) = s_\alpha \otimes 1$ and we can use this to identify $M_0 = D \otimes_{\mathbb{Z}_p} W$. In general, the generic fiber of T_x has a section constructed using $\mathbb{D}(\rho)$ and β_0. Since the G-torsor T_x is trivial there is $\beta : M \xrightarrow{\sim} D'_W$ such that $\beta^\otimes(t_\alpha(W')) = s_\alpha \otimes 1$. Using $\mathbb{D}(\rho)$, we can identify

$$M_{\mathbb{Q}} = M_0 \otimes_W K' = D \otimes_{\mathbb{Z}_p} K'$$

and therefore think of $M \subset M_{\mathbb{Q}}$ as a W'-lattice in $M_0 \otimes_W K' = D \otimes_{\mathbb{Z}_p} K'$. Under this identification, the choice of β is equivalent to picking $g \in G(K')$ such that $M = g \cdot (D \otimes_{\mathbb{Z}_p} W')$; then β is given

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by left multiplication by g^{-1}. Notice that the coset $gG(W')$ is independent of the choice of β_0 and β and so we have a well-defined map

$$\mathrm{RZ}_G^{\mathrm{nilp}}(k') \to G(K')/G(W')$$

given by $(X, \rho, (t_\alpha)) \mapsto gG(W')$.

By Zink's theory, as in § 2.1.5 (see especially [Zin01, Theorem 4]), and using the inclusion

$$\mathrm{RZ}_G^{\mathrm{nilp}}(k') \subset \mathrm{RZ}(X_0)(k'),$$

we see that the W'-lattice $M \subset M_0 \otimes_W W'[1/p]$ uniquely determines the point $(X, \rho, (t_\alpha)) \in \mathrm{RZ}_G^{\mathrm{nilp}}(k')$. On the other hand,

$$M = g \cdot (M_0 \otimes_W W') \subset M_0 \otimes_W W'[1/p]$$

is uniquely determined by $gG(W')$, and so this construction gives an injection

$$\pi : \mathrm{RZ}_G^{\mathrm{nilp}}(k') \hookrightarrow G(K')/G(W').$$

We have to show that the image of this injection is exactly the refined affine Deligne–Lusztig set $X_{G,b,\mu^\sigma,\sigma}(k')$.

This amounts to showing that M is the W'-window corresponding to a point in $\mathrm{RZ}_G^{\mathrm{nilp}}(k')$ if and only if we can pick $g \in G(K')$ such that $M = g \cdot (M_0 \otimes_W W')$ and $g^{-1} b\sigma(g)\mu^\sigma(p)^{-1} \in G(W')$.

Suppose that M indeed corresponds to the point $x \in \mathrm{RZ}_G^{\mathrm{nilp}}(k')$ as above. Then

$$\mathbb{D}(X)(k') = M \otimes_{W'} k' = M/pM$$

with Hodge filtration given by $\mathrm{Fil}^1(X) = M_1/pM \subset M/pM$. Since the Hodge filtration is a G-filtration of type μ by Definition 2.3.3(iii), we can pick a trivialization $\bar\beta : M/pM \xrightarrow{\sim} D'_k$ that preserves the tensors (i.e. a k'-section of the trivial torsor T_x above), and such that

$$M_1/pM \subset M/pM \cong D'_k$$

is actually the filtration given by μ. (Indeed, since G_W is split, the quotient morphism $G_W \to G_W/P_\mu$ splits locally for the Zariski topology over W. Therefore, $G(k')$ acts transitively on the set of G-filtrations of D'_k of type μ.) By Hensel's lemma, we can lift $\bar\beta$ to $\beta : M \xrightarrow{\sim} D'_W$. Then $\beta(M_1) = p\mu(p)^{-1} D'_W$. As before, β corresponds to $g \in G(K')$ such that

$$M = g \cdot (M_0 \otimes_W W') = g \cdot D'_W$$

and then β is multiplication by g^{-1}. Hence,

$$M = g \cdot D'_W, \quad M_1 = g \cdot p\mu(p)^{-1} D'_W.$$

The quadruple $(M, M_1, F = b \circ \sigma)$ defines a W'-window if and only if the W'-submodule $\langle p^{-1}F(M_1)\rangle$ of $M[1/p]$ generated by $p^{-1}F(M_1)$ is equal to M. Indeed, if $(M, M_1, F = b \circ \sigma)$ is a W'-window this condition follows from the definition in § 2.1.5. Conversely, assume that $\langle p^{-1}F(M_1)\rangle = M$. Then

$$pM \subset M_1 \subset F^{-1}(pM) \subset M$$

and $\langle F(M_1)\rangle = M = \langle F(F^{-1}(pM))\rangle$. Hence $M_1 = F^{-1}(pM)$, and then $(M, M_1, F = b \circ \sigma)$ gives a W'-module by the observation in § 2.1.5 (the nilpotence condition also follows by a slope argument).

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The condition that $\langle p^{-1}F(M_1)\rangle = M$ reads

$$\langle p^{-1}b\sigma(gp\mu(p)^{-1}\cdot D\otimes_{\mathbb{Z}_p} W')\rangle = g\cdot(D\otimes_{\mathbb{Z}_p} W'),$$

which translates to

$$\langle g^{-1}b\sigma(g)\mu^{\sigma}(p)^{-1}\cdot(D\otimes_{\mathbb{Z}_p}\sigma(W'))\rangle = D\otimes_{\mathbb{Z}_p} W'. \tag{2.4.10.1}$$

Set $u = gb\sigma(g)\mu^{\sigma}(p)^{-1}\in G(K')$. Since

$$\langle h\cdot(D\otimes_{\mathbb{Z}_p}\sigma(W'))\rangle = h\cdot(D\otimes_{\mathbb{Z}_p} W'),$$

for any $h\in \mathrm{GL}_n(K')$, the equation (2.4.10.1) above amounts to $u\in\mathrm{GL}_n(W')$. We obtain that u is in $G(W') = G(K')\cap\mathrm{GL}_n(W')$, and thus $g\in X_{G,b,\mu^{\sigma},\sigma}(k')$. Therefore, the image of π is contained in $X_{G,b,\mu^{\sigma},\sigma}(k')$.

Let us now discuss the converse. Start with $g\in X_{G,b,\mu^{\sigma},\sigma}(k')$, and set

$$M = g\cdot D'_W,\quad M_1 = gp\mu(p)^{-1}\cdot D'_W.$$

By the argument above, $(M, M_1, b\circ\sigma)$ is a W'-window. By Zink's theory, there is a corresponding p-divisible group X over k' with a quasi-isogeny $\rho : X_0\times_k k' \dashrightarrow X$, and (X,ρ) gives a k'-point of the Rapoport–Zink space $\mathrm{RZ}(X_0)$.

Since the underlying reduced scheme $\mathrm{RZ}(X_0)^{\mathrm{red}}$ is locally of finite type, we can find a smooth domain R over k as above with $k' = \mathrm{Frac}(R)$ and a p-divisible group with quasi-isogeny (X_R,ρ_R) over R that extends (X,ρ). By replacing R by a localization we can further assume that R has a lift $\widetilde{R}$ such that $\sigma(\widetilde{R})\subset\widetilde{R}$ and that $\mathbb{D}(X_R)(\widetilde{R})$ is $\widetilde{R}$-free. We have

$$\mathbb{D}(X_R)(\widetilde{R})\otimes_{\widetilde{R}} W'\cong \mathbb{D}(X)(W')\cong M = g\cdot(M_0\otimes_W W').$$

We will now produce a corresponding R-valued point $(X_R,\rho_R,(t_\alpha))$ of $\mathrm{RZ}_G^{\mathrm{nilp}}$. For this, we will construct a morphism of crystals $t_\alpha : \mathbf{1}\to\mathbb{D}(X_R)^{\otimes}$ such that $t_\alpha:\mathbf{1}\to\mathbb{D}(X_R)^{\otimes}[1/p]$ are Frobenius invariant and then check that t_α satisfy (i), (ii), and (iii) of Definition 2.3.3.

By [BM90, Proposition 1.3.3] or [deJ95, Corollary 2.2.3], to give t_α as above, it suffices to give $t_\alpha(\widetilde{R})(1)\in\mathbb{D}(X_R)(\widetilde{R})^{\otimes}$ which are horizontal for the connection and are Frobenius invariant in $\mathbb{D}(X_R)(\widetilde{R})^{\otimes}[1/p]$. Consider the images t_α of the 'constant' tensors $s_\alpha\otimes 1$ under the isomorphism of isocrystals

$$\mathbb{D}(\rho)^{-1} : \mathbb{D}(X_{0,R})^{\otimes}[1/p]\xrightarrow{\sim}\mathbb{D}(X_R)^{\otimes}[1/p]$$

induced by the quasi-isogeny ρ. Since ρ is defined over R, we obtain $t_\alpha(\widetilde{R})\in\mathbb{D}(X_R)(\widetilde{R})^{\otimes}[1/p]$. However, since g is in $G(K')$, we actually have $t_\alpha(\widetilde{R}) = s_\alpha\otimes 1$ in

$$M^{\otimes}[1/p] = \mathbb{D}(X_R)(\widetilde{R})^{\otimes}\otimes_{\widetilde{R}} W'[1/p],$$

and these lie in

$$M^{\otimes} = \mathbb{D}(X_R)(\widetilde{R})^{\otimes}\otimes_{\widetilde{R}} W'.$$

Since $\mathbb{D}(X_R)(\widetilde{R})$ is $\widetilde{R}$-free and $\widetilde{R}[1/p]\cap W' = \widetilde{R}$, we can see that our tensors $t_\alpha(\widetilde{R}) = s_\alpha\otimes 1$ lie in $\mathbb{D}(X_R)(\widetilde{R})^{\otimes}$. They are horizontal and Frobenius invariant since this is true over W', and $\widetilde{R}\subset W'$. Moreover, conditions (i), (ii), and (ii) can now be seen to be satisfied after possibly further localizing R. We have now produced an R-valued point in $\mathrm{RZ}_G^{\mathrm{nilp}}$.

These two constructions are inverses of each other. This shows that $\varinjlim_R \mathrm{RZ}_G^{\mathrm{nilp}}(R) = \mathrm{RZ}_G^{\mathrm{nilp}}(k')$ and the image of π is $X_{G,b,\mu^{\sigma},\sigma}(k')$. $\square$

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Remark 2.4.11. Observe that $\mathrm{RZ}_G^{\mathrm{fsm}}(R)=\mathrm{RZ}_G^{\mathrm{nilp}}(R)$, and so Theorem 2.4.10 also gives

$$\varinjlim_R \mathrm{RZ}_G^{\mathrm{fsm}}(R)\xrightarrow{\sim} X_{G,b,\mu^\sigma,\sigma}(k').$$

This will be important for the application to Rapoport–Zink formal schemes. As we will see, these are defined via the functor $\mathrm{RZ}_G^{\mathrm{fsm}}$ which can be evaluated at R as above but not at k'.

We can directly obtain a bijection $\mathrm{RZ}_G^{\mathrm{nilp}}(k')\xrightarrow{\sim} X_{G,b,\mu^\sigma,\sigma}(k')$, without assuming that k'/k is finitely generated, by a simpler version of the above argument.

3. Shimura varieties and representability

3.1 Integral models of Shimura varieties of Hodge type

Here, we recall results of [Kis10] about integral models of Shimura varieties of Hodge type. We actually follow the set-up of [Kis13, (1.3)], to which the reader is referred for more details.

3.1.1 Let $(G,\mathcal{H})$ be a Hodge type Shimura datum in the sense of [Del79], so that G is a connected reductive group over $\mathbb{Q}$ and $\mathcal{H}=\{h\}$ is the $G(\mathbb{R})$-conjugacy class of a Deligne cocharacter $h:\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m\to G_{\mathbb{R}}$. Define $\mu_h:\mathbb{G}_{m\mathbb{C}}\to G_{\mathbb{C}}$, as usual, by $\mu_h(z)=h_{\mathbb{C}}(z,1)$. The reflex field $E\subset\bar{\mathbb{Q}}\subset\mathbb{C}$ is the field of definition of the conjugacy class $\{\mu_h\}$.

The condition of Hodge type means that there is an algebraic group embedding

$$\iota: G\hookrightarrow \mathrm{GSp}_{2g} \tag{3.1.1.1}$$

over $\mathbb{Q}$ inducing a morphism of Shimura data $(G,\mathcal{H})\to(\mathrm{GSp}_{2g},\mathcal{H}_{2g})$. Here $\mathcal{H}_{2g}$ is the union of the usual Siegel upper and lower half-spaces of genus g. The composition $\iota\circ\mu_h$ is conjugate to the standard minuscule cocharacter $\mu_{\mathrm{std}}:\mathbb{G}_m\to\mathrm{GSp}_{2g}$ given by $\mu_{\mathrm{std}}(a)=\mathrm{diag}(a^{(g)},1^{(g)})$. As $G(\mathbb{R})$ contains the image of the weight homomorphism w_h, we see that G has to contain the torus of scalars (diagonal matrices) of GSp_{2g}.

We assume that G extends to a connected reductive group over $\mathbb{Z}_{(p)}$, which we again denote by G. As in [Kis10, Lemma (2.3.1)], this implies that there is a rank $2g$ symplectic space (C,ψ) over $\mathbb{Z}_{(p)}$ and a closed immersion $\iota:G\hookrightarrow\mathrm{GL}(C)$ of reductive groups over $\mathbb{Z}_{(p)}$ whose generic fiber factors through the subgroup $\mathrm{GSp}(C_{\mathbb{Q}},\psi)\subset\mathrm{GL}(C_{\mathbb{Q}})$ and induces (3.1.1.1) after fixing an identification $\mathrm{GSp}(C_{\mathbb{Q}},\psi)=\mathrm{GSp}_{2g}$. By Zarhin's trick, after replacing C by $\mathrm{Hom}_{\mathbb{Z}_{(p)}}(C,C)^{\oplus 4}$ and enlarging g, we may assume that C is self-dual with respect to ψ. We then have a closed immersion of reductive group schemes

$$\iota: G\hookrightarrow \mathrm{GSp}(C,\psi) \tag{3.1.1.2}$$

over $\mathbb{Z}_{(p)}$ with generic fiber (3.1.1.1).

As in (1.1.1.3), let D be the G-representation contragredient to C. By [Kis10, (1.3.2)] there is finite list (s_α) of tensors $s_\alpha\in C^\otimes=D^\otimes$ that cut out $G\subset\mathrm{GL}(C)$, in the sense that

$$G(R)=\{g\in\mathrm{GL}(C_R): g\cdot(s_\alpha\otimes 1)=(s_\alpha\otimes 1),\forall\alpha\},$$

for all $\mathbb{Z}_{(p)}$-algebras R. In particular, $G(W)=G(K)\cap\mathrm{GL}(C_W)$. In what follows, we take the set of tensors (s_α) to always include the tensor corresponding to the perfect symplectic form ψ; see the proof of Theorem 3.2.1 below.

A choice of field embedding $\bar{\mathbb{Q}}\hookrightarrow\bar{K}$ determines a place $v\mid p$ of E. The completion E_v is the field of definition of $\{\mu_h\}$, now regarded as a $G(\bar{K})$-conjugacy class of cocharacters $\mathbb{G}_{m\bar{K}}\to G_{\bar{K}}$.

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Let

$$\mathrm{Sh}_{U_p}(G,\mathcal{H}) = \varprojlim_{U^p} \mathrm{Sh}_{U^pU_p}(G,\mathcal{H})$$

be the canonical model over $E \subset K$ of the corresponding Shimura variety for the hyperspecial subgroup

$$U_p = G(\mathbb{Z}_p) \subset G(\mathbb{Q}_p).$$

Here the limit is over compact open subgroups U^p of $G(\mathbb{A}_f^p)$.

3.1.2 For a $\mathbb{Z}_{(p)}$-scheme S and an abelian scheme $A \to S$ we set

$$\mathrm{Ta}^p(A) = \varprojlim_{p\nmid n} A[n],$$

viewed as an étale local system on S, and write

$$\mathrm{Ta}^p(A)_{\mathbb{Q}} = \mathrm{Ta}^p(A) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Consider the category obtained from the category of abelian schemes over S by tensoring the Hom groups by $\mathbb{Z}_{(p)}$. An object in this category will be called an *abelian scheme over S up to prime-to-p-isogeny*. An isomorphism in this category will be called a *p'-quasi-isogeny*. Note that $\mathrm{Ta}^p(A)_{\mathbb{Q}}$ is functorial for p'-quasi-isogenies.

If A is an abelian scheme up to prime-to-p-isogeny, we write $A^\vee$ for the dual abelian scheme up to prime-to-p-isogeny. A *weak polarization* of A is an equivalence class of p'-quasi-isogenies $\lambda : A \xrightarrow{\sim} A^\vee$ such that some $\mathbb{Z}_{(p)}^\times$-multiple of λ is a polarization. Here two such λ are equivalent if they differ by multiplication by an element of $\mathbb{Z}_{(p)}^\times$.

Let $U'^p \subset \mathrm{GSp}_{2g}(\mathbb{A}_f^p)$ be any compact open subgroup, let

$$U'_p = \mathrm{GSp}(C,\psi)(\mathbb{Z}_p) \subset \mathrm{GSp}_{2g}(\mathbb{Q}_p)$$

be the hyperspecial subgroup determined by the self-dual symplectic space (C,ψ) over $\mathbb{Z}_{(p)}$, and set

$$U' = U'^p U'_p \subset \mathrm{GSp}_{2g}(\mathbb{A}_f).$$

Assume (A,λ) is an abelian scheme up to prime-to-p-isogeny with a weak polarization. A *U'^p-level structure* on (A,λ) is a global section

$$\epsilon^p_{U'} \in \Gamma(S, \underline{\mathrm{Isom}}(C\otimes \mathbb{A}_f^p, \mathrm{Ta}^p(A)_{\mathbb{Q}})/U'^p).$$

Here $\underline{\mathrm{Isom}}(C\otimes \mathbb{A}_f^p, \mathrm{Ta}^p(A)_{\mathbb{Q}})/U'^p$ is the étale sheaf on S of U'^p-orbits of isomorphisms

$$C \otimes \mathbb{A}_f^p \xrightarrow{\sim} \mathrm{Ta}^p(A)_{\mathbb{Q}}$$

identifying the symplectic pairings induced by ψ and λ, up to a $(\mathbb{A}_f^p)^\times$-scalar.

For U'^p sufficiently small, the functor that assigns to S the set of isomorphism classes of triples $(A,\lambda,\epsilon^p_{U'})$ as above, is representable by a smooth $\mathbb{Z}_{(p)}$-scheme $\mathscr{A}_{g,U'}$, whose generic fiber

$$\mathscr{A}_{g,U'} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Q} \xrightarrow{\sim} \mathrm{Sh}_{U'}(\mathrm{GSp}_{2g},\mathcal{H}_{2g}) \tag{3.1.2.1}$$

is identified with the Siegel Shimura variety; see [Kot92]. We will always assume that U'^p is sufficiently small in what follows.

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3.1.3 Let us denote by $\mathscr{S}_{U_p}(G,\mathcal{H})$ the canonical smooth model over $\mathcal{O}_{E,(v)}$ of the Shimura variety

$$\mathrm{Sh}_{U_p}(G,\mathcal{H}) = \varprojlim_{U^p} \mathrm{Sh}_{U^pU_p}(G,\mathcal{H})$$

constructed in [Kis10] for the hyperspecial subgroup $U_p = G(\mathbb{Z}_p)$. Thus

$$\mathscr{S}_{U_p}(G,\mathcal{H}) = \varprojlim_{U^p} \mathscr{S}_{U^pU_p}(G,\mathcal{H}),$$

where for sufficiently small subgroups $U_1^p \subset U_2^p \subset G(\mathbb{A}_f^p)$ the transition morphism

$$\mathscr{S}_{U_1^pU_p}(G,\mathcal{H}) \to \mathscr{S}_{U_2^pU_p}(G,\mathcal{H})$$

is finite étale. In fact, for $U = U_pU^p$ with U^p sufficiently small, the integral model $\mathscr{S}_U(G,\mathcal{H})$ is smooth over $\mathcal{O}_{E,(v)}$, and is constructed as the normalization of the Zariski closure $\mathscr{S}_U(G,\mathcal{H})^-$ of the image of the morphism

$$\mathrm{Sh}_U(G,\mathcal{H}) \to \mathrm{Sh}_{U'}(\mathrm{GSp}_{2g},\mathcal{H}_{2g}) \otimes_{\mathbb{Q}} E \to \mathscr{A}_{g,U'} \otimes_{\mathbb{Z}_{(p)}} \mathcal{O}_{E,(v)}$$

induced by (3.1.1.2) and (3.1.2.1) for a suitable choice of level structure U'^p. In particular, there are finite morphisms

$$\iota : \mathscr{S}_U(G,\mathcal{H}) \xrightarrow{\text{normalization}} \mathscr{S}_U(G,\mathcal{H})^- \to \mathscr{A}_g \otimes_{\mathbb{Z}_{(p)}} \mathcal{O}_{E,(v)},$$

where now we suppress the level structure U' on the Siegel space from the notation. This should not lead to confusion, as the particular choice of level U' will play little part in our arguments.

3.1.4 Now let us pick a point $x_0 \in \mathscr{S}_U(G,\mathcal{H})(k)$. Consider the contravariant Dieudonné module $\mathbb{D}(X_0)(W)$ of the p-divisible group $X_0 = A_{x_0}[p^\infty]$ of the abelian scheme A_{x_0} over k determined by the point $\iota(x_0) \in \mathscr{A}_g(k)$. By [Kis10, Corollary (1.4.3)], there are crystalline tensors

$$t_{\alpha,0} = t_{\alpha,0}^{\mathrm{cr}} \in \mathbb{D}(X_0)(W)^{\otimes}$$

that are fixed by the action of Frobenius on $\mathbb{D}(X_0)(W)[1/p]$, and satisfy

$$t_{\alpha,0}(k) \in \mathrm{Fil}^0(\mathbb{D}(X_0)^{\otimes}(k)).$$

By [Kis10], there is an isomorphism of W-modules

$$\beta_0 : D \otimes_{\mathbb{Z}_p} W \xrightarrow{\sim} \mathbb{D}(X_0)(W)$$

identifying $s_\alpha \otimes 1$ with $t_{\alpha,0}$. After choosing such an isomorphism, the Frobenius on $\mathbb{D}(X_0)(W)$ has the form $F = b_{x_0} \circ \sigma$ for some $b_{x_0} \in G(K)$.

Consider the Hodge filtration

$$\mathrm{Fil}^1(X_0) \subset \mathbb{D}(X_0)(k) \cong \mathrm{H}^1_{\mathrm{dR}}(A_{x_0}/k).$$

By [Kis10, Corollary (1.4.3) (4)] this filtration is given by a G_k-valued cocharacter, and we pick any lift to a cocharacter

$$\mu_{x_0} : \mathbb{G}_{mW} \to G_W.$$

By the argument in the proof of [Kis13, Lemma (1.1.9)] the $G(W)$-conjugacy class of μ_{x_0} is independent of the choice of β_0. In fact, any such cocharacter satisfies

$$b_{x_0} \in G(W)\mu_{x_0}^{\sigma}(p)G(W),$$

and lies in the $G(\bar{K})$-conjugacy class defined by μ_h^{-1} (see [Kis13, (1.1.9)]), whose local reflex field is $E_v \subset K$.

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In this way, each point $x_0 \in \mathscr{S}_{U_p}(G,\mathcal{H})(k)$ produces a local Shimura–Hodge datum, which, for ease of notation, we abbreviate to

$$(G,b,\mu,C) = (G_{\mathbb{Z}_p}, b_{x_0}, \mu_{x_0}, C_{\mathbb{Z}_p}). \tag{3.1.4.1}$$

Note that the p-divisible group of Lemma 2.2.5 is $X_0 = A_{x_0}[p^\infty]$.

For the rest of § 3 we fix the local Shimura–Hodge datum (3.1.4.1) given by a point $x_0 \in \mathscr{S}_{U_p}(G,\mathcal{H})(k)$ by the above procedure. We also fix the tensors $s_\alpha \in C^\otimes = D^\otimes$ cutting out the subgroup $G \subset \mathrm{GL}(C)$. It is essential in what follows that both the local Shimura–Hodge datum (G,b,μ,C) and the tensors (s_α) arise from a global point $x_0 \in \mathscr{S}_{U_p}(G,\mathcal{H})(k)$, and from tensors defined on C, not just on $C_{\mathbb{Z}_p}$.

3.1.5 As the Shimura datum $(G,\mathcal{H})$ will remain fixed, in what follows we will often abbreviate $\mathscr{S}_U$ or just $\mathscr{S}$ instead of $\mathscr{S}_U(G,\mathcal{H})$. We will also often write $\mathscr{S}_{U_p}$ instead of $\mathscr{S}_{U_p}(G,\mathcal{H})$.

Let $f : A \to \mathscr{S}$ be the universal abelian scheme over $\mathscr{S}$, and denote by

$$\widehat{f} : \widehat{A} \to \widehat{\mathscr{S}}$$

the corresponding morphism of smooth formal schemes over $\mathbb{Z}_p$ obtained by p-adic completion.

We have the crystals $\mathbb{D}(X^{\mathrm{univ}})$ and $\mathbb{D}(X^{\mathrm{univ}})^\otimes$, and Frobenius isocrystals $\mathbb{D}(X^{\mathrm{univ}})[1/p]$ and $\mathbb{D}(X^{\mathrm{univ}})^\otimes[1/p]$, over $\widehat{\mathscr{S}}$, where $X^{\mathrm{univ}} = \widehat{A}[p^\infty]$ is the p-divisible group of the universal abelian scheme. By [BBM82], we have

$$\mathbb{D}(X^{\mathrm{univ}}) = \mathrm{R}^1\widehat{f}_{\mathrm{cris},*}\mathcal{O}_{\widehat{A}/\mathbb{Z}_p},$$

and there is a natural isomorphism

$$\mathrm{H}^1_{\mathrm{dR}}(\widehat{A}/\widehat{\mathscr{S}}) \xrightarrow{\sim} (\mathrm{R}^1\widehat{f}_{\mathrm{cris},*}\mathcal{O}_{\widehat{A}/\mathbb{Z}_p})_{\widehat{\mathscr{S}}}$$

of coherent sheaves $\mathcal{O}_{\widehat{\mathscr{S}}}$-modules, where the left-hand side is the first relative de Rham cohomology of $\widehat{A}$, and the right-hand side is the pull-back of the crystal to the Zariski site.

By [Kis10, Corollary (2.3.9)], there are de Rham tensors

$$t^{\mathrm{univ}}_{\alpha,\mathrm{dR}} : \mathbf{1} \to \mathrm{H}^1_{\mathrm{dR}}(\widehat{A}/\widehat{\mathscr{S}})^\otimes,$$

which are horizontal for the Gauss–Manin connection. Since $\mathscr{S}$ is smooth over $\mathbb{Z}_p$, we obtain $\mathcal{O}_{\widehat{\mathscr{S}}/\mathbb{Z}_p}$-morphisms

$$t^{\mathrm{univ}}_{\alpha} : \mathbf{1} \to (\mathrm{R}^1\widehat{f}_{\mathrm{cris},*}\mathcal{O}_{\widehat{A}/\mathbb{Z}_p})^\otimes = \mathbb{D}(X^{\mathrm{univ}})^\otimes$$

of crystals over $\widehat{\mathscr{S}}$. By the construction in [Kis10], the tensors t^{univ}_α restrict to $t_{\alpha,0}$ by pulling back via $x_0 : \mathrm{Spec}(k) \to \widehat{\mathscr{S}}$, and

$$t^{\mathrm{univ}}_\alpha[1/p] : \mathbf{1}[1/p] \to \mathbb{D}(X^{\mathrm{univ}})^\otimes[1/p]$$

are Frobenius equivariant.

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3.1.6 Recall from § 2.2.1 the homogeneous space

$$M_{G,\mu} \otimes_{\mathcal{O}_{E,v}} W \xrightarrow{\sim} G_W/P_\mu$$

over W, where $P_\mu \subset G_W$ is the parabolic subgroup defined by $\mu : \mathbb{G}_{mW} \to G_W$. Denote by $U^\mu = U^\mu_G$ the unipotent radical of the opposite to P_μ parabolic subgroup of G_W and by $U^{\mu,\wedge}_G$ the formal completion of U^μ_G at its identity section over W. Then $U^{\mu,\wedge}_G$ can be identified with the formal completion of $M_{G,\mu} \otimes_{\mathcal{O}_E} W$ at the section given by $1 \cdot P_\mu/P_\mu$.

Let $R = \widehat{\mathcal{O}}_{\mathscr{S}_W,x_0}$ be the completion of the local ring of $\mathscr{S}_W$ at x_0. By [Kis10, Proposition (2.3.5)] and its proof, we may identify $\mathrm{Spf}(R)$ with $U^{\mu_{x_0},\wedge}_G$ and identify $R = W[\![x_1,\ldots,x_d]\!]$ in such a way that the W-point given by $x_1 = \cdots = x_d = 0$ has Hodge filtration given by μ_{x_0}. In fact, by a result of Faltings [Fal99], we see, as in [Kis10, (1.5)] (see also [Moo98, § 4]), that this identification can be chosen in such a way that the Dieudonné crystal of the universal p-divisible group X^{univ} with its tensors t_α over R is

$$\mathbb{D}(X^{\mathrm{univ}})(R) = \mathbb{D}(X_0)(W) \otimes_W R = D \otimes_{\mathbb{Z}_p} R,$$

(as R-modules) with filtration

$$\mathrm{Fil}^1(X^{\mathrm{univ}}) = \mathrm{Fil}^1(X_0) \otimes_W R,$$

and $t_\alpha = t_{\alpha,0} \otimes 1 = s_\alpha \otimes 1$, while the semi-linear Frobenius

$$F : \mathbb{D}(X^{\mathrm{univ}})(R) \to \mathbb{D}(X^{\mathrm{univ}})(R)$$

is given by $F = u \cdot (b_{x_0} \otimes \phi_R)$. Here u is the universal R-point of the completion $U^{\mu_{x_0},\wedge}_G$ of the unipotent subgroup $U^{\mu_{x_0}}_G$ at the identity section and ϕ_R is the lift of Frobenius such that $\phi_R(x_i) = x_i^p$.

3.2 A global construction of Rapoport–Zink formal schemes

A more general version of the following representability result appears in work of Kim [Kim13]. Kim does not assume that the local Shimura datum $(G, [b], \{\mu\})$ of Hodge type is obtained from a point on a Shimura variety. Although we were very much inspired by Kim's work, our arguments are quite different and independent of [Kim13].

Theorem 3.2.1. *Let the unramified local Shimura–Hodge datum (G, b, μ, C) and the finite set of tensors $(s_\alpha) \in C^\otimes$ that cut out $G \subset \mathrm{GL}(C)$ be as in (3.1.4.1) above. Suppose $X_0 = X_0(G, b, \mu, C)$ is the p-divisible group over k defined in Lemma 2.2.5.*

(i) *There exists a formal scheme $\mathrm{RZ}_G = \mathrm{RZ}_{G,b,\mu,C,(s_\alpha)}$ over $\mathrm{Spf}(W)$, formally smooth and formally locally of finite type, that represents the functor $\mathrm{RZ}^{\mathrm{fsm}}_G$ on $\mathrm{ANilp}^{\mathrm{fsm}}_W$ defined in § 2.3.6.*

(ii) *The action of $J_b(\mathbb{Q}_p)$ on $\mathrm{RZ}^{\mathrm{fsm}}_G$ given by (2.3.7.1) is induced by a left $J_b(\mathbb{Q}_p)$-action on the formal scheme RZ_G.*

(iii) *The formal scheme RZ_G is a closed formal subscheme of $\mathrm{RZ}(X_0)$.*

Remark 3.2.2. The formal scheme RZ_G is characterized as the unique formally smooth and locally formally of finite type over W that represents the functor $\mathrm{RZ}^{\mathrm{fsm}}_G$, and this functor is determined by $(X_0, (t_{\alpha,0}))$. (The definition of the functor also involves the conjugacy class $\{\mu\}$, but this is determined by X_0 and its tensors.) In particular, it follows that RZ_G agrees, up to isomorphism, with the Rapoport–Zink formal scheme of Kim [Kim13].

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Corollary 3.2.3. *Under the above assumptions, if k'/k is a finitely generated field extension, the construction in the proof of Theorem 2.4.10 provides a bijection*

$$\pi : \mathrm{RZ}_G(k') \xrightarrow{\sim} X_{G,b,\mu^\sigma,\sigma}(k').$$

Proof. By Theorem 3.2.1(i), the underlying reduced scheme $\mathrm{RZ}_G^{\mathrm{red}}$ is locally of finite type over k, and so

$$\mathrm{RZ}_G(k') = \varinjlim_R \mathrm{RZ}_G(R) = \varinjlim_R \mathrm{RZ}_G^{\mathrm{nilp}}(R),$$

where the limit is as in Proposition 2.4.8. The corollary then follows from Theorem 2.4.10 after taking into account Remark 2.4.11. □

3.2.4 We will now turn to the proof of Theorem 3.2.1, but, before we do that, we sketch the argument. We first construct the formal scheme RZ_G over W using the integral model of a Shimura variety and then we show, using also certain results from [Kis13], that it represents the functor $\mathrm{RZ}_G^{\mathrm{fsm}}$.

Roughly, the construction of RZ_G is done in two steps. First, we form a fiber product $\mathrm{RZ}_G^\diamond$ of the classical Rapoport–Zink space associated to the p-divisible group X_0 with the p-adic completion of the integral model of the Shimura variety; the fiber product is over a Siegel moduli space. Over this fiber product, we have crystalline tensors obtained by pulling back the universal crystalline tensors t_α^{univ} over the integral model constructed by Kisin (see § 3.1.5); we also have corresponding crystalline tensors $t_{\alpha,0}$ obtained by pulling back the crystalline tensors on the base point X_0 via the universal quasi-isogeny. The second step is to show that the locus where these two tensors agree, i.e. with $t_\alpha^{\mathrm{univ}} = t_{\alpha,0}$, is given by a closed and open formal subscheme of the fiber product. This formal subscheme is the desired formal scheme RZ_G.

Proof of Theorem 3.2.1. It follows directly from the definition that the usual Rapoport–Zink formal scheme $\mathrm{RZ}(X_0)$ of § 2.3.1 represents

$$\mathrm{RZ}^{\mathrm{fsm}}_{\mathrm{GL}(C)} := \mathrm{RZ}^{\mathrm{fsm}}_{\mathrm{GL}(C),b,\mu,C,\emptyset}.$$

Here we take the set of tensors to be empty. Also, we can show that the symplectic Rapoport–Zink formal scheme $\mathrm{RZ}(X_0, \lambda_0)$ represents

$$\mathrm{RZ}^{\mathrm{fsm}}_{\mathrm{GSp}(C)} := \mathrm{RZ}^{\mathrm{fsm}}_{\mathrm{GSp}(C,\psi),b,\mu,C,(s_{\mathrm{sympl}})}.$$

Here, $\lambda_0 : X_0 \xrightarrow{\sim} X_0^\vee$ is the principal polarization deduced from the symplectic form ψ on C, and the single tensor s_{sympl} is defined as follows. Denote by $\eta : \mathrm{GSp}(C, \psi) \to \mathbb{G}_m$ the similitude character. The symplectic pairing $\psi : C \otimes C \to \mathbb{Z}_p(\eta)$ defining $\mathrm{GSp}(C, \psi)$ induces an isomorphism $C \xrightarrow{\sim} D(\eta)$, which allows us to view the dual

$$\psi^* : \mathbb{Z}_p(\eta^{-1}) \to (C \otimes C)^* \cong D \otimes D$$

as a map $\psi^* : \mathbb{Z}_p(\eta) \to C \otimes C$. Now define

$$s_{\mathrm{sympl}} \in \mathrm{End}(C \otimes C) \cong C^{\otimes 2} \otimes D^{\otimes 2} \subset C^{\otimes}$$

as the composition

$$C \otimes C \xrightarrow{\psi} \mathbb{Z}_p(\eta) \xrightarrow{\psi^*} C \otimes C.$$

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The group $\mathrm{GSp}(C,\psi)\subset\mathrm{GL}(C)$ is the stabilizer of s_{sympl}. One can easily show that $\mathrm{RZ}(X_0,\lambda_0)$ represents $\mathrm{RZ}^{\mathrm{fsm}}_{\mathrm{GSp}(C)}$ by using duality and the full faithfulness of the Dieudonné crystal functor (see [BM90, 4.1 and 4.3] and [deJ95]) for p-divisible groups over bases in $\mathrm{ANilp}^{\mathrm{fsm}}_W$.

In accordance with the above general notation scheme, we can now denote

$$\mathrm{RZ}_{\mathrm{GL}(C)}=\mathrm{RZ}(X_0),\quad \mathrm{RZ}_{\mathrm{GSp}(C)}=\mathrm{RZ}(X_0,\lambda_0).$$

If C is clear from the context we will simply write $\mathrm{RZ}_{\mathrm{GL}}$, $\mathrm{RZ}_{\mathrm{GSp}}$, instead.

As in [RZ96, Theorem 6.21], there is a canonical morphism

$$\Theta:\mathrm{RZ}_{\mathrm{GSp}(C)}\to\widehat{\mathscr{A}_{g,W}}:=\widehat{\mathscr{A}_g}\widehat{\otimes}_{\mathbb{Z}_p}W.$$

(Here again, $\widehat{\mathscr{A}_g}$ denotes the completion of $\mathscr{A}_g\to\mathrm{Spec}(\mathbb{Z}_{(p)})$ along its special fiber.) This morphism sends a triple (X,λ,ρ) to the point corresponding to the unique principally polarized abelian scheme A whose p-divisible group is X, and for which there exists a quasi-isogeny $A_{x_0}\dashrightarrow A$ respecting polarizations up to $\mathbb{Q}^\times$-scaling, inducing an isomorphism on ℓ-divisible groups for all $\ell\neq p$, and inducing the quasi-isogeny

$$A_{x_0}[p^\infty]=X_0\overset{\rho}{\dashrightarrow}X=A[p^\infty].$$

We now let $\mathrm{RZ}^\diamond_G$ be the formal scheme over $\mathrm{Spf}(W)$ defined by the fiber product

$$\begin{tikzcd}
\mathrm{RZ}^\diamond_G \arrow[r] \arrow[d, "\Theta^\diamond_G"'] & \mathrm{RZ}_{\mathrm{GSp}(C)} \arrow[d, "\Theta"] \\
\widehat{\mathscr{S}_W} \arrow[r, "\iota"] & \widehat{\mathscr{A}_{g,W}}.
\end{tikzcd}$$

Here, $\mathscr{S}=\mathscr{S}_{U^pU_p}$, in which we fix a choice of a sufficiently small prime-to-p level U^p. We will see eventually that all such choices produce the same Rapoport–Zink formal scheme.

The formal scheme $\mathrm{RZ}^\diamond_G$ represents the functor that assigns to each W-scheme S in Nilp_W the set of quadruples $(X,\lambda,\rho,f:S\to\mathscr{S}_W)$, where (X,λ,ρ) is an S-valued point of $\mathrm{RZ}_{\mathrm{GSp}}$ such that the composition

$$S\xrightarrow{f}\mathscr{S}_W\xrightarrow{\mathrm{norm}}\mathscr{S}^-_W\hookrightarrow\mathscr{A}_{g,W}$$

gives the corresponding point $\Theta((X,\lambda,\rho))\in\mathscr{A}_{g,W}(S)$.

In this situation, the Dieudonné crystal $\mathbb{D}(X)$ of the p-divisible group over S supports tensors $t_\alpha:\mathbf{1}\to\mathbb{D}(X)^\otimes$ which are obtained as $t_\alpha=f^*(t_\alpha^{\mathrm{univ}})$, i.e. by pulling back the universal crystalline tensors over $\widehat{\mathscr{S}}$.

Proposition 3.2.5. *The morphism $\mathrm{RZ}^\diamond_G\to\mathrm{RZ}_{\mathrm{GSp}(C)}$ is finite and the formal scheme $\mathrm{RZ}^\diamond_G$ is formally smooth and locally formally of finite type over* $\mathrm{Spf}(W)$.

Proof. Since ι is finite, the same is true for $\mathrm{RZ}^\diamond_G\to\mathrm{RZ}_{\mathrm{GSp}(C)}$, and so $\mathrm{RZ}^\diamond_G$ is locally formally of finite type over W. By [Kis10, Proposition (2.3.5)] and the proof of [Kis10, Theorem (2.3.8)] (see also § 3.1.6), the morphism ι induces a closed immersion between the formal completions of $\mathscr{S}_W$ and $\mathscr{A}_{g,W}$ at each closed point $s\in\mathscr{S}_W$. Moreover, the scheme $\mathscr{S}_W$ is smooth. By the Serre–Tate theorem, the formal completion of $\mathrm{RZ}_{\mathrm{GSp}(C)}$ at any closed point can be identified with the formal completion of $\mathscr{A}_{g,W}$ at the corresponding point and formal smoothness follows. In fact, for each closed point s of $\mathrm{RZ}^\diamond_G$ the morphism $\Theta^\diamond_G$ gives an isomorphism

$$\widehat{\mathrm{RZ}}^\diamond_{G,s}\xrightarrow{\sim}\widehat{\mathscr{S}}_{W,s}$$

between formal completions. $\square$

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Suppose that S is a scheme in Nilp_W and $a \in \mathrm{RZ}_G^{\diamond}(S)$. We have crystals $\mathbb{D}(X)$ and $\mathbb{D}(X)^{\otimes}$, and isocrystals $\mathbb{D}(X)[1/p]$ and $\mathbb{D}(X)^{\otimes}[1/p]$, over S. By pulling back the universal crystalline tensors $t_{\alpha}^{\mathrm{univ}}$ via $\Theta^{\diamond} \circ a : S \to \mathscr{S}$, we obtain $t_{\alpha} : \mathbf{1} \to \mathbb{D}(X)^{\otimes}$ over S.

Definition 3.2.6. For each S in Nilp_W, denote by $\mathrm{RZ}_G(S) \subset \mathrm{RZ}_G^{\diamond}(S)$ the subset consisting of all points $a \in \mathrm{RZ}_G^{\diamond}(S)$ with the following property: for every field extension k'/k and every point $y \in S(k')$, the isomorphism

$$\mathbb{D}(\rho) : \mathbb{D}(X_y)^{\otimes}(W')[1/p] \xrightarrow{\sim} \mathbb{D}(X_0)^{\otimes}[1/p] \otimes_{W[1/p]} W'[1/p]$$

identifies the tensors $t_{\alpha}(W')$ with $t_{\alpha,0}(W) \otimes 1$. In other words,

$$\mathbb{D}(\rho)(t_{\alpha}(W')) = t_{\alpha,0}(W) \otimes 1. \tag{3.2.6.1}$$

(Recall that W' is the Cohen ring of k'.)

Proposition 3.2.7. (i) *The subfunctor RZ_G is represented by a closed and open formal subscheme of $\mathrm{RZ}_G^{\diamond}$.*

(ii) *The formal scheme RZ_G is formally smooth and locally formally of finite type over $\mathrm{Spf}(W)$.*

Proof. The proof relies on the following lemma.

Lemma 3.2.8. *Assume that S in Nilp_W is connected, $a \in \mathrm{RZ}_G^{\diamond}(S)$, and that there is a field k' and a point $y \in S(k')$ such that the condition (3.2.6.1) is satisfied at y. Then $\mathbb{D}(\rho)(t_{\alpha}) = t_{\alpha,0} \otimes 1$, where*

$$\mathbb{D}(\rho) : \mathbb{D}(X_{\bar{S}})^{\otimes}[1/p] \xrightarrow{\sim} \mathbb{D}(X_0 \times_k \bar{S})^{\otimes}[1/p]$$

is the morphism of Frobenius isocrystals over $\bar{S}$ induced by the quasi-isogeny ρ. In particular, condition (3.2.6.1) is satisfied at all field-valued points of S, and so $a \in \mathrm{RZ}_G(S)$.

Proof. Note that, as $\mathrm{RZ}_G^{\diamond}$ is locally formally of finite type, for any $a \in \mathrm{RZ}_G^{\diamond}(S)$, there exist a locally finite type scheme S' in Nilp_W, a morphism $\omega : S \to S'$, and $b \in \mathrm{RZ}_G^{\diamond}(S')$ such that $a = b \circ \omega$. Since S is connected, we can assume that S' is connected and so we reduce to showing the statement above for S connected and locally of finite type.

We will first show that (3.2.6.1) holds for all field-valued points of S. All such points factor through the underlying reduced scheme S_{red}; hence, we can further assume that S is reduced and is actually affine of finite type over k. Now the argument of [Mad16, Lemma 5.10] implies that condition (3.2.6.1) is satisfied for all field-valued points of S, and in particular for all closed points $s \in S$, taking $k' = k(s)$ to be the residue field. This already gives that a is in $\mathrm{RZ}_G(S)$.

To show the rest, observe that, by Berthelot's construction [Ber96, Theorem (2.4.2)], the Frobenius crystal $\mathbb{D}(X)$ over S determines a convergent Frobenius isocrystal $M = \mathbb{D}(X)[1/p]^{\mathrm{an}}$ over S/W. Similarly, we have a convergent Frobenius isocrystal M_0 given by base-changing $\mathbb{D}(X_0)[1/p]^{\mathrm{an}}$ to a convergent Frobenius isocrystal over S/W. The quasi-isogeny ρ induces a morphism of convergent Frobenius isocrystals

$$\mathbb{D}(\rho)^{\mathrm{an}} : M^{\otimes} \xrightarrow{\sim} M_0^{\otimes}.$$

By [Ogu84, Theorem 4.1] (see [Ber96, Remark 2.3.4]), we have $\mathbb{D}(\rho)^{\mathrm{an}}(t_{\alpha}) = t_{\alpha,0} \otimes 1$ in $M^{\otimes}$ since, by the above, this is true at a closed point. By [Ber96, Theorem (2.4.2)] the functor from Frobenius crystals up to isogeny over S to convergent Frobenius isocrystals over S/W is faithful, and the result follows. □

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Consider the union of connected components of $\mathrm{RZ}_G^{\diamond}$ which have a field-valued point such that (3.2.6.1) is satisfied. We can now see that this union represents the functor RZ_G. The second statement now follows from the first and the previous proposition. □

The following proposition gives a main part of Theorem 3.2.1.

Proposition 3.2.9. *The formal scheme* RZ_G *constructed above represents the functor*

$$\mathrm{RZ}_G^{\mathrm{fsm}} : \mathrm{ANilp}_W^{\mathrm{fsm}} \to \mathrm{Sets}$$

defined in § 2.3.6.

Proof. Suppose that R is in $\mathrm{ANilp}_W^{\mathrm{fsm}}$. Denote by I an ideal of definition of R with $pR \subset I$. We will establish a functorial bijection

$$\mathrm{RZ}_G(R) \xrightarrow{\sim} \mathrm{RZ}_G^{\mathrm{fsm}}(R).$$

Suppose first that we are given

$$(X, \lambda, \rho, f : \mathrm{Spf}(R) \to \widehat{\mathscr{S}_W}) \in \mathrm{RZ}_G(R).$$

By Remark 2.3.5(c), we have morphisms of crystals $t_\alpha : \mathbf{1} \to \mathbb{D}(X)^{\otimes}$ over $\mathrm{Spec}(R)$ obtained by pulling back the universal crystalline tensors on $\widehat{\mathscr{S}}$. Recall that the tensors (t_α) include the crystalline tensor that corresponds to the polarization λ. By the definition of RZ_G, for any k'-valued point of R, where k' is any field, we have the identity (3.2.6.1). Property (i) of Definition 2.3.3 follows from Lemma 3.2.8. To show that properties (ii) and (iii) are satisfied, we reduce by fppf descent to the case that R is replaced by its completion $\widehat{R}_x$ at an arbitrary closed k-valued point x. Since R is formally smooth, we can assume $R = W/p^m[\![x_1, \ldots, x_n]\!]$ for some m and n. Then the morphism

$$f : \mathrm{Spf}(W/p^m[\![x_1, \ldots, x_n]\!]) \to \widehat{\mathscr{S}_W}$$

factors through the completion $\mathrm{Spf}(\hat{\mathcal{O}}_{\mathscr{S}, f(x)})$; this completion is described in § 3.1.6, from which properties (ii) and (iii) follow for t_α and the Hodge filtration over $\widehat{\mathcal{O}}_{\mathscr{S}, f(x)}$, and therefore also over R.

Conversely, suppose

$$(X, \rho, (t_\alpha)) \in \mathrm{RZ}_G^{\mathrm{fsm}}(R).$$

Since (t_α) include the polarization tensor t_{sympl}, it follows, by using duality and the full faithfulness of the Dieudonné crystal functor over R (see [BM90, 4.1 and 4.3] and also [deJ95]), that these data also produce a principal polarization λ on the p-divisible group X. We thus obtain $\Theta((X, \lambda, \rho))$, a $\mathrm{Spf}(R)$-valued point of $\mathscr{A}_{g,W}$ which, by the standard algebraization theorems, corresponds to

$$y : \mathrm{Spec}(R) \to \mathscr{A}_{g,W}. \tag{3.2.9.1}$$

It is enough to show that y factors through $\mathscr{S}_W$.

This is true when $R = k$, but this is already quite deep. It follows from Theorem 2.4.10 (with $k' = k$), together with [Kis13, Proposition (1.4.4)] and its proof (this uses the main result of [CKV15]).

Let us now deal with more general R. Assume first $R \xrightarrow{\sim} W[\![x_1, \ldots, x_n]\!]$, for some $n \geqslant 0$, with the obvious extension of a notion of an R-valued point of $\mathrm{RZ}_G^{\mathrm{fsm}}$ (take $I = (p, x_1, \ldots, x_n)$ as an ideal of definition in § 2.3.6). As just explained, we know that

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$\mathrm{Spec}(k) \to \mathscr{A}_W$ given $x_1 = \cdots = x_n = p = 0$ gives a point $x \in \mathscr{S}_W(k)$. Use property (ii) of Definition 2.3.3 to choose a trivialization

$$\mathbb{D}(X)(R) \cong D \otimes_{\mathbb{Z}_p} R$$

that matches the tensors $t_\alpha(R)$ with the standard tensors $s_\alpha \otimes 1$. By Faltings' construction of the universal deformation of the polarized p-divisible group (X_x, λ_x) as in § 3.1.6, and the Serre–Tate theorem, we can identify the completed local ring of $\mathscr{A}_{g,W}$ with the completion $U^{\mu_{\mathrm{std}},\wedge}_{\mathrm{GSp(C)}}$ of the opposite unipotent at the identity section. Our conditions now imply that the tensors $t_\alpha = s_\alpha \otimes 1$ are 'Tate tensors' over R, therefore, by [Moo98, 4.8], the corresponding R-valued point of $U^{\mu_{\mathrm{std}},\wedge}_{\mathrm{GSp(C)}}$ factors through

$$U^{\mu_x,\wedge}_G \subset U^{\mu_{\mathrm{std}},\wedge}_{\mathrm{GSp(C)}}.$$

As in (3.1.6), $U^{\mu_x,\wedge}_G$ is identified with the completion of $\mathscr{S}_W$ at x and the result for $R \xrightarrow{\sim} W[\![x_1, \ldots, x_n]\!]$ follows.

We now consider the case of a general R in $\mathrm{ANilp}^{\mathrm{fsm}}_W$. Evaluate t_α on the PD lift $\widetilde{R}$ of § 2.1.3 to obtain $t_\alpha(\widetilde{R})$, and a corresponding $\widetilde{R}$-scheme $T_{\widetilde{R}}$ of trivializations of $(\mathbb{D}(X)(\widetilde{R}), (t_\alpha(\widetilde{R})))$.

Lemma 3.2.10. *$T_{\widetilde{R}}$ is a $G_{\widetilde{R}}$-torsor.*

Proof. Notice that $\widetilde{R}/p^n\widetilde{R} \to R$ is a nilpotent PD thickening, for all $n \geqslant 1$. The claim follows from the local criterion of flatness and the definition of $\mathrm{RZ}^{\mathrm{fsm}}_G$ (see, in particular, § 2.3.6 and condition (ii) of Definition 2.3.3), by an argument as in the proof of Lemma 2.4.6. ☐

As in [Kis10, Proposition (1.1.5)], the scheme $M_{G,\mu}$ of G-split filtrations of type μ is smooth over $\mathcal{O}_E$. Hence, so is its twist

$$M^T_{G,\mu} = T_{\widetilde{R}} \times^{G_{\widetilde{R}}} M_{G,\mu} \tag{3.2.10.1}$$

by the $G_{\widetilde{R}}$-torsor $T_{\widetilde{R}}$; this classifies filtrations in $\mathbb{D}(X)(\widetilde{R})$ which are, locally for the étale topology, induced by a cocharacter which is G-conjugate to μ. Since $\widetilde{R}$ is p-adically complete, we can lift the R-valued point of (3.2.10.1) corresponding to the Hodge filtration $\mathrm{Fil}^1(X) \subset \mathbb{D}(X)(R)$ to an $\widetilde{R}$-valued point corresponding to a filtration in $\mathbb{D}(X)(\widetilde{R})$ as above. By Grothendieck–Messing theory, this lift of the Hodge filtration gives a morphism

$$\widetilde{y} : \mathrm{Spec}(\widetilde{R}) \to \mathscr{A}_{g,W} \tag{3.2.10.2}$$

extending the point y of (3.2.9.1). Now suppose that x is a k-valued point of $\bar{R} = \widetilde{R}/p\widetilde{R}$ and consider

$$y^\wedge_x : \mathrm{Spec}(\widetilde{R}^\wedge_x) \to \mathrm{Spec}(\widetilde{R}) \to \mathscr{A}_{g,W}.$$

Since $\widetilde{R}^\wedge_x \xrightarrow{\sim} W[\![x_1, \ldots, x_n]\!]$, we obtain, by the result above, that $y^\wedge_x$ factors through $\mathscr{S}_W$. It follows that (3.2.10.2) factors through the Zariski closure $\mathscr{S}^-_W$ of the generic fiber of $\mathscr{S}_W$ in $\mathscr{A}_{g,W}$; since $\widetilde{R}$ is integrally closed in $\widetilde{R}[1/p]$, we see that $\widetilde{y}$ factors through the normalization $\mathscr{S}_W$. Therefore, the morphism $y : \mathrm{Spec}(R) \to \mathscr{A}_{g,W}$ also factors through $\mathscr{S}_W$. This completes the proof of Proposition 3.2.9. ☐

This completes the proof of (i) and (ii) of Theorem 3.2.1. Indeed, the statement about the action of $J_b(\mathbb{Q}_p)$ can be easily deduced from the rest. Part (iii) follows from the fact that

$$\mathrm{RZ}_{\mathrm{GSp}} = \mathrm{RZ}(X_0, \lambda_0) \hookrightarrow \mathrm{RZ}_{\mathrm{GL}} = \mathrm{RZ}(X_0)$$

is a closed immersion, together with the following proposition.

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Proposition 3.2.11. *The morphism* $\mathrm{RZ}_G \to \mathrm{RZ}_{\mathrm{GSp}}$ *obtained by composing* $\mathrm{RZ}_G \hookrightarrow \mathrm{RZ}_G^{\diamond}$ *and* $\mathrm{RZ}_G^{\diamond} \to \mathrm{RZ}_{\mathrm{GSp}}$ *is a closed immersion.*

Proof. Recall that by Proposition 3.2.5, $\mathrm{RZ}_G \to \mathrm{RZ}_{\mathrm{GSp}}$ is finite. By the previous proposition, we can identify $\mathrm{RZ}_G^{\mathrm{fsm}}(k) = \mathrm{RZ}_G(k)$. Let $(X, \lambda, \rho) \in \mathrm{RZ}_{\mathrm{GSp}}(k)$, and let

$$x = (X, \rho, (t_\alpha)) \in \mathrm{RZ}_G^{\mathrm{fsm}}(k)$$

be a preimage. The crystalline tensors $t_\alpha : \mathbf{1} \to \mathbb{D}(X)^{\otimes}$ are uniquely determined by $t_\alpha(W) \in \mathbb{D}(X)(W)^{\otimes}$. The condition (3.2.6.1) shows that these are then also uniquely determined by the rest of the data, so

$$\mathrm{RZ}_G^{\mathrm{fsm}}(k) = \mathrm{RZ}_G(k) \to \mathrm{RZ}_{\mathrm{GSp}}(k)$$

is injective. The proof of Proposition 3.2.5 now implies that $\mathrm{RZ}_G \to \mathrm{RZ}_{\mathrm{GSp}}$ induces a closed immersion $\widehat{\mathrm{RZ}}_{G,x} \hookrightarrow \widehat{\mathrm{RZ}}_{\mathrm{GSp},x}$ on formal completions. Since $\mathrm{RZ}_G \to \mathrm{RZ}_{\mathrm{GSp}}$ is finite the result easily follows, for example by using Nakayama's lemma. $\square$

This completes the proof of Theorem 3.2.1. $\square$

Proposition 3.2.12. *The formal scheme* RZ_G *depends only on the local Shimura–Hodge datum* (G, b, μ, C) *and not on the choice of the tensors* $(s_\alpha) \subset C^{\otimes}$, *as in § 3.1.1, that cut out* G.

Proof. By Proposition 3.2.11, RZ_G is a closed formal subscheme of the undecorated Rapoport–Zink formal scheme $\mathrm{RZ}_{\mathrm{GL}} = \mathrm{RZ}(X_0)$. By Theorem 2.4.10, the choice of the base point $x_0 = (X_0, (t_{\alpha,0}))$, together with an isomorphism of its Dieudonné module with $(D_W, b \circ \sigma, (s_\alpha \otimes 1))$, determines bijections

$$\mathrm{RZ}_G(k) = \mathrm{RZ}_G^{\mathrm{fsm}}(k) \xrightarrow{\sim} X_{G,b_0,\mu_0^\sigma}(k), \quad \mathrm{RZ}_{\mathrm{GL}}(k) \xrightarrow{\sim} X_{\mathrm{GL},i(b_0),i(\mu_0^\sigma)}(k).$$

In fact, these bijections are compatible with the maps $\mathrm{RZ}_G \hookrightarrow \mathrm{RZ}_{\mathrm{GL}}$ and

$$X_{G,b_0,\mu_0^\sigma}(k) \hookrightarrow X_{\mathrm{GL},i(b_0),i(\mu_0^\sigma)}(k)$$

determined by $i : G \hookrightarrow \mathrm{GL}(C)$. Moreover, for each $x \in \mathrm{RZ}_G(k)$, the formal completions $\widehat{\mathrm{RZ}}_{G,x} \subset \widehat{\mathrm{RZ}}_{\mathrm{GL},x}$ at x can be identified with $U_G^{\mu_x,\wedge} \subset U_{\mathrm{GL}}^{i(\mu_x),\wedge}$, where $\mu_x : \mathbb{G}_{mW} \to G_W$ gives a filtration that lifts the Hodge filtration for x. Therefore, both the set of k-valued points and the formal completions at each point of the closed formal subscheme $\mathrm{RZ}_G \subset \mathrm{RZ}_{\mathrm{GL}}$ do not depend on the choice of tensors $(s_\alpha) \subset C^{\otimes}$. Hence, we deduce that the closed formal subscheme $\mathrm{RZ}_G \subset \mathrm{RZ}_{\mathrm{GL}}$ also does not depend on the choice of tensors (s_α). Combining with the above, this now implies that RZ_G depends only on the local Shimura–Hodge datum (G, b, μ, C). $\square$

Remark 3.2.13. According to [Kim13], the Rapoport–Zink formal scheme only depends, up to isomorphism, on the datum $(G, [b], \{\mu\})$. Then by the above, RZ_G also only depends, up to isomorphism, on $(G, [b], \{\mu\})$ and not on the local Hodge embedding. However, this independence does not follow directly from our construction without appealing to [Kim13].

Remark 3.2.14. Define the W-morphism

$$\Theta : \mathrm{RZ}_G \to \widehat{\mathscr{S}}_W = \widehat{\mathscr{S}}_{U,W}$$

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to be the composition of $\mathrm{RZ}_G \hookrightarrow \mathrm{RZ}_G^{\diamond}$ with $\Theta_G^{\diamond} : \mathrm{RZ}_G^{\diamond} \to \widehat{\mathscr{S}_W}$. The morphisms

$$\Theta : \mathrm{RZ}_G \to \widehat{\mathscr{S}_W} = \widehat{\mathscr{S}_{U,W}}$$

commute with the projections $\mathscr{S}_{U_1^p U_p, W} \to \mathscr{S}_{U_2^p U_p, W}$ for $U_1^p \subset U_2^p$. Hence, they combine to also provide a morphism

$$\Theta : \mathrm{RZ}_G \to \widehat{\mathscr{S}_{U_p,W}} := \varprojlim_{U^p} \widehat{\mathscr{S}_{U^p U_p, W}}.$$

Remark 3.2.15. Recall from § 3.1.4 that the local Shimura–Hodge datum (G, b, μ, C) was constructed from a point $x_0 \in \mathscr{S}(k)$. Fix $g \in G(K)$ and $h \in G(W)$, set

$$b' = g^{-1} b \sigma(g), \quad \mu' = h \mu h^{-1},$$

and assume that $b' \in G(W)\mu'^{\sigma}(p)G(W)$. Then there is a point $x_0' \in \mathscr{S}(k)$ such that the unramified local Shimura–Hodge datum (G, b', μ', C) is constructed, in the sense of § 3.1.4, from x_0'.

To see this, notice that the above condition on b' implies that $g \in X_{G,b,\mu^{\sigma}}(k)$, and we may take x_0' to be the image of g under the composition

$$X_{G,b,\mu^{\sigma}}(k) \xrightarrow{\pi^{-1}} \mathrm{RZ}_G(k) \xrightarrow{\Theta} \mathscr{S}(k).$$

Note that we then obtain an isomorphism

$$\mathrm{RZ}_{G,b,\mu,C,(s_\alpha)} \xrightarrow{\sim} \mathrm{RZ}_{G,b',\mu',C,(s_\alpha)}$$

by composing the quasi-isogeny ρ in the definition of the Rapoport–Zink functor (Definition 2.3.3) with the quasi-isogeny

$$X_0(G, b', \mu', C) \dashrightarrow X_0(G, b, \mu, C)$$

determined by g.

3.3 Formal uniformization of the basic locus

By our construction, RZ_G comes with a W-morphism

$$\Theta : \mathrm{RZ}_G \to \widehat{\mathscr{S}_W},$$

where $\mathscr{S} = \mathscr{S}_U$ is the integral model of the Shimura variety given by our choice of a global Shimura datum. Such a morphism is one of the main ingredients of the uniformization theorems of [RZ96, Theorem 6.2] and [Kim14]. In our approach, Θ is essentially part of the definition of RZ_G. We can directly show a version of uniformization (Theorem 3.3.2) by combining the above with results of Kisin [Kis13].

3.3.1 For simplicity, we will only discuss the uniformization when $b \in G(K)$ is *basic*. We assume this is the case for the rest of this section.

We fix a sufficiently small compact open subgroup U^p of $G(\mathbb{A}_f^p)$, again set $U = U^p U_p$, and again abbreviate $\mathscr{S} = \mathscr{S}_U(G, \mathcal{H})$ for the smooth integral model over $\mathcal{O}_{E,(v)}$ of the Shimura variety $\mathrm{Sh}_U(G, \mathcal{H})$.

We continue as in § 3.1.4. In particular, we assume that $X_0 = X_0(G, b, \mu, C)$ arises as the p-divisible group $A_{x_0}[p^\infty]$ with tensors attached to a point $x_0 \in \mathscr{S}_{U_p}(k)$. We will denote also by x_0 the image of x_0 in $\mathscr{S}(k)$. Denote by

$$\mathscr{S}_b \subset \mathscr{S} \otimes_{\mathcal{O}_{E,(v)}} k \tag{3.3.1.1}$$

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the Newton stratum determined by b in the geometric special fiber of $\mathscr{S}$. By definition, $\mathscr{S}_b$ consists of all points x such that there is a quasi-isogeny

$$X_0 \otimes_k k(x) = A_{x_0}[p^\infty] \otimes_k k(x) \dashrightarrow A_x[p^\infty]$$

of p-divisible groups whose corresponding morphism of contravariant Dieudonné isocrystals identifies $t_{\alpha,x} := i_x^*(t_\alpha^{\mathrm{univ}})$ with $t_{\alpha,0} \otimes 1$. Obviously, $x_0 \in \mathscr{S}_b$.

By [RR96] and our assumption that b is basic, the stratum (3.3.1.1) is closed. Denote by $(\widehat{\mathscr{S}_W})_{/\mathscr{S}_b}$ the completion of $\mathscr{S}_W$ along $\mathscr{S}_b$.

Theorem 3.3.2 (Kim [Kim14]). *The morphism Θ extends to a $G(\mathbb{A}_f^p)$-equivariant morphism*

$$\Theta : \mathrm{RZ}_G \times G(\mathbb{A}_f^p) \to \widehat{\mathscr{S}_{U_p,W}} := \varprojlim_{U^p} \widehat{\mathscr{S}_{U^pU_p,W}}$$

which induces an isomorphism of formal schemes

$$\Theta^b : I(\mathbb{Q})\backslash \mathrm{RZ}_G \times G(\mathbb{A}_f^p)/U^p \xrightarrow{\sim} (\widehat{\mathscr{S}_W})_{/\mathscr{S}_b}.$$

Here I is a reductive group over $\mathbb{Q}$, which is an inner form of G, and is such that $I_{\mathbb{R}}$ is anisotropic modulo center. Moreover, there are natural identifications

$$I_{\mathbb{Q}_\ell} = \begin{cases} J_b & \text{if } \ell = p, \\ G_{\mathbb{Q}_\ell} & \text{otherwise.} \end{cases}$$

The quotient is for the action of $I(\mathbb{Q})$ obtained by combining the (discrete) embedding $I(\mathbb{Q}) \subset J_b(\mathbb{Q}_p) \times G(\mathbb{A}_f^p)$ given by the above identifications, with the actions of $J_b(\mathbb{Q}_p)$ on RZ_G, and of $G(\mathbb{A}_f^p)$ on $G(\mathbb{A}_f^p)/K_p$ by left multiplication.

There is a more general result for non-basic b, which is more complicated to state. Compare, for example, with [RZ96, Theorem 6.23] or [Kim14]. Also, the uniformization isomorphism descends to an isomorphism over a finite unramified extension of $\mathbb{Q}_p$; again, we omit this discussion.

Proof. Given the existence of the morphism Θ, this closely follows [Kim14] and [RZ96]; for the convenience of the reader we sketch the proof here.

The morphism

$$\Theta : \mathrm{RZ}_G \to \widehat{\mathscr{S}_{U_p,W}}$$

is given using Remark 3.2.14 and, by its construction, sends the base point $(X_0, \mathrm{id}, (t_{\alpha,0}))$ to the point x_0. Note that $G(\mathbb{A}_f^p)$ acts on the projective system $\mathscr{S}_{U_p} := \varprojlim_{U^p} \mathscr{S}_{U_pU^p}$ on the right (this is the prime-to-p Hecke action) and so this gives a morphism

$$\Theta : \mathrm{RZ}_G \times G(\mathbb{A}_f^p) \to \widehat{\mathscr{S}_{U_p,W}}.$$

After taking the quotient by U^p we obtain

$$\Theta : \mathrm{RZ}_G \times G(\mathbb{A}_f^p)/U^p \to \widehat{\mathscr{S}_W}.$$

The prime-to-p Hecke action on $\mathscr{S}_{U_p}$ preserves the p-divisible groups, and we can easily see that this morphism factors through $(\widehat{\mathscr{S}_W})_{/\mathscr{S}_b}$.

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By their construction (see also § 3.1.1), the integral model $\mathscr{S}$ supports a lisse $\mathbb{A}_f^p$-sheaf $\mathrm{Ta}^p(A)_{\mathbb{Q}}$ given by the Tate $\mathbb{A}_f^p$-module of the universal abelian scheme A, and sections

$$t^p_{\alpha,\mathrm{et}} : \mathbb{A}_f^p \to \mathrm{Ta}^p(A)_{\mathbb{Q}}^{\otimes}.$$

We say that two points x and x' of $\mathscr{S}(k)$ are in the same isogeny class if there is a quasi-isogeny $f : A_x \dashrightarrow A_{x'}$ of the corresponding abelian schemes, respecting weak polarizations, such that the induced maps

$$\mathbb{D}(A_{x'})[1/p] \xrightarrow{\sim} \mathbb{D}(A_x)[1/p]$$

and

$$\mathrm{Ta}^p(A_x)_{\mathbb{Q}} \xrightarrow{\sim} \mathrm{Ta}^p(A_{x'})_{\mathbb{Q}}$$

send $t_{\alpha,x'} \mapsto t_{\alpha,x}$ and $t^p_{\alpha,\mathrm{et},x} \mapsto t^p_{\alpha,\mathrm{et},x'}$, respectively; compare with [Kis13, Proposition (1.4.15)].

Kisin [Kis13, (2.1)] associates to the isogeny class $\phi \subset \mathscr{S}(k)$ of x_0 an algebraic group $I = I_\phi$ with rational points $I(\mathbb{Q})$ given by the self-quasi-isogenies $A_{x_0} \dashrightarrow A_{x_0}$ that preserve $t^p_{\alpha,\mathrm{et},x_0}$ and t_{α,x_0}. By its very definition, $I(\mathbb{Q})$ is a subgroup of $J_b(\mathbb{Q}_p) \times G(\mathbb{A}_f^p)$, and an argument as in [RZ96, p. 289] shows that this subgroup is discrete.

Moreover, as in [RZ96], Θ factors as

$$\Theta : I(\mathbb{Q})\backslash \mathrm{RZ}_G \times G(\mathbb{A}_f^p)/U^p \to (\widehat{\mathscr{S}_W})_{/\mathscr{S}_b}. \tag{3.3.2.1}$$

We continue to assume that U^p is sufficiently small. Using Corollary 3.2.3 and [Kis13, Proposition (2.3.1)], we can see that (3.3.2.1) gives an injection on k-valued points. By the proof of Proposition 3.2.5, we then see that Θ induces an isomorphism between the formal completions at such points. It then also follows that, for U^p sufficiently small, the quotient

$$I(\mathbb{Q})\backslash \mathrm{RZ}_G \times G(\mathbb{A}_f^p)/U^p$$

is representable by a formal scheme over W.

It remains to show that the group I has the properties in the statement of Theorem 3.3.2, and that (3.3.2.1) is an isomorphism.

Note that the point $x_0 \in \mathscr{S}(k)$ is actually defined over a finite field of cardinality p^r. By [Kis13, (2.3) and Corollary (2.3.5)], one sees that there is an element $\gamma_0 \in G(\mathbb{Q})$ such that I is an inner form of the centralizer I_0 of a sufficiently divisible power of γ_0. In fact, γ_0 is a part of a so-called *Kottwitz triple*

$$\mathfrak{k} = (\gamma_0, (\gamma_\ell)_{\ell \neq p}, \delta)$$

as in [Kis13, (4.3) and (4.4.6)]. Here, γ_ℓ belongs to $G(\mathbb{Q}_\ell)$, for all $\ell \neq p$. Also δ belongs to $G(\mathbb{Q}_{p^r})$, with $\mathbb{Q}_{p^r} \subset K$ a finite unramified field extension of $\mathbb{Q}_p$, and is σ-conjugate to b. There are reductive groups I_p over $\mathbb{Q}_p$, and I_ℓ over $\mathbb{Q}_\ell$ for $\ell \neq p$, associated to $\mathfrak{k}$. By [Kis13, Corollary (2.3.2)] we have isomorphisms

$$I \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \xrightarrow{\sim} I_\ell, \quad I \otimes_{\mathbb{Q}} \mathbb{Q}_p \xrightarrow{\sim} I_p.$$

As we assume that b is basic, a power of the element $\gamma_p := \delta\sigma(\delta)\cdots\sigma^{r-1}(\delta)$ is central. Therefore $I_p = J_\delta = J_b$ and they are both inner forms of G. It follows from the definition of Kottwitz triple that a power of γ_0 is also central, and hence $I_0 = G$, and $I_\ell = G_{\mathbb{Q}_l}$ for $\ell \neq p$. The statements about the group I follow from this and the results of Kisin [Kis13, (2.3)] mentioned above.

In fact, the isogeny class ϕ is independent of our choice of (basic) point x_0. More precisely, we have the following proposition.

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Proposition 3.3.3. *Suppose that $x \in \mathscr{S}_b(k)$; in other words, assume that $\mathbb{D}(A_x[p^\infty])(W)$ has Frobenius $F = b_x \circ \sigma$, where $b_x \in G(K)$ is σ-conjugate to $b = b_{x_0}$ in $G(K)$. Then x and x_0 are in the same isogeny class.*

Proof. This again follows from [Kis13]. As above, a power of $\gamma_{p,x}$ obtained from δ_x as above is central. Using this we can easily see that there is a unique equivalence class of Kottwitz triples $\mathfrak{k} = (\gamma_0, (\gamma_\ell)_{\ell \neq p}, \delta_x)$ with δ_x σ-conjugate to b. (See [Kis13, 4.3.1] for the definition of the equivalence relation.) Now, by [Kis13, Proposition 4.4.13], the set of isogeny classes which produce the same Kottwitz triple $\mathfrak{k}$ is in bijection with the abelian group $\text{Ш}_G(\mathbb{Q}, I)$. However, since b is basic, $I_0 = G$ as above, and we can see from its definition [Kis13, §§ 4.4.9 and 4.4.7] that $\text{Ш}_G(\mathbb{Q}, I)$ is trivial. This concludes the proof. □

Since the image of (3.3.2.1) on k-points is the isogeny class of x_0, Proposition 3.3.3 implies that (3.3.2.1) surjects onto $\mathscr{S}_b(k)$. Given the above, the remaining claim that (3.3.2.1) is an isomorphism can be proven quickly by following the arguments in [RZ96, ch. 6]. □

4. Rapoport–Zink spaces for spinor similitude groups

We now turn to the description of a special class of Hodge type Rapoport–Zink formal schemes: those associated with the Shimura varieties for spinor similitude groups. Throughout §§ 4–6 we work purely locally. The Shimura varieties themselves will not appear until § 7.

Fix a non-degenerate quadratic space (V, Q) of rank $n \geqslant 3$ over $\mathbb{Z}_p$ and define a bilinear form on V by (1.2.1.1). We assume that V is self-dual, in the sense that the bilinear form induces an isomorphism $V \xrightarrow{\sim} \operatorname{Hom}(V, \mathbb{Z}_p)$. The space V will remain fixed throughout §§ 4–6.

4.1 Quadratic spaces, Clifford algebras, and spinor similitudes

For details on quadratic spaces, Clifford algebras, and spinor similitude groups we refer the reader to [Bas74, Mad16, Shi10].

4.1.1 The *Hasse invariant* of $V_{\mathbb{Q}_p}$ is the product of Hilbert symbols

$$\epsilon(V_{\mathbb{Q}_p}) = \prod_{i<j} (a_i, a_j)_p,$$

where $e_1, \ldots, e_n \in V_{\mathbb{Q}_p}$ is an orthogonal basis and $a_i = Q(e_i)$. The *determinant*

$$\det(V_{\mathbb{Q}_p}) = 2^n a_1 \cdots a_n$$

is the determinant of the matrix of inner products $[e_i, e_j]$. It is well defined up to multiplication by a square in $\mathbb{Q}_p^\times$. The self-duality hypothesis on V implies that $\epsilon(V_{\mathbb{Q}_p}) = 1$ and

$$\operatorname{ord}_p(\det(V_{\mathbb{Q}_p})) \equiv 0 \pmod 2.$$

4.1.2 The Clifford algebra of V is a $\mathbb{Z}/2\mathbb{Z}$-graded $\mathbb{Z}_p$-algebra denoted by

$$C(V) = C^+(V) \oplus C^-(V).$$

It is free of rank 2^n over $\mathbb{Z}_p$, generated as an algebra by the image of a canonical injection $V \hookrightarrow C^-(V)$ satisfying $v \cdot v = Q(v)$. The *canonical involution* on $C(V)$ is the $\mathbb{Z}_p$-linear endomorphism $c \mapsto c^*$ characterized by $(v_1 \cdots v_d)^* = v_d \cdots v_1$ for all $v_1, \ldots, v_d \in V$.

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For some faithfully flat $\mathbb{Z}_p$-algebra R there is an isomorphism

$$C(V_R) \xrightarrow{\sim} \begin{cases} M_{2^k}(R) & \text{if } n = 2k, \\ M_{2^k}(R) \times M_{2^k}(R) & \text{if } n = 2k+1. \end{cases}$$

The *reduced trace* $\mathrm{Trd} : C(V) \to \mathbb{Z}_p$ is the unique $\mathbb{Z}_p$-linear map which induces, under any such isomorphism, the usual trace on $M_{2^k}(R)$ when $n = 2k$, and the sum of the usual traces when $n = 2k+1$.

The center $Z(V) \subset C(V)$ is easy to determine: if n is even then $Z(V) = \mathbb{Z}_p$, while if n is odd then $Z(V)$ is either $\mathbb{Z}_{p^2}$ or $\mathbb{Z}_p \times \mathbb{Z}_p$, depending on the determinant of $V_{\mathbb{Q}_p}$. In all cases the natural map

$$C(V) \otimes_{Z(V)} C(V)^{op} \to \mathrm{End}_{Z(V)}(C(V))$$

is an isomorphism.

4.1.3 For a $\mathbb{Z}_p$-algebra R, the tensor product $V_R = V \otimes_{\mathbb{Z}_p} R$ is a non-degenerate quadratic space over R with Clifford algebra $C(V_R) = C(V) \otimes_{\mathbb{Z}_p} R$. The *spinor similitude group* $G = \mathrm{GSpin}(V)$ is the reductive group over $\mathbb{Z}_p$ with R-points

$$G(R) = \{g \in C^+(V_R)^\times : gV_Rg^{-1} = V_R, g^*g \in R^\times\},$$

and the *spinor similitude* $\eta_G : G \to \mathbb{G}_m$ is the character $\eta_G(g) = g^*g$.

The conjugation action of G on $C(V)$ leaves invariant the $\mathbb{Z}_p$-submodule V, and this action of G on V is denoted by $g \bullet v = gvg^{-1}$. There is a short exact sequence of group schemes

$$1 \to \mathbb{G}_m \to G \xrightarrow{g \mapsto g\bullet} \mathrm{SO}(V) \to 1$$

over $\mathbb{Z}_p$, and the restriction of η_G to the central $\mathbb{G}_m$ is $z \mapsto z^2$.

4.1.4 If we fix any $\delta \in C(V)^\times$ with $\delta^* = -\delta$, then

$$\psi_\delta(c_1, c_2) := \mathrm{Trd}(c_1 \delta c_2^*)$$

is a perfect symplectic form on $C(V)$. The group G, being a subgroup of $C(V)^\times$, acts on $C(V)$ by left multiplication, yielding a closed immersion $G \hookrightarrow \mathrm{GSp}(C(V), \psi_\delta)$. Under this embedding the symplectic similitude character restricts to the spinor similitude on G.

4.1.5 As in (1.1.1.3), we denote by $D = \mathrm{Hom}_{\mathbb{Z}_p}(C(V), \mathbb{Z}_p)$ the contragredient representation. It follows from § 4.1.2 that there is an isomorphism

$$C(V)^{op} \otimes_{Z(V)} C(V) \xrightarrow{\sim} \mathrm{End}_{Z(V)}(D)$$

defined by $((c_1 \otimes c_2)d)(c) = d(c_1cc_2)$. Note that the contragredient action of G on D commutes with the action of $C(V)$, but not with that of $C(V)^{op}$.

However, the inclusion $V \subset C(V)^{op}$ allows us to view

$$V \subset \mathrm{End}_{\mathbb{Z}_p}(D). \tag{4.1.5.1}$$

These are the *special endomorphisms* of D. Again, they do not commute with the G action; rather, they satisfy the relation $g \circ v \circ g^{-1} = g \bullet v$ as endomorphisms of D, for any $g \in G(\mathbb{Z}_p)$ and $v \in V$.

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4.2 The GSpin local Shimura datum

From the quadratic space V we will construct an *unramified local Shimura–Hodge* datum $(G, b, \mu, C(V))$ in the sense of Definition 2.2.4.

4.2.1 Fix a $\mathbb{Z}_p$-basis $x_1, \dots, x_n \in V$ for which the matrix of inner products has the form

$$([x_i, x_j]) = \begin{pmatrix} 0 & 1 & & & & \\ 1 & 0 & & & & \\ & & * & & & \\ & & & * & & \\ & & & & \ddots & \\ & & & & & * \end{pmatrix}$$

(the matrix is diagonal except for the upper left 2×2 block). This choice of basis determines a cocharacter $\mu : \mathbb{G}_m \to G$ by

$$\mu(t) = t^{-1} x_1 x_2 + x_2 x_1,$$

where the arithmetic on the right-hand side takes place in $C(V)$.

Under the representation $G \to \mathrm{SO}(V)$, we have

$$\mu(t) \bullet x_i = \begin{cases} t^{-1} x_i & \text{if } i = 1, \\ t x_i & \text{if } i = 2, \\ x_i & \text{if } 3 \leqslant i \leqslant n. \end{cases}$$

The relation $x_1 x_2 + x_2 x_1 = [x_1, x_2] = 1$ implies that $C(V) = x_1 C(V) \oplus x_2 C(V)$, and under the representation $G \to \mathrm{GSp}(C(V), \psi_\delta)$, we have

$$\mu(t) \cdot z = \begin{cases} t^{-1} z & \text{if } z \in x_1 C(V), \\ z & \text{if } z \in x_2 C(V). \end{cases}$$

The following lemma will be needed in the proof of Proposition 6.2.2.

Lemma 4.2.2. *For self-dual W-lattices $A, A^\sharp \subset V_K$, the following are equivalent:*

(i) $(A + A^\sharp)/A \xrightarrow{\sim} W/pW$;

(ii) *there is a $g \in G(K)$ such that $A^\sharp = g \bullet V_W$ and $A = g\mu(p^{-1}) \bullet V_W$.*

Proof. First assume (i) holds. As self-dual lattices are necessarily maximal, Theorem A.1.3 implies that A and $A^\sharp$ have the form

$$A = We \oplus Wf \oplus B_1, \quad A^\sharp = Wpe \oplus Wp^{-1} f \oplus B_1,$$

for some isotropic $e, f \in V_K$ with $[e, f] = 1$, and some W-submodule $B_1 \subset V_K$ orthogonal to both e and f.

Now consider the self-dual W-lattice $V_W \subset V_K$. The calculations of § 4.2.1 imply that

$$\big(\mu(p^{-1}) \bullet V_W + V_W\big) / \big(\mu(p^{-1}) \bullet V_W\big) \xrightarrow{\sim} W/pW, \tag{4.2.2.1}$$

and so there is a similar decomposition

$$\begin{aligned} \mu(p^{-1}) \bullet V_W &= W\widetilde{e} \oplus W\widetilde{f} \oplus B_2, \\ V_W &= Wp\widetilde{e} \oplus Wp^{-1}\widetilde{f} \oplus B_2 \end{aligned}$$

for some W-submodule $B_2 \subset V_K$.

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Witt's cancellation theorem implies that $B_{1K} \xrightarrow{\sim} B_{2K}$ as K-quadratic spaces. As B_1 and B_2 are self-dual (hence maximal), Theorem A.1.2 implies that B_1 and B_2 are isomorphic as W-quadratic spaces. It follows that there is a $g \in \mathrm{SO}(V_K)$ such that $g\widetilde{e} = e$, $g\widetilde{f} = f$, and $gB_2 = B_1$. Choosing any lift $g \in G(K)$ yields the element required in (ii).

The reverse implication is clear from (4.2.2.1). □

4.2.3 As G acts on both V and D, any $b \in G(K)$ determines isocrystals

$$(V_K, \Phi = b \circ \sigma) \quad \text{and} \quad (D_K, F = b \circ \sigma).$$

These will play a central role in everything that follows.

Recall from § 2.2.8 that $b \in G(K)$ is *basic* if its slope cocharacter $\nu_b : \mathbb{T}_K \to G_K$ factors through the center of G_K. As $\mathbb{T}_K$ is an inverse limit of connected group schemes, this is equivalent to the slope cocharacter factoring through the connected component of the center, which is

$$\mathbb{G}_m = \ker(G \to \mathrm{SO}(V)).$$

Thus any basic $b \in G(K)$ determines a rational number

$$\nu_b \in \mathrm{Hom}(\mathbb{T}_K, \mathbb{G}_m) = \mathbb{Q},$$

which depends only on the σ-conjugacy class of b.

Lemma 4.2.4. *For each $b \in G(K)$, the following are equivalent:*

(i) *b is basic;*
(ii) *the isocrystal $(V_K, \Phi = b \circ \sigma)$ is isoclinic of slope 0;*
(iii) *the isocrystal $(D_K, F = b \circ \sigma)$ is isoclinic.*

When these equivalent conditions hold, the slope of the isocrystal D_K is $-\nu_b$.

Proof. The equivalence of the first two statements follows from the fact that the central $\mathbb{G}_m \subset G_K$ is the kernel of the representation $G_K \to \mathrm{SO}(V_K)$. The equivalence of the first and third follows from the observation that the representation $G_K \to \mathrm{GL}(D_K)$ identifies the central $\mathbb{G}_m \subset G_K$ with the torus of scalars in $\mathrm{GL}(D_K)$. Moreover, $\mathbb{G}_m \subset G_K$ acts on D_K through the character $t \mapsto t^{-1}$, proving the final claim. □

Proposition 4.2.5. *Every basic $b \in G(K)$ satisfies*

$$\nu_b = \frac{\mathrm{ord}_p(\eta_G(b))}{2}, \tag{4.2.5.1}$$

and $b \mapsto \nu_b$ establishes a bijection

$$\{\text{basic } b \in G(K)\}/\sigma\text{-conjugacy} \xrightarrow{\sim} \tfrac{1}{2}\mathbb{Z}. \tag{4.2.5.2}$$

Moreover, for any basic $b \in G(K)$, the $\mathbb{Q}_p$-quadratic space

$$V_K^{\Phi} = \{x \in V_K : \Phi x = x\}$$

has the same dimension and determinant as $V_{\mathbb{Q}_p}$, and has Hasse invariant $\epsilon(V_K^{\Phi}) = (-1)^{2\nu_b}$.

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Proof. As in [Asg02, § 2], the derived group of G is the kernel of the spinor similitude, which is just the usual spin double cover of $\mathrm{SO}(V)$. In particular, the derived group is simply connected, and results of Kottwitz (combine [Kot85, Proposition 5.4] and [Kot85, (2.4.1)]) imply that

$$\frac{\mathrm{ord}_p \circ \eta_G}{2} : G(K) \to \frac{1}{2}\mathbb{Z}$$

induces a bijection

$$\{\text{basic } b \in G(K)\}/\sigma\text{-conjugacy} \xrightarrow{\sim} \tfrac{1}{2}\mathbb{Z}.$$

Recalling the basis $x_1, \ldots, x_n$ of V of § 4.2.1, we now set

$$b = x_3(p^{-1}x_1 + x_2) \in G(\mathbb{Q}_p). \tag{4.2.5.3}$$

A simple calculation gives

$$b^2 = -p^{-1}Q(x_3) \in p^{-1} \cdot \mathbb{Z}_p^{\times}. \tag{4.2.5.4}$$

As $\mathbb{G}_m \subset G$ acts on D_K via $t \mapsto t^{-1}$, the relation (4.2.5.4) implies that b^k makes D_K into an isoclinic isocrystal of slope $k/2$. Thus b^k is basic with $\nu_{b^k} = -k/2$ by Lemma 4.2.4.

As $\eta_G(p) = p^2$, the relation (4.2.5.4) implies $\mathrm{ord}_p(\eta_G(b)) = -1$, and so the powers of b form a complete set of representatives for the basic σ-conjugacy classes. As these satisfy

$$\nu_{b^k} = -\frac{k}{2} = \frac{\mathrm{ord}_p(\eta_G(b^k))}{2},$$

we have now proved both (4.2.5.1) and (4.2.5.2).

As b^2 is a scalar, it lies in the kernel of $G(K) \to \mathrm{SO}(V)(K)$. Thus the isocrystal structure on V_K defined by $\Phi = b^k \circ \sigma$ depends only on $k \pmod 2$. If $k = 0$ then $V_K^{\Phi} = V_{\mathbb{Q}_p}$ as subspaces of V_K, and so they have the same dimension, determinant, and Hasse invariant.

On the other hand, if $k = 1$ then direct calculation shows that the isocrystal V_K defined by $\Phi = b \circ \sigma$ satisfies

$$\Phi x_1 = -px_2, \quad \Phi x_2 = -p^{-1}x_1, \quad \Phi x_3 = -x_3, \tag{4.2.5.5}$$

and $\Phi x_i = x_i$ for $i > 3$. If we define subspaces $M = \mathbb{Q}_p x_1 + \mathbb{Q}_p x_2 + \mathbb{Q}_p x_3$ and $N = M^{\perp}$ in $V_{\mathbb{Q}_p}$, then there are orthogonal decompositions

$$V_{\mathbb{Q}_p} = M \oplus N, \quad V_K^{\Phi} = M_K^{\Phi} \oplus N,$$

and an elementary calculation (as in the proof of [HP14, Proposition 2.6]) shows that M and M_K^{Φ} have the same dimension and determinant, but different Hasse invariants. Hence the same is true of $V_{\mathbb{Q}_p}$ and V_K^{Φ}. □

PROPOSITION 4.2.6. *If we let*

- $\mu : \mathbb{G}_m \to G$ *be as in § 4.2.1;*
- $b \in G(K)$ *be defined by (4.2.5.3), so that* $\nu_b = -1/2$;
- $G \to \mathrm{GSp}(C(V), \psi_\delta)$ *be the representation of § 4.1.4,*

then $b \in G(W)\mu^{\sigma}(p)G(W)$, *and the action of* $\mathbb{G}_m$ *on* $C(V)$ *determined by*

$$\mathbb{G}_m \xrightarrow{\mu} G \to \mathrm{GSp}(C(V), \psi_\delta)$$

has the form

$$t \mapsto \begin{pmatrix} t^{-1} I_{2^{n-1}} & \\ & I_{2^{n-1}} \end{pmatrix}$$

for some choice of basis of $C(V)$. *In particular,* $(G, b, \mu, C(V))$ *is a local unramified Shimura–Hodge datum in the sense of Definition 2.2.4.*

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Proof. The calculations of § 4.2.1 show that the action of $\mathbb{G}_m$ on $C(V)$ has the stated form. Thus, as $\mu = \mu^\sigma$, it suffices to prove $b \in \mu(p)G(W)$. Comparing the calculations of § 4.2.1 with (4.2.5.5) shows that

$$b \bullet V_W = \Phi(V_W) = \mu(p) \bullet V_W$$

as lattices in V_K. Thus $\mu(p^{-1})b$ lies in $p^{\mathbb{Z}}G(W)$, the stabilizer in $G(K)$ of the lattice V_W. But

$$\mathrm{ord}_p(\eta_G(b)) = -1 = \mathrm{ord}_p(\eta_G(\mu(p))),$$

and so in fact $\mu(p^{-1})b \in G(W)$. □

Remark 4.2.7. In general, given G and $\{\mu\}$ as in § 2.2.1 with $\{\mu\}$ minuscule, there is a unique basic σ-conjugacy class $[b]$ such that $(G, [b], \{\mu\})$ is a local unramified Shimura datum. This follows from [RR96, Theorem 1.15(i)] and the description of the set $B(G_{\mathbb{Q}_p}, \{\mu\})$ given there; see also [Wor13, § 5.2].

4.3 The GSpin Rapoport–Zink space

The local Hodge–Shimura datum $(G, b, \mu, C(V))$ of Proposition 4.2.6 will remain fixed throughout the remainder of § 4, and throughout §§ 5 and 6.

4.3.1 By Lemma 4.2.4, the isocrystals

$$(V_K, \Phi = b \circ \sigma), \quad (D_K, F = b \circ \sigma)$$

have slopes 0 and 1/2, respectively, and Lemma 2.2.5 implies that there is a p-divisible group

$$X_0 = X_0(G, b, \mu, C(V))$$

over k whose contravariant Dieudonné module is the lattice $D_W \subset D_K$. The perfect symplectic form ψ_δ on $C(V)$ determines a principal polarization $\lambda_0 : X_0 \to X_0^\vee$, and the inclusion $C(V)^{op} \subset \mathrm{End}(C(V))$ by right multiplication defines an action of $C(V)^{op}$ on X_0. Let $\mathbb{D}(X_0)$ be the contravariant crystal of X_0, so that $\mathbb{D}(X_0)(W) = D_W$.

Tensoring (4.1.5.1) with K yields a subspace $V_K \subset \mathrm{End}_K(D_K)$ of special endomorphisms, on which the operators Φ and F are related by $\Phi x = F \circ x \circ F^{-1}$. In particular, the Φ-fixed vectors commute with F, and so determine a distinguished $\mathbb{Q}_p$-subspace

$$V_K^\Phi \subset \mathrm{End}(X_0)_\mathbb{Q}$$

of *special quasi-endomorphisms* of X_0. The restriction to V_K^Φ of the K-valued quadratic form on V_K then satisfies $x \circ x = Q(x) \cdot \mathrm{id}$. By Proposition 4.2.5 the space V_K^Φ has the same dimension and determinant as $V_{\mathbb{Q}_p}$, but has Hasse invariant $\epsilon(V_K^\Phi) = -\epsilon(V_{\mathbb{Q}_p}) = -1$.

4.3.2 We now have all the ingredients needed to attach a Rapoport–Zink formal scheme to the quadratic space (V, Q), using the general constructions of §§ 2 and 3. By Theorem 3.2.1, the quadruple $(G, b, \mu, C(V))$ determines a formal W-scheme

$$\mathrm{RZ} = \mathrm{RZ}_{G,b,\mu,C(V),(s_\alpha)} \tag{4.3.2.1}$$

together with a closed immersion $\mathrm{RZ} \hookrightarrow \mathrm{RZ}(X_0, \lambda_0)$. Here $\mathrm{RZ}(X_0, \lambda_0)$ is the symplectic Rapoport–Zink space as in § 2.3.1. By Proposition 3.2.12, the formal scheme (4.3.2.1) does not depend on the collection of tensors $(s_\alpha) \subset C(V)^\otimes = D^\otimes$ that cut out the subgroup $G \subset \mathrm{GL}(C(V))$.

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For this to make sense, we must explain why the datum $(G, b, \mu, C(V))$ has the form (3.1.4.1); in other words, why this quadruple (along with some choice of tensors s_α) agrees with the one coming from a k-point on an integral canonical model of a Shimura variety. The quadratic space (V, Q) over $\mathbb{Z}_p$ can be realized as the p-adic completion of a quadratic space over $\mathbb{Z}_{(p)}$ of signature $(n-2, 2)$, and hence the reductive group G and its representation $C(V)$ also arise from analogous objects over $\mathbb{Z}_{(p)}$. The tensors (s_α) may then be chosen to come from this $\mathbb{Z}_{(p)}$-model of $C(V)$, and the existence of the desired point x on a global Shimura variety then follows from Proposition 7.2.3 below.

An explicit list of tensors (s_α) that cut out $G \subset \mathrm{GL}(C(V))$ can be found in [Mad16, Lemma 1.4(3)]. It will be convenient to fix, once and for all, such a list, and assume that it includes the tensor induced by the symplectic form ψ_δ on $C(V)$, as in the proof of Theorem 3.2.1, and that it includes a set of $\mathbb{Z}_p$-algebra generators for the subring

$$C(V)^{op} \subset \mathrm{End}(C(V)) = C(V) \otimes C(V)^*$$

defined by right multiplication (equivalently, a set of generators for the subring $C(V) \subset \mathrm{End}(D)$ of § 4.1.5).

4.3.3 The restriction of the universal object via $\mathrm{RZ} \hookrightarrow \mathrm{RZ}(X_0, \lambda_0)$ is a pair (X, ρ), in which X is a p-divisible group over RZ, and

$$\rho : X_0 \times_{\mathrm{Spf}(k)} \overline{\mathrm{RZ}} \dashrightarrow X \times_{\mathrm{RZ}} \overline{\mathrm{RZ}}$$

is a quasi-isogeny of p-divisible groups over

$$\overline{\mathrm{RZ}} = \mathrm{RZ} \times_{\mathrm{Spf}(W)} \mathrm{Spf}(k).$$

As we have chosen our list of tensors to include generators of the subalgebra $C(V)^{op} \subset \mathrm{End}(C(V))$, the universal X is endowed not only with a principal polarization $\lambda : X \to X^\vee$, but also with an action of $C(V)^{op}$. The universal quasi-isogeny is $C(V)^{op}$-linear. The action of $C(V)^{op}$ on the universal object will play little part in what follows; it will be used only in the proof of Proposition 6.1.2 below.

The universal quasi-isogeny also respects the polarizations λ and λ_0 up to scaling, and hence, Zariski locally on RZ, we have $\rho^*\lambda = c(\rho)^{-1}\lambda_0$ for some $c(\rho) \in \mathbb{Q}_p^\times$. For each $\ell \in \mathbb{Z}$, let $\mathrm{RZ}^{(\ell)} \subset \mathrm{RZ}$ be the open and closed formal subscheme on which $\mathrm{ord}_p(c(\rho)) = \ell$, so that

$$\mathrm{RZ} = \bigsqcup_{\ell \in \mathbb{Z}} \mathrm{RZ}^{(\ell)}.$$

4.3.4 The algebraic group $J_b = \mathrm{GSpin}(V_K^\Phi)$ has $\mathbb{Q}_p$-points

$$J_b(\mathbb{Q}_p) = \{g \in G(K) : gb = b\sigma(g)\},$$

and acts as automorphisms of the isocrystal D_K. This realizes $J_b(\mathbb{Q}_p) \subset \mathrm{End}(X_0)^\times_{\mathbb{Q}}$, and, as in (2.3.7.1), there is an induced action of $J_b(\mathbb{Q}_p)$ on RZ. Each $g \in J_b(\mathbb{Q}_p)$ restricts to an isomorphism

$$g : \mathrm{RZ}^{(\ell)} \to \mathrm{RZ}^{(\ell + \mathrm{ord}_p(\eta_b(g)))}.$$

In particular, the subgroup $p^{\mathbb{Z}} \subset J_b(\mathbb{Q}_p)$ acts on RZ, and, as $\eta_b(p) = p^2$,

$$p^{\mathbb{Z}} \backslash \mathrm{RZ} \xrightarrow{\sim} \mathrm{RZ}^{(0)} \sqcup \mathrm{RZ}^{(1)}. \tag{4.3.4.1}$$

The surjectivity of the spinor similitude $\eta_b : J_b(\mathbb{Q}_p) \to \mathbb{Q}_p^\times$ implies that the $\mathrm{RZ}^{(\ell)}$ for various ℓ are (non-canonically) isomorphic.

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4.3.5 In this paper we do not discuss the very interesting general theory of the p-adic symmetric domain and the period morphism for the p-analytic spaces associated to the formal schemes RZ_G. See [RZ96, ch. 5], and for the Hodge type case [Kim13].

We will just mention briefly, and without details, an elegant description of the p-adic symmetric domains that relates to the basic Rapoport–Zink spaces for $\mathrm{GSpin}(V)$ treated here.

In this case, the corresponding flag variety is simply the quadric $\mathcal{Q} \subset \mathbb{P}(V)$ of isotropic lines $L \subset V$. If F is a finite extension of K, the F-valued points $\mathcal{Q}^{\mathrm{wa}}(F)$ of the (weakly) admissible locus $\mathcal{Q}^{\mathrm{wa}} \subset \mathcal{Q}_K^{\mathrm{rig}}$ in the rigid analytic quadric are those isotropic F-lines $L \subset V_F$ which are not contained in any isotropic F-subspace of V_F which is $\mathbb{Q}_p$-rational; here 'rational' is for the $\mathbb{Q}_p$-vector space structure on V_F given by $V_F = V_K^{\Phi} \otimes_{\mathbb{Q}_p} F$.

This description can be obtained by first reducing consideration to the corresponding p-adic symmetric domain for the group $\mathrm{SO}(V) = \mathrm{GSpin}(V)/\mathbb{G}_m$ (see, for example, [DOR10, Corollary 9.2.22]), and then by working through the definitions of [RZ96, ch. 1] for $\mathrm{SO}(V)$. We leave the details to the reader.

5. Vertex lattices and special lattices

The section contains mostly linear algebra. We study the family of *vertex lattices* $\Lambda \subset V_K^{\Phi}$, and the family of *special lattices* $L \subset V_K$.

5.1 Vertex lattices

In this subsection we introduce the vertex lattices and study their combinatorial properties (compare with [HP14, Vol10, VW11]). Later, in § 6, we will express the reduced scheme underlying the spinor similitude Rapoport–Zink formal scheme (4.3.2.1) as a union of closed subschemes indexed by these vertex lattices.

Definition 5.1.1. A *vertex lattice* is a $\mathbb{Z}_p$-lattice $\Lambda \subset V_K^{\Phi}$ satisfying

$$p\Lambda \subset \Lambda^{\vee} \subset \Lambda,$$

where $\Lambda^{\vee}$ is the dual lattice in the sense of Definition A.1.1. The *type* of Λ is

$$t_{\Lambda} = \dim_{\mathbb{F}_p}(\Lambda/\Lambda^{\vee}).$$

Proposition 5.1.2. *Let Λ be a vertex lattice, and recall the integer $t_{\max}$ of (1.2.3.1). The type t_{Λ} is even and satisfies $2 \leqslant t_{\Lambda} \leqslant t_{\max}$. Furthermore, every vertex lattice is contained in a vertex lattice of type $t_{\max}$.*

Proof. Recall from § 4.1.1 and Proposition 4.2.5 that

$$\mathrm{ord}_p(\det(V_K^{\Phi})) \equiv \mathrm{ord}_p(\det(V_{\mathbb{Q}_p})) \equiv 0 \pmod 2. \tag{5.1.2.1}$$

It follows that the type of a vertex lattice is even. The type cannot be 0, for then V_K^{Φ} would contain a self-dual lattice, contradicting the Hasse invariant calculation $\epsilon(V_K^{\Phi}) = -1$.

Let **Lat** be the set of all $\mathbb{Z}_p$-lattices $\Lambda \subset V_K^{\Phi}$ satisfying $[\Lambda, \Lambda] \subset p^{-1}\mathbb{Z}_p$. In other words, **Lat** is the set of all lattices for which $p\Lambda \subset \Lambda^{\vee}$, and so **Lat** contains all vertex lattices. Let Λ be any lattice which is maximal (with respect to inclusion) among all elements of **Lat**. We will prove that Λ is a vertex lattice of type $t_{\max}$, from which the proposition follows immediately.

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The lattice Λ is a maximal lattice (in the sense of Definition A.1.1) with respect to the rescaled quadratic form pQ on V_K^Φ, and so by Theorems A.1.2 and A.1.3 there is a decomposition

$$\Lambda = \mathrm{Span}_{\mathbb{Z}_p}\{e_1, f_1, \ldots, e_r, f_r\} \oplus Z$$

in which $Z_{\mathbb{Q}_p}$ is anisotropic,

$$Z = \{x \in Z_{\mathbb{Q}_p} : Q(x) \in p^{-1}\mathbb{Z}_p\}, \tag{5.1.2.2}$$

and the vectors e_i and f_j satisfy

$$[Z, e_i] = [Z, f_i] = 0, \quad [e_i, e_j] = [f_i, f_j] = 0,$$

and $[e_i, f_j] = p^{-1}\delta_{i,j}$. As every quadratic space over $\mathbb{Q}_p$ of dimension greater than 4 contains an isotropic vector, we also have $\dim(Z_{\mathbb{Q}_p}) \leqslant 4$.

The relation $p\Lambda \subset \Lambda^\vee$ implies $pZ \subset Z^\vee$. We cannot have $Z^\vee = Z$, for then

$$\mathrm{Span}_{\mathbb{Z}_p}\{pe_1, f_1, \ldots, pe_r, f_r\} \oplus Z \subset V_K^\Phi$$

would be a self-dual lattice, contradicting $\epsilon(V_K^\Phi) = -1$. In particular, $Z \neq 0$.

If $\dim(Z_{\mathbb{Q}_p}) = 1$ then $Z_{\mathbb{Q}_p} \xrightarrow{\sim} \mathbb{Q}_p$ with the quadratic form $Q(x) = cx^2$ for some $c \in \mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^2$. We cannot have $\mathrm{ord}_p(c)$ even, for then (5.1.2.2) implies $Z = Z^\vee$, contradicting what was said above. But we also cannot have $\mathrm{ord}_p(c)$ odd, for then

$$\mathrm{ord}_p(\det(V_{\mathbb{Q}_p})) = \mathrm{ord}_p(\det(V_K^\Phi)) = 2r + \mathrm{ord}_p(\det(Z_{\mathbb{Q}_p}))$$

is odd, contradicting (5.1.2.1). Thus $\dim(Z_{\mathbb{Q}_p}) \in \{2, 3, 4\}$.

Suppose $\dim(Z_{\mathbb{Q}_p}) = 2$. Let $\mathbb{Q}_{p^2}$ be the unramified quadratic extension of $\mathbb{Q}_p$, and let $x \mapsto \bar{x}$ be its non-trivial Galois automorphism. For some $c \in \mathbb{Q}_p^\times/\mathrm{Nm}(\mathbb{Q}_{p^2}^\times)$ there is an isomorphism $Z_{\mathbb{Q}_p} \xrightarrow{\sim} \mathbb{Q}_{p^2}$ identifying the quadratic form Q with $Q(x) = cx\bar{x}$. If $\mathrm{ord}_p(c)$ is even then, as above, $Z = Z^\vee$ yields a contradiction. Thus $\mathrm{ord}_p(c)$ is odd, and simple calculation shows that $Z^\vee \subset Z$, $\dim(Z/Z^\vee) = 2$ and $\det(Z_{\mathbb{Q}_p}) = -u$ for a non-square $u \in \mathbb{Q}_p^\times$. This implies that $\Lambda^\vee \subset \Lambda$ with $\dim(\Lambda/\Lambda^\vee) = 2r + 2 = n$, and

$$\det(V_{\mathbb{Q}_p}) = \det(V_K^\Phi) = (-1)^r \det(Z_{\mathbb{Q}_p}) = (-1)^{n/2} u \neq (-1)^{n/2}.$$

Thus Λ is a vertex lattice of type $n = t_{\max}$.

Suppose that $\dim(Z_{\mathbb{Q}_p}) = 3$. Let B denote the quaternion division algebra over $\mathbb{Q}_p$, with its main involution $x \mapsto \bar{x}$. The subspace of traceless elements $B^0 = \{x \in B : x + \bar{x} = 0\}$ has dimension 3, and the reduced norm $\mathrm{Nrd}(x) = x\bar{x}$ restricts to an anisotropic quadratic form on B^0 with $\mathrm{ord}_p(\det(B^0))$ even. In fact

$$(B^0, \mathrm{Nrd}) \xrightarrow{\sim} (\mathbb{Q}_p^3, -ux_1^2 - px_2^2 + upx_3^2)$$

for any non-square $u \in \mathbb{Z}_p^\times$. There are exactly four anisotropic quadratic spaces over $\mathbb{Q}_p$ of dimension 3, and they are the spaces $(B_0, c\mathrm{Nrd})$ with $c \in \mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^2$. If $\mathrm{ord}_p(c)$ is odd then $\mathrm{ord}_p(\det(V_K^\Phi))$ is also odd, contradicting (5.1.2.1). Thus $\mathrm{ord}_p(c)$ is even, and one can easily check from (5.1.2.2) that $Z^\vee \subset Z$ with $Z/Z^\vee$ of dimension 2. Thus $\Lambda^\vee \subset \Lambda$ and

$$\dim(\Lambda/\Lambda^\vee) = 2r + 2 = n - 1.$$

In other words, Λ is a vertex lattice of type $n - 1 = t_{\max}$.

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Finally, suppose $\dim(Z_{\mathbb{Q}_p}) = 4$. By [Ser73, Corollary IV.2.3], the only anisotropic quadratic space of dimension 4 is $Z_{\mathbb{Q}_p} \xrightarrow{\sim} B$ with its reduced norm form. In particular, $\det(Z_{\mathbb{Q}_p}) = 1$ and $Z \xrightarrow{\sim} \mathfrak{m}^{-1}$, where $\mathcal{O}_B \subset B$ is the unique maximal order and $\mathfrak{m} \subset \mathcal{O}_B$ is its unique maximal ideal. The dual lattice is $Z^\vee = \mathcal{O}_B$, and it follows that $Z^\vee \subset Z$ with $\dim(Z/Z^\vee) = 2$. This implies that $\Lambda^\vee \subset \Lambda$, $\dim(\Lambda/\Lambda^\vee) = 2r + 2 = n - 2$, and

$$\det(V_{\mathbb{Q}_p}) = \det(V_K^\Phi) = (-1)^r \det(Z_{\mathbb{Q}_p}) = (-1)^{n/2}.$$

Thus Λ is a vertex lattice of type $n - 2 = t_{\max}$. □

There is a natural notion of adjacency between vertex lattices, which makes them into the vertices of a connected graph, as we now explain.

Definition 5.1.3. Two vertex lattices $\Lambda_1, \Lambda_2 \subset V_K^\Phi$ are *adjacent* if either $\Lambda_1 \subsetneq \Lambda_2$ or $\Lambda_2 \subsetneq \Lambda_1$.

We write $\Lambda_1 \sim \Lambda_2$ to indicate that Λ_1 and Λ_2 are adjacent. Adjacent lattices have different types, and the inclusion between them is always the lattice of smaller type inside the lattice of larger type.

Proposition 5.1.4. *Let $\Lambda \subset V_K^\Phi$ be a vertex lattice of type t_Λ and suppose $t \neq t_\Lambda$ is any even integer with $2 \leqslant t \leqslant t_{\max}$. There is a vertex lattice of type t adjacent to Λ.*

Proof. First suppose that $t < t_\Lambda$. The quadratic form $q(x) = pQ(x)$ makes $\Lambda/\Lambda^\vee$ into a non-degenerate quadratic space over $\mathbb{F}_p$ of rank $t_\Lambda \geqslant 4$, and [Ser73, Corollary 1.2.2] implies the existence of an isotropic line $\ell \subset \Lambda/\Lambda^\vee$. The orthogonal $\ell^\perp \subset \Lambda/\Lambda^\vee$ determines a vertex lattice $\Lambda' = \Lambda^\vee + \ell^\perp \subset \Lambda$ of type $t_{\Lambda'} = t_\Lambda - 2$, and repeating this process yields a vertex lattice of type t contained in Λ.

Now suppose that $t_\Lambda < t$. By Proposition 5.1.2 there is a vertex lattice $\Lambda_{\max}$ of maximal type $t_{\max}$ satisfying $\Lambda_{\max}^\vee \subsetneq \Lambda^\vee \subset \Lambda \subsetneq \Lambda_{\max}$. The subspace $\Lambda^\vee/\Lambda_{\max}^\vee \subset \Lambda_{\max}/\Lambda_{\max}^\vee$ is totally isotropic, and for any codimension 1 subspace $\ell \subset \Lambda^\vee/\Lambda_{\max}^\vee$ the orthogonal

$$\Lambda/\Lambda_{\max}^\vee \subset \ell^\perp \subset \Lambda_{\max}/\Lambda_{\max}^\vee$$

determines a vertex lattice $\Lambda' = \Lambda_{\max}^\vee + \ell^\perp$ of type $t_\Lambda + 2$ containing Λ. Repeating this process yields a vertex lattice of type t containing Λ. □

The following proposition, which proves the connectedness of the graph of vertex lattices, will be used in the proof of Theorem 6.4.1 to show that $\mathrm{RZ}^{(\ell)}$ is connected.

Proposition 5.1.5. *Given any two vertex lattices $\Lambda', \Lambda'' \subset V_K^\Phi$, there is a sequence of adjacent vertex lattices*

$$\Lambda' \sim \Lambda_1 \sim \Lambda_2 \sim \cdots \sim \Lambda_s \sim \Lambda''.$$

Proof. As in the proof of Proposition 5.1.2, let **Lat** be the set of all $\mathbb{Z}_p$-lattices $\Lambda \subset V_K^\Phi$ satisfying $[\Lambda, \Lambda] \subset p^{-1}\mathbb{Z}_p$. Recall that **Lat** contains all vertex lattices. Recall also that any maximal (with respect to inclusion) element $\Lambda \in \mathbf{Lat}$ is necessarily a vertex lattice of type $t_{\max}$, and is a maximal lattice with respect to the rescaled quadratic form pQ.

Pick maximal elements $\underline{\Lambda}', \underline{\Lambda}'' \in \mathbf{Lat}$ with $\Lambda' \subset \underline{\Lambda}'$ and $\Lambda'' \subset \underline{\Lambda}''$. In particular, $\Lambda' \sim \underline{\Lambda}'$ and $\Lambda'' \sim \underline{\Lambda}''$. Using the maximality of $\underline{\Lambda}'$ and $\underline{\Lambda}''$ with respect to pQ, Theorem A.1.3 implies that there are decompositions

$$\underline{\Lambda}'' = \mathrm{Span}_{\mathbb{Z}_p}\{e_1, f_1, \ldots, e_r, f_r\} \oplus Z$$

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and

$$\underline{\Lambda}' = \mathrm{Span}_{\mathbb{Z}_p}\{p^{a_1}e_1, p^{-a_1}f_1, \dots, p^{a_r}e_r, p^{-a_r}f_r\} \oplus Z,$$

where all e_i and f_i are isotropic, $[e_i, f_j] = p^{-1}\delta_{ij}$, each $a_i \geqslant 0$, and Z is orthogonal to all e_i and f_i and satisfies $pZ \subset Z^\vee \subset Z$.

From these decompositions it is elementary to construct a chain of adjacent vertex lattices

$$\Lambda' \sim \underline{\Lambda}' \sim \Lambda_1 \sim \Lambda_2 \sim \cdots \sim \Lambda_s \sim \underline{\Lambda}'' \sim \Lambda''.$$

For example, set

$$\Lambda_1 = \mathrm{Span}_{\mathbb{Z}_p}\{p^{a_1}e_1, p^{-a_1+1}f_1, p^{a_2}e_2, p^{-a_2}f_2, \dots, p^{a_r}e_r, p^{-a_r}f_r\} \oplus Z$$

so that $\Lambda_1 \subsetneq \underline{\Lambda}'$ is a vertex lattice of type $t_{\max} - 2$, and then set

$$\Lambda_2 = \mathrm{Span}_{\mathbb{Z}_p}\{p^{a_1-1}e_1, p^{-a_1+1}f_1, p^{a_2}e_2, p^{-a_2}f_2, \dots, p^{a_r}e_r, p^{-a_r}f_r\} \oplus Z$$

so that $\Lambda_2 \supsetneq \Lambda_1$ is a vertex lattice of type $t_{\max}$. Repeat until all the exponents reach 0. □

5.2 Special lattices

We now define a family of special lattices in V_K. In § 6 we will show that these special lattices are in bijection with the set $p^{\mathbb{Z}}\backslash \mathrm{RZ}(k)$.

In fact, we will need a similar result for any finitely generated extension k' of k. Let W' be the Cohen ring of k', let $K' = W'[1/p]$ be its fraction field, and let $\sigma : K' \to K'$ be any lift of Frobenius. Define a σ-linear operator $\Phi = b \circ \sigma$ on $V_{K'}$. If $L \subset V_{K'}$ is any W'-submodule, let $\Phi_*(L) \subset V_{K'}$ be the W'-submodule generated by $\Phi(L)$.

DEFINITION 5.2.1. A *special lattice* $L \subset V_{K'}$ is a self-dual W'-lattice such that

$$(L + \Phi_*(L))/L \xrightarrow{\sim} W'/pW'.$$

The following proposition implies that for every special lattice $L \subset V_K$ there is a vertex lattice $\Lambda \subset V_K^\Phi$ with $\Lambda_W^\vee \subset L \subset \Lambda_W$. In fact, there is a unique minimal such Λ, denoted $\Lambda(L)$. The proof is identical to that of [RTW14, Proposition 4.1] and [Vol10, Lemma 2.1], and so is omitted here.

PROPOSITION 5.2.2. *Let $L \subset V_K$ be a special lattice. If we define*

$$L^{(r)} = L + \Phi(L) + \cdots + \Phi^r(L),$$

then there is a (necessarily unique) integer $1 \leqslant d \leqslant t_{\max}/2$ such that

$$L = L^{(0)} \subsetneq L^{(1)} \subsetneq \cdots \subsetneq L^{(d)} = L^{(d+1)}.$$

Moreover, the W-module $L^{(r+1)}/L^{(r)}$ has length 1 for all $r < d$, and

$$\Lambda(L) = \{x \in L^{(d)} : \Phi(x) = x\} \subset V_K^\Phi$$

is a vertex lattice of type $2d$ satisfying $\Lambda(L)^\vee = \{x \in L : \Phi(x) = x\}$.

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5.3 The variety S_Λ

We next attach to a vertex lattice $\Lambda \subset V_K^\Phi$ a k-variety S_Λ parametrizing certain special lattices.

5.3.1 Define an $\mathbb{F}_p$-vector space $\Omega_0 = \Lambda/\Lambda^\vee$ of dimension t_Λ. The quadratic form pQ on Λ is $\mathbb{Z}_p$-valued, and its reduction modulo p makes Ω_0 into a non-degenerate quadratic space over $\mathbb{F}_p$. Set

$$\Omega := \Omega_0 \otimes_{\mathbb{F}_p} k \xrightarrow{\sim} \Lambda_W/\Lambda_W^\vee$$

with its Frobenius operator $\mathrm{id} \otimes \sigma = \Phi$. Note that Ω_0 cannot admit a Lagrangian (= totally isotropic of dimension $t_\Lambda/2$) subspace. Indeed, if such a subspace $\mathscr{L} \subset \Omega_0$ existed, then $\Lambda^\vee + \mathscr{L} \subset V_K^\Phi$ would be a vertex lattice of type 0, contradicting Proposition 5.1.2. In fact, Ω_0 is characterized up to isomorphism as the unique non-degenerate quadratic space of dimension t_Λ that does *not* admit a Lagrangian subspace.

The *orthogonal Grassmannian* $\mathrm{OGr}(\Omega)$ is the moduli space of Lagrangian subspaces $\mathscr{L} \subset \Omega$. More precisely, an R-point of $\mathrm{OGr}(\Omega)$ is a totally isotropic local direct summand $\mathscr{L} \subset \Omega \otimes_k R$ of rank $t_\Lambda/2$. Denote by $S_\Lambda \subset \mathrm{OGr}(\Omega)$ the reduced closed subscheme with k-points

$$\begin{aligned} S_\Lambda(k) \;&=\; \left\{ \text{Lagrangians } \mathscr{L} \subset \Omega : \dim_k(\mathscr{L} + \Phi(\mathscr{L})) = \frac{t_\Lambda}{2} + 1 \right\} \\ &\xrightarrow{\sim} \{\text{special lattices } L \subset V_K : \Lambda_W^\vee \subset L \subset \Lambda_W\}. \end{aligned}$$

Proposition 5.3.2. *The k-schemeS_Λ has two connected components $S_\Lambda = S_\Lambda^+ \sqcup S_\Lambda^-$. The two components are isomorphic, and each is projective and smooth of dimension $(t_\Lambda/2) - 1$.*

Proof. All of the claims are included in [HP14, Proposition 3.6], except for the isomorphism $S_\Lambda^+ \xrightarrow{\sim} S_\Lambda^-$. Pick any $g \in \mathrm{O}(\Omega_0)(\mathbb{F}_p)$ with $\det(g) = -1$. The natural action of g on $\mathrm{OGr}(\Omega)$ leaves S_Λ invariant, and the discussion of [HP14, § 3.2] shows that g interchanges S_Λ^+ with S_Λ^-. □

6. Structure of the spinor similitude Rapoport–Zink space

We will determine explicitly the structure of the reduced k-scheme $\mathrm{RZ}^{\mathrm{red}}$ underlying the formal W-scheme RZ of § 4.3.2. More precisely, we will express $\mathrm{RZ}^{\mathrm{red}}$ as a union of closed subschemes $\mathrm{RZ}_\Lambda^{\mathrm{red}}$ indexed by vertex lattices, and then relate each $\mathrm{RZ}_\Lambda^{\mathrm{red}}$ to the variety S_Λ of § 5.3.1.

6.1 Closed subschemes defined by vertex lattices

Recall from §§ 4.3.1 and 4.3.3 the $\mathbb{Q}_p$-quadratic space of special quasi-endomorphisms $V_K^\Phi \subset \mathrm{End}(X_0)_\mathbb{Q}$, and the universal quasi-isogeny

$$\rho : X_0 \times_{\mathrm{Spf}(k)} \overline{\mathrm{RZ}} \dashrightarrow X \times_{\mathrm{RZ}} \overline{\mathrm{RZ}}.$$

6.1.1 Fix a vertex lattice $\Lambda \subset V_K^\Phi$, and denote by $\mathrm{RZ}_\Lambda \subset \mathrm{RZ}$ the closed [RZ96, Proposition 2.9] formal subscheme defined by the condition

$$\rho \circ \Lambda^\vee \circ \rho^{-1} \subset \mathrm{End}(X).$$

In other words, RZ_Λ is the locus where the quasi-endomorphisms $\rho \circ \Lambda^\vee \circ \rho^{-1}$ of X are actually integral. As in § 4.3.4, the subgroup $p^\mathbb{Z} \subset J_b(\mathbb{Q}_p)$ acts on RZ_Λ. Set

$$\mathrm{RZ}_\Lambda^{(\ell)} = \mathrm{RZ}_\Lambda \cap \mathrm{RZ}^{(\ell)},$$

so that $p^\mathbb{Z} \backslash \mathrm{RZ}_\Lambda \xrightarrow{\sim} \mathrm{RZ}_\Lambda^{(0)} \sqcup \mathrm{RZ}_\Lambda^{(1)}$, exactly as in (4.3.4.1).

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Proposition 6.1.2. *The reduced k-scheme underlying $\mathrm{RZ}_\Lambda^{(\ell)}$ is projective.*

Proof. Abbreviate $Z = Z(V)$ for the center of $C(V)$. Using the isomorphism

$$C(V)^{op} \otimes_Z C(V) \xrightarrow{\sim} \mathrm{End}_Z(D) \qquad (6.1.2.1)$$

of § 4.1.4, and the inclusion $\Lambda^\vee \subset V_K \subset C(V_K)^{op}$, we denote by

$$R \subset \mathrm{End}_{Z_K}(D_K)$$

the W-subalgebra generated by $\Lambda^\vee \otimes_{\mathbb{Z}_p} C(V) \subset \mathrm{End}_{Z_K}(D_K)$. The isomorphism (6.1.2.1) implies that R generates $\mathrm{End}_{Z_K}(D_K)$ as a K-vector space. Fix any maximal Z_W-order $\tilde{R}$ with

$$R \subset \tilde{R} \subset \mathrm{End}_{Z_K}(D_K).$$

As R and $\tilde{R}$ are both W-lattices in $\mathrm{End}_K(D_K)$, we have $p^m\tilde{R} \subset R$ for some positive integer m. Fix a W-lattice $\tilde{M} \subset D_K$ stable under the action of $\tilde{R}$. It is unique up to scaling by $Z_K^\times$.

Suppose $y \in \mathrm{RZ}_\Lambda^{(\ell)}(k)$, and use the quasi-isogeny $\rho_y : X_0 \dashrightarrow X_y$ to view

$$M_y = \mathbb{D}(X_y)(W)$$

as a W-lattice in $D_K = \mathbb{D}(X_0)(W)[1/p]$. On one hand, M_y is stable under the action of $C(V)$ defined by (6.1.2.1). Indeed, this action in precisely the action on M_y induced by the action of $C(V)^{op}$ on X_y and contravariant functoriality; see § 4.3.3. On the other hand, $\Lambda^\vee \subset \mathrm{End}(X_y)$ by the very definition of $\mathrm{RZ}_\Lambda^{(\ell)}$. Combining these, we see that M_y is stable under the action of R, and so we may define the $\tilde{R}$-stable lattice $\tilde{M}_y = \tilde{R} \cdot M_y$. By the uniqueness of $\tilde{M}$ up to scaling, there is an $a(y) \in Z_K^\times$ such that $\tilde{M}_y = a(y)\tilde{M}$, and so

$$p^m a(y)\tilde{M} \subset M_y \subset a(y)\tilde{M}.$$

First suppose that n is even, so that $Z_K = K^\times$. The perfect symplectic form ψ_δ on $C(V_W)$ induces a dual form on D_W, which satisfies

$$p^\ell \psi_\delta(D_W, D_W) = \psi_\delta(\tilde{M}_y, \tilde{M}_y) = a(y)^2 \psi_\delta(\tilde{M}, \tilde{M}).$$

Thus the p-adic valuation of $a(y)$ is constant as y varies, and we may choose $a = a(y)$ to be independent of y.

Now suppose that n is odd, so that $Z_K = K \times K$. Let $\epsilon_1, \epsilon_2 \in Z_K$ be the orthogonal idempotents. In this case the dual form on D_W satisfies

$$p^\ell \psi_\delta(\epsilon_i D_W, \epsilon_i D_W) = \psi_\delta(\epsilon_i \tilde{M}_y, \epsilon_i \tilde{M}_y) = a_i(y)^2 \psi_\delta(\epsilon_i \tilde{M}, \epsilon_i \tilde{M}),$$

where $a_i(y) = \epsilon_i(y)a(y)$. Again this shows that $a_i(y)$ has constant p-adic valuation as y varies, and we may take $a = a(y)$ to be independent of y.

In either case $ap^m\tilde{M} \subset M_y \subset a\tilde{M}$ for all y. Combining this bound on M_y and the projectivity result of [RZ96, Corollary 2.29], we see that the closed immersion $\mathrm{RZ} \hookrightarrow \mathrm{RZ}(X_0, \lambda_0)$ realizes the reduced scheme underlying $\mathrm{RZ}_\Lambda^{(\ell)}$ as a closed subscheme of a projective k-scheme. □

6.2 Special lattices and the points of RZ_Λ

Let k'/k be a finitely generated field extension. Let W' be the Cohen ring of k', set $K' = W'[1/p]$, and let $\sigma : W' \to W'$ be a lift of Frobenius chosen as in Proposition 2.4.8. This choice of σ determines an operator $F = b \circ \sigma$ on $D_{K'}$, and an operator $\Phi = b \circ \sigma$ on $V_{K'}$.

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6.2.1 As in the proof of Proposition 6.1.2, each $y \in \mathrm{RZ}(k')$ determines a W'-lattice

$$M_y = \mathbb{D}(X_y)(W') \subset D_{K'},$$

and a W'-submodule $M_{1,y} = F^{-1}(pM_y)$ as in § 2.1.5. Using the inclusion $V_{K'} \subset \mathrm{End}_{K'}(D_{K'})$ obtained from (4.1.5.1), define W'-lattices

$$\begin{aligned} L_y &= \{x \in V_{K'} : xM_{1,y} \subset M_{1,y}\}, \\ L_y^{\sharp} &= \{x \in V_{K'} : xM_y \subset M_y\}, \\ L_y^{\sharp\sharp} &= \{x \in V_{K'} : xM_{1,y} \subset M_y\}. \end{aligned}$$

The action of $p^{\mathbb{Z}}$ on $\mathrm{RZ}(k')$ rescales the lattices M_y and $M_{1,y}$, and hence the three lattices defined above depend only on the image of y in $p^{\mathbb{Z}}\backslash \mathrm{RZ}(k')$.

Proposition 6.2.2. *For every $y \in \mathrm{RZ}(k')$ the lattice L_y is special (in the sense of § 5.2), and satisfies $\Phi_*(L_y) = L_y^{\sharp}$ and $L_y + L_y^{\sharp} = L_y^{\sharp\sharp}$. Moreover, $y \mapsto L_y$ establishes bijections*

$$\begin{aligned} p^{\mathbb{Z}}\backslash \mathrm{RZ}(k') &\xrightarrow{\sim} \{\textit{special lattices } L \subset V_{K'}\}, \\ p^{\mathbb{Z}}\backslash \mathrm{RZ}_{\Lambda}(k') &\xrightarrow{\sim} \{\textit{special lattices } L \subset V_{K'} : \Lambda_{W'}^{\vee} \subset L \subset \Lambda_{W'}\}. \end{aligned}$$

Proof. As in § 2.4 we have the refined affine Deligne–Lusztig set

$$X_{G,b,\mu^{\sigma},\sigma}(k') = \big\{g \in G(K') : g^{-1}b\sigma(g)\mu^{\sigma}(p)^{-1} \in G(W')\big\}/Q(W'),$$

where

$$Q(W') = G(W') \cap \mu(p^{-1})G(W')\mu(p).$$

Recalling the action $G \to \mathrm{SO}(V)$ defined by $g \bullet v = gvg^{-1}$, for each $g \in X_{G,b,\mu^{\sigma},\sigma}(k')$ define self-dual W'-lattices

$$L_g^{\sharp} = g \bullet V_{W'} \quad \text{and} \quad L_g = g\mu(p^{-1}) \bullet V_{W'}.$$

As the action of $p\bullet$ is trivial, these lattices depend only on the image of g modulo $p^{\mathbb{Z}}$.

First we show that $g \mapsto L_g$ establishes a bijection

$$p^{\mathbb{Z}}\backslash X_{G,b,\mu^{\sigma},\sigma}(k') \xrightarrow{\sim} \{\text{special lattices in } V_{K'}\}.$$

Given a $g \in X_{G,b,\mu^{\sigma},\sigma}(k')$, Lemma 4.2.2 (which holds with W replaced by W' throughout) implies

$$(L_g + L_g^{\sharp})/L_g \xrightarrow{\sim} W'/pW'.$$

Moreover, $g^{-1}b\sigma(g)\mu^{\sigma}(p^{-1}) \in G(W')$ implies

$$\Phi_*(L_g) = L_g^{\sharp}, \tag{6.2.2.1}$$

and so L_g is special. To prove injectivity, assume $L_g = L_h$. Applying Φ_* to both sides and using (6.2.2.1) shows that $L_g^{\sharp} = L_h^{\sharp}$. It follows that $h^{-1}g$ lies in the intersection in $G(K')$ of the stabilizers of $V_{W'}$ and $\mu(p^{-1}) \bullet V_{W'}$, which is $p^{\mathbb{Z}}Q(W')$. Thus $g = h$ in $p^{\mathbb{Z}}\backslash X_{G,b,\mu^{\sigma},\sigma}(k')$. For surjectivity, suppose L is a special lattice. Lemma 4.2.2 implies the existence of a $g \in G(K')$ such that

$$(\Phi_*(L), L) = (g \bullet V_{W'}, g\mu(p^{-1}) \bullet V_{W'}).$$

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This equality implies that $g^{-1}b\sigma(g)\mu^\sigma(p^{-1})$ stabilizes $V_{W'}$, and so lies in $p^{\mathbb{Z}}G(W')$. The relation $b \in G(W)\mu^\sigma(p)G(W)$ of Proposition 4.2.6 implies that

$$\eta_G(g^{-1}b\sigma(g)\mu^\sigma(p^{-1})) \in (W')^\times,$$

and so in fact $g^{-1}b\sigma(g)\mu^\sigma(p^{-1}) \in G(W')$. Thus we have found a $g \in X_{G,b,\mu^\sigma,\sigma}(k')$ with $L = L_g$.

By Corollary 3.2.3, there is bijection $\mathrm{RZ}(k') \xrightarrow{\sim} X_{G,b,\mu^\sigma,\sigma}(k')$, defined by sending the point $y \in \mathrm{RZ}(k')$ to the unique $g \in X_{G,b,\mu^\sigma,\sigma}(k')$ satisfying both

$$M_y = g \cdot D_{W'} \quad \text{and} \quad M_{1,y} = g \cdot p\mu(p^{-1})D_{W'}.$$

Assuming that y and g are related in this way, we claim that

$$(L_y^\sharp, L_y) = (L_g^\sharp, L_g). \tag{6.2.2.2}$$

To prove this, let $B = \{x \in V_{K'} : xD_{W'} \subset D_{W'}\}$. The inclusion $V_{W'} \subset B$ is obvious. For the other inclusion note that any $x \in B$ must have $Q(x) = x \circ x \in W'$, and so $V_{W'} \subset B \subset B^\vee \subset (V_{W'})^\vee$. The self-duality of $V_{W'}$ implies that equality holds throughout, and so

$$V_{W'} = \{x \in V_{K'} : xD_{W'} \subset D_{W'}\}.$$

Applying $g\bullet$ to both sides of this equality proves $L_y^\sharp = L_g^\sharp$, while applying $g\mu(p^{-1})\bullet$ to both sides proves $L_y = L_g$.

We have now established bijections

$$p^{\mathbb{Z}}\backslash \mathrm{RZ}(k') \xrightarrow{\sim} p^{\mathbb{Z}}\backslash X_{G,b,\mu^\sigma,\sigma}(k') \xrightarrow{\sim} \{\text{special lattices } L \subset V_{K'}\}.$$

The relation $\Phi_*(L_y^\sharp) = L_y$ follows from (6.2.2.1) and (6.2.2.2). We verify $L_y + L_y^\sharp = L_y^{\sharp\sharp}$ as follows. Using the calculations of § 4.2.1, one can show

$$\mu(p^{-1}) \bullet V_{W'} + V_{W'} = \{x \in V_{K'} : x\mu(p^{-1})D_{W'} \subset D_{W'}\}. \tag{6.2.2.3}$$

If $y \in \mathrm{RZ}(k')$ corresponds to $g \in X_{G,b,\mu^\sigma,\sigma}(k')$ under the bijection above, then applying $g\bullet$ to both sides of (6.2.2.3) yields

$$\begin{aligned} L_y^\sharp + L_y = L_g^\sharp + L_g &= \{x \in V_{K'} : (g^{-1}xg)\mu(p^{-1}) \cdot D_{W'} \subset D_{W'}\} \\ &= \{x \in V_{K'} : xM_y^1 \subset M_y\} \\ &= L_y^{\sharp\sharp}. \end{aligned}$$

Finally, a point $y \in p^{\mathbb{Z}}\backslash \mathrm{RZ}(k')$ lies in the subset $p^{\mathbb{Z}}\backslash \mathrm{RZ}_\Lambda(k')$ if and only if the quasi-endomorphisms $\Lambda^\vee \subset \mathrm{End}(D_{K'})$ stabilize both lattices $M_{1,y} \subset M_y$. This is equivalent to the condition $\Lambda^\vee \subset L_y \cap L_y^\sharp$, and so

$$\begin{aligned} p^{\mathbb{Z}}\backslash \mathrm{RZ}_\Lambda(k') &\xrightarrow{\sim} \{\text{special lattices } L \subset V_{K'} : \Lambda^\vee \subset L \cap \Phi_*(L)\} \\ &= \{\text{special lattices } L \subset V_{K'} : \Lambda^\vee \subset L\} \\ &= \{\text{special lattices } L \subset V_{K'} : \Lambda_{W'}^\vee \subset L \subset \Lambda_{W'}\}. \end{aligned}$$

Here we have used first the fact that all elements of $\Lambda^\vee$ are fixed by Φ, and then the fact that special lattices are self-dual. This completes the proof of Proposition 6.2.2. ☐

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Corollary 6.2.3. *We have*

$$\mathrm{RZ}(k) = \bigcup_{\substack{\Lambda \\ t_\Lambda = t_{\max}}} \mathrm{RZ}_\Lambda(k).$$

Proof. Suppose $y \in \mathrm{RZ}(k)$. Let $L_y \subset V_K$ be the corresponding special lattice of Proposition 6.2.2, and let $\Lambda(L_y)$ be the vertex lattice of Proposition 5.2.2. By Proposition 5.1.2 there is a vertex lattice $\Lambda \supset \Lambda(L_y)$ with $t_\Lambda = t_{\max}$, and clearly

$$\Lambda^\vee \subset \Lambda(L_y)^\vee = \{x \in L_y : \Phi(x) = x\} \subset L_y.$$

The self-duality of L_y implies $\Lambda_W^\vee \subset L_y \subset \Lambda_W$, and so $y \in \mathrm{RZ}_\Lambda(k)$. □

Corollary 6.2.4. *For any vertex lattices Λ_1 and Λ_2, we have*

$$\mathrm{RZ}_{\Lambda_1}(k) \cap \mathrm{RZ}_{\Lambda_2}(k) = \begin{cases} \mathrm{RZ}_{\Lambda_1 \cap \Lambda_2}(k) & \text{if } \Lambda_1 \cap \Lambda_2 \text{ is a vertex lattice,} \\ \emptyset & \text{otherwise.} \end{cases}$$

Proof. The proof is the same as [RTW14, Proposition 4.3(ii)]. □

6.3 Comparison of RZ_Λ and S_Λ

Fix a vertex lattice $\Lambda \subset V_K^\Phi$. Comparing Proposition 6.2.2 with the bijection of § 5.3.1 yields bijections

$$p^{\mathbb{Z}} \backslash \mathrm{RZ}_\Lambda(k) \xrightarrow{\sim} \{\text{special lattices } L \subset V_K : \Lambda_W^\vee \subset L \subset \Lambda_W\} \xrightarrow{\sim} S_\Lambda(k),$$

and similarly for any finitely generated field extension k'/k.

Theorem 6.3.1. *Let $\mathrm{RZ}_\Lambda^{\mathrm{red}}$ be the reduced k-scheme underlying RZ_Λ. There is a unique isomorphism of k-schemes*

$$p^{\mathbb{Z}} \backslash \mathrm{RZ}_\Lambda^{\mathrm{red}} \xrightarrow{\sim} S_\Lambda$$

inducing the above bijection on k-points.

Proof. First we construct a morphism $\mathrm{RZ}_\Lambda^{\mathrm{red}} \to S_\Lambda$. Suppose we are given an R-point $y \in \mathrm{RZ}_\Lambda^{\mathrm{red}}(R)$ for some reduced k-algebra R of finite type. Pulling back the universal object of § 4.3.3 yields a triple (X_y, ρ_y, λ_y) over R in which X_y is a p-divisible group, λ_y is a principal polarization, and $\rho_y : X_{0/R} \dashrightarrow X_y$ is a quasi-isogeny. Moreover, $x \mapsto \rho \circ x \circ \rho^{-1}$ defines a $\mathbb{Z}_p$-module map

$$\Lambda^\vee \to \rho \circ \Lambda^\vee \circ \rho^{-1} \subset \mathrm{End}(X_y).$$

Let $\mathscr{D}_y = \mathbb{D}(X_y)(R)$ be the contravariant crystal of X_y evaluated at the trivial divided power thickening $R \to R$, and let $\mathrm{Fil}^1(\mathscr{D}_y) \subset \mathscr{D}_y$ be the Hodge filtration. The locally free R-modules $\mathrm{Fil}^1(\mathscr{D}_y) \subset \mathscr{D}_y$ depend functorially on X_y, and so $\Lambda^\vee \to \mathrm{End}(X_y)$ induces R-module maps

$$\phi^\sharp : (\Lambda^\vee / p\Lambda^\vee) \otimes_{\mathbb{F}_p} R \to \mathrm{End}_R(\mathscr{D}_y)$$

and

$$\phi^{\sharp\sharp} : (\Lambda^\vee / p\Lambda^\vee) \otimes_{\mathbb{F}_p} R \to \mathrm{End}_R(\mathscr{D}_y^1)$$

with $\ker(\phi^\sharp) \subset \ker(\phi^{\sharp\sharp})$.

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The bilinear form on $\Lambda^\vee$ induces an R-valued bilinear form on $(\Lambda^\vee/p\Lambda^\vee)\otimes_{\mathbb{F}_p} R$, and any $x_1, x_2 \in (\Lambda^\vee/p\Lambda^\vee)\otimes_{\mathbb{F}_p} R$ satisfy

$$x_1 \circ x_2 + x_2 \circ x_1 = [x_1, x_2] \in R$$

as endomorphisms of $\mathscr{D}_y$. In particular, if $x_1 \in \ker(\phi^{\sharp\sharp})$ then $[x_1, x_2] = 0$, as the value of the scalar $[x_1, x_2]$ can be computed from its action on $\mathrm{Fil}^1(\mathscr{D}_y)$, which is obviously trivial. This shows that $\ker(\phi^{\sharp\sharp})$ is contained in the radical of the quadratic space $(\Lambda^\vee/p\Lambda^\vee)\otimes_{\mathbb{F}_p} R$, which is $(p\Lambda/p\Lambda^\vee)\otimes_{\mathbb{F}_p} R$. Recalling the k-quadratic space $\Omega = (\Lambda/\Lambda^\vee)\otimes_{\mathbb{F}_p} k$ from § 5.3.1, let

$$\mathscr{L}_y^\sharp \subset \mathscr{L}_y^{\sharp\sharp} \subset \Omega \otimes_k R$$

be the images of $\ker(\phi^\sharp) \subset \ker(\phi^{\sharp\sharp})$ under the isomorphism

$$(p\Lambda/p\Lambda^\vee)\otimes_{\mathbb{F}_p} R \xrightarrow{p^{-1}\otimes \mathrm{id}} \Omega \otimes_k R.$$

Suppose for the moment that $R = k$. Recalling from § 6.2.1 (with $k' = k$) the W-modules $M_{1,y} \subset M_y$, there is an isomorphism $\mathscr{D}_y \xrightarrow{\sim} M_y/pM_y$ identifying $\mathrm{Fil}^1(\mathscr{D}_y) \xrightarrow{\sim} M_{1,y}/pM_y$. The subspaces

$$\mathscr{L}_y^\sharp \subset \mathscr{L}_y^{\sharp\sharp} \subset \Omega \xrightarrow{\sim} (\Lambda_W/\Lambda_W^\vee)$$

correspond to lattices $\Lambda_W^\vee \subset L_y^\sharp \subset L_y^{\sharp\sharp} \subset \Lambda_W$, and tracing through the definitions shows that these are none other than the lattices

$$L_y^\sharp = \{x \in V_K : xM_y = M_y\} \quad \text{and} \quad L_y^{\sharp\sharp} = \{x \in V_K : xM_{1,y} \subset M_y\}$$

appearing in § 6.2.1. Comparison with Proposition 6.2.2 shows that $L_y^{\sharp\sharp} = L_y + L_y^\sharp$, where L_y is the special lattice

$$L_y = \{x \in V_K : xM_y^1 \subset M_y^1\}$$

satisfying $\Phi(L_y) = L_y^\sharp$. Noting that $\Lambda_W^\vee \subset L_y \subset \Lambda_W$, we denote by $\mathscr{L}_y \subset \Omega$ the k-subspace corresponding to L_y.

The self-duality of the W-lattices L_y and $L_y^\sharp$ implies that the corresponding k-subspaces $\mathscr{L}_y$ and $\mathscr{L}_y^\sharp$ of Ω are maximal isotropic, and so have dimension $t_\Lambda/2$. The specialness of L_y also implies that $\mathscr{L}_y^{\sharp\sharp} = \mathscr{L}_y + \mathscr{L}_y^\sharp$ has dimension $(t_\Lambda/2)+1$. It follows that $(\mathscr{L}_y^{\sharp\sharp})^\perp \subset \mathscr{L}_y^{\sharp\sharp}$ with codimension 2, and that the quotient $\mathscr{L}_y^{\sharp\sharp}/(\mathscr{L}_y^{\sharp\sharp})^\perp$ is a hyperbolic plane over k. The subspaces $\mathscr{L}_y/(\mathscr{L}_y^{\sharp\sharp})^\perp$ and $\mathscr{L}_y^\sharp/(\mathscr{L}_y^{\sharp\sharp})^\perp$ are its unique isotropic lines.

Now return to a general reduced R of finite type. The submodule $\mathscr{L}_y^\sharp \subset \Omega\otimes_k R$ is a totally isotropic local direct summand of rank $t_\Lambda/2$, and $\mathscr{L}_y^{\sharp\sharp} \subset \Omega\otimes_k R$ is a local direct summand of rank $(t_\Lambda/2)+1$. Indeed, by [Lan02, Exercise X.16] it suffices to check these properties fiber by fiber at the closed points of $\mathrm{Spec}(R)$, which is precisely what we did in the $R = k$ case above.

By similar reasoning the quotient $\mathscr{L}_y^{\sharp\sharp}/(\mathscr{L}_y^{\sharp\sharp})^\perp$ is a hyperbolic plane over R, and so contains exactly two isotropic local direct summands of rank 1. One of them is $\mathscr{L}_y^\sharp/(\mathscr{L}_y^{\sharp\sharp})^\perp$, and the other has the form $\mathscr{L}_y/(\mathscr{L}_y^{\sharp\sharp})^\perp$ for a uniquely determined Lagrangian $\mathscr{L}_y \subset \Omega\otimes_k R$. By again reducing to the $R = k$ case treated above, we see that $\Phi(\mathscr{L}_y) = \mathscr{L}_y^\sharp$, and so

$$\mathscr{L}_y + \Phi(\mathscr{L}_y) = \mathscr{L}_y + \mathscr{L}_y^\sharp = \mathscr{L}_y^{\sharp\sharp}$$

is a local direct summand of rank $(t_\Lambda/2)+1$. In other words, $\mathscr{L}_y \in S_\Lambda(R)$.

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The k-scheme $\mathrm{RZ}_\Lambda^{\mathrm{red}}$ is itself reduced and locally of finite type, and so the rule $y \mapsto \mathscr{L}_y$ defines (at last) the promised morphism $\mathrm{RZ}_\Lambda^{\mathrm{red}} \to S_\Lambda$. It is clear from the construction that the morphism descends to

$$p^{\mathbb{Z}}\backslash \mathrm{RZ}_\Lambda^{\mathrm{red}} \to S_\Lambda \tag{6.3.1.1}$$

and induces the desired bijection on k-points. In fact, the generality of Proposition 6.2.2 shows that this morphism induces a bijection

$$p^{\mathbb{Z}}\backslash \mathrm{RZ}_\Lambda^{\mathrm{red}}(k') \xrightarrow{\sim} S_\Lambda(k')$$

for any extension field k'/k. In particular (6.3.1.1) is birational and quasi-finite. It is a proper morphism, as Proposition 6.1.2 implies that $p^{\mathbb{Z}}\backslash \mathrm{RZ}_\Lambda^{\mathrm{red}}$ is projective. The variety S_Λ is smooth by Proposition 5.3.2, and so Zariski's main theorem implies that (6.3.1.1) is an isomorphism. □

Recall from Proposition 5.3.2 that S_Λ has two connected components. The two components are isomorphic, and are labeled (arbitrarily) as S_Λ^+ and S_Λ^-.

Corollary 6.3.2. *The reduced scheme $\mathrm{RZ}_\Lambda^{(\ell),\mathrm{red}}$ underlying $\mathrm{RZ}_\Lambda^{(\ell)}$ is connected and non-empty, and is isomorphic to $S_\Lambda^\pm$.*

Proof. The action of $p^{\mathbb{Z}}$ on RZ_Λ identifies $\mathrm{RZ}_\Lambda^{(\ell)} \xrightarrow{\sim} \mathrm{RZ}_\Lambda^{(\ell+2)}$, and so it suffices to assume $\ell \in \{0,1\}$. Moreover, we know from Proposition 6.2.2 that

$$\mathrm{RZ}_\Lambda^{(0),\mathrm{red}} \sqcup \mathrm{RZ}_\Lambda^{(1),\mathrm{red}} \xrightarrow{\sim} p^{\mathbb{Z}}\backslash \mathrm{RZ}_\Lambda^{\mathrm{red}} \xrightarrow{\sim} S_\Lambda^+ \sqcup S_\Lambda^-.$$

This leaves two possibilities: either each of $\mathrm{RZ}_\Lambda^{(0),\mathrm{red}}$ and $\mathrm{RZ}_\Lambda^{(1),\mathrm{red}}$ is connected and isomorphic to $S_\Lambda^\pm$, or one of them is empty and the other has two connected components. To complete the proof of the corollary, it therefore suffices to show that $\mathrm{RZ}_\Lambda^{(0),\mathrm{red}}$ and $\mathrm{RZ}_\Lambda^{(1),\mathrm{red}}$ are non-empty.

First suppose that Λ has type $t_\Lambda = 2$. In this case one can easily check that S_Λ consists of two points, and so the same is true of $p^{\mathbb{Z}}\backslash \mathrm{RZ}_\Lambda^{\mathrm{red}}$. There is a W-basis $e_1, \dots, e_n$ of Λ such that the matrix of the bilinear form is

$$\begin{pmatrix} u_1p^{-1} & & & & \\ & u_2p^{-1} & & & \\ & & u_3 & & \\ & & & \ddots & \\ & & & & u_n \end{pmatrix}$$

for some $u_1, \dots, u_n \in \mathbb{Z}_p^\times$.

Let $r_i \in \mathrm{O}(V_K^\Phi)(\mathbb{Q}_p)$ be the reflection with $e_i \mapsto -e_i$ and $e_j \mapsto e_j$ for all $j \neq i$. The *spinor norm* of r_1r_3, in the sense of [Kit93], is

$$\frac{u_1}{2p} \cdot \frac{u_3}{2} = Q(e_1)Q(e_3) \in \mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^2.$$

The spinor norm of [Kit93] is compatible with the spinor similitude of § 4.1.3, in the sense that any lift of $r_1r_3 \in \mathrm{SO}(V_K^\Phi)(\mathbb{Q}_p)$ to

$$g \in \mathrm{GSpin}(V_K^\Phi)(\mathbb{Q}_p) \xrightarrow{\sim} J_b(\mathbb{Q}_p)$$

satisfies $\eta_b(g) = Q(e_1)Q(e_3)$ up to scaling by $(\mathbb{Q}_p^\times)^2$. Thus

$$\mathrm{ord}_p(\eta_b(g)) \equiv 1 \pmod 2.$$

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By this calculation and the discussion of § 4.3.4, g acts on

$$p^{\mathbb{Z}}\backslash \mathrm{RZ}_\Lambda^{\mathrm{red}} \xrightarrow{\sim} \mathrm{RZ}_\Lambda^{(0),\mathrm{red}} \sqcup \mathrm{RZ}_\Lambda^{(1),\mathrm{red}}$$

and interchanges the two subsets on the right. Thus each is non-empty, and in fact each is a single reduced point.

For general Λ, Proposition 5.1.4 allows us to pick a type 2 vertex lattice $\Lambda_2 \subset \Lambda$. Combining Corollary 6.2.4 with the paragraph above shows that $\emptyset \neq \mathrm{RZ}_{\Lambda_2}^{(\ell),\mathrm{red}} \subset \mathrm{RZ}_\Lambda^{(\ell),\mathrm{red}}$. □

6.4 The main result

We can now prove our main result on the structure of the reduced scheme

$$\mathrm{RZ}^{\mathrm{red}} = \bigsqcup_{\ell \in \mathbb{Z}} \mathrm{RZ}^{(\ell),\mathrm{red}}.$$

For each vertex lattice Λ, recall that $\mathrm{RZ}_\Lambda^{(\ell),\mathrm{red}}$ is the reduced k-scheme underlying the formal W-scheme $\mathrm{RZ}_\Lambda^{(\ell)} = \mathrm{RZ}_\Lambda \cap \mathrm{RZ}^{(\ell)}$.

Theorem 6.4.1. *For each ℓ, the k-scheme $\mathrm{RZ}^{(\ell),\mathrm{red}}$ is connected. Each closed subscheme $\mathrm{RZ}_\Lambda^{(\ell),\mathrm{red}}$ is projective and smooth of dimension $(t_\Lambda/2) - 1$, and is isomorphic to $S_\Lambda^{\pm}$. The irreducible components of $\mathrm{RZ}^{(\ell),\mathrm{red}}$ are precisely the closed subschemes $\mathrm{RZ}_\Lambda^{(\ell),\mathrm{red}}$ as Λ runs over the vertex lattices of maximal type $t_\Lambda = t_{\max}$, and, in particular, $\mathrm{RZ}^{\mathrm{red}}$ is equidimensional with*

$$\dim(\mathrm{RZ}^{\mathrm{red}}) = \frac{1}{2}\begin{cases} n-4 & \text{if } n \text{ is even and } \det(V_{\mathbb{Q}_p}) = (-1)^{n/2}, \\ n-3 & \text{if } n \text{ is odd}, \\ n-2 & \text{if } n \text{ is even and} \det(V_{\mathbb{Q}_p}) \neq (-1)^{n/2}. \end{cases}$$

Proof. For any vertex lattice $\Lambda \subset V_K^{\Phi}$, Corollary 6.3.2 and Proposition 5.3.2 tell us that

$$\mathrm{RZ}_\Lambda^{(\ell),\mathrm{red}} \xrightarrow{\sim} S_\Lambda^{\pm}$$

is irreducible, projective, and smooth of dimension $(t_\Lambda/2) - 1$.

Corollary 6.2.3 implies that

$$\mathrm{RZ}^{(\ell),\mathrm{red}} = \bigcup_{\substack{\Lambda \\ t_\Lambda = t_{\max}}} \mathrm{RZ}_\Lambda^{(\ell),\mathrm{red}}, \tag{6.4.1.1}$$

and so the irreducible components of $\mathrm{RZ}^{(\ell),\mathrm{red}}$ are precisely the $\mathrm{RZ}_\Lambda^{(\ell),\mathrm{red}}$ with Λ of maximal type $t_{\max}$. This proves all parts of the claim, except for the connectedness of $\mathrm{RZ}^{(\ell),\mathrm{red}}$.

Suppose that $\Lambda_1 \sim \Lambda_2$ are adjacent vertex lattices. If $\Lambda_1 \subset \Lambda_2$ then Corollary 6.2.4 implies that $\mathrm{RZ}_{\Lambda_1}^{(\ell),\mathrm{red}}$ and $\mathrm{RZ}_{\Lambda_2}^{(\ell),\mathrm{red}}$ lie on the same connected component of $\mathrm{RZ}^{(\ell)}$. Of course similar remarks hold if $\Lambda_2 \subset \Lambda_1$. Proposition 5.1.5 shows that any two vertex lattices are connected by a chain of adjacent vertex lattices, and so all of the closed subschemes $\mathrm{RZ}_\Lambda^{(\ell),\mathrm{red}}$ lie on the same connected component of $\mathrm{RZ}^{(\ell),\mathrm{red}}$. The equality (6.4.1.1) now shows that $\mathrm{RZ}^{(\ell),\mathrm{red}}$ is connected. □

Remark 6.4.2. When n is odd, the center of $C^+(V)$ is $\mathbb{Z}_p$. When n is even, the center of $C^+(V)$ is the maximal $\mathbb{Z}_p$-order in $F = \mathbb{Q}_p[x]/(x^2 - \Delta)$, where $\Delta = (-1)^{n/2}\det(V_{\mathbb{Q}_p})$. Thus for n even,

$$\dim(\mathrm{RZ}^{\mathrm{red}}) = \begin{cases} (n/2) - 2 & \text{if } F \xrightarrow{\sim} \mathbb{Q}_p \times \mathbb{Q}_p, \\ (n/2) - 1 & \text{if } F \xrightarrow{\sim} \mathbb{Q}_{p^2}. \end{cases}$$

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Remark 6.4.3. The dimension formula of Theorem 6.4.1 verifies a case of a conjecture of Chai and of Rapoport [GHKR06, Rap05]. According to this conjecture, we should have

$$\dim(\mathrm{RZ}^{\mathrm{red}}) = \langle \rho, \mu - \nu_b \rangle - \tfrac{1}{2}\mathrm{def}_G(b).$$

Here, μ is assumed to be a dominant representative of the conjugacy class $\{\mu\}$, ρ is the half sum of all absolute positive roots of G, and, by definition,

$$\mathrm{def}_G(b) = \mathrm{rank}_{\mathbb{Q}_p}(G) - \mathrm{rank}_{\mathbb{Q}_p}(J_b).$$

In our case, $\langle \rho, \mu - \nu_b \rangle = \langle \rho, \mu \rangle = (n-2)/2$, while we have $\mathrm{def}_G(b) = 2$, 1, or 0, in the three cases listed in the theorem (in that order). Indeed, $\mathrm{def}_G(b)$ is the difference between the Witt indices of $V_{\mathbb{Q}_p}$ and V_K^{Φ}, and this can be determined as in the proof of Proposition 5.1.2. The above dimension formula has recently been shown, for all the (unramified) Rapoport–Zink spaces of Hodge type defined in this paper, by Hamacher [Ham16] and by Zhang [Zha15].

6.5 The Bruhat–Tits stratification

Using the collection of closed subschemes $\mathrm{RZ}^{\mathrm{red}}_{\Lambda}$ of $\mathrm{RZ}^{\mathrm{red}}$, we explain how to define a stratification of $\mathrm{RZ}^{\mathrm{red}}$ in which each stratum is the Deligne–Lusztig variety determined by a Coxeter element in a special orthogonal group over $\mathbb{F}_p$.

6.5.1 Recall from Corollary 6.2.4 that $\Lambda' \subset \Lambda$ implies $\mathrm{RZ}^{\mathrm{red}}_{\Lambda'} \subset \mathrm{RZ}^{\mathrm{red}}_{\Lambda}$. For each vertex lattice Λ define the *Bruhat–Tits stratum*

$$\mathrm{BT}_{\Lambda} = \mathrm{RZ}^{\mathrm{red}}_{\Lambda} \smallsetminus \bigcup_{\Lambda' \subsetneq \Lambda} \mathrm{RZ}^{\mathrm{red}}_{\Lambda'}.$$

It is an open and dense subscheme of $\mathrm{RZ}^{\mathrm{red}}_{\Lambda}$, and

$$\mathrm{RZ}^{\mathrm{red}}_{\Lambda} = \bigcup_{\Lambda' \subset \Lambda} \mathrm{BT}_{\Lambda}$$

defines a stratification of $\mathrm{RZ}^{\mathrm{red}}_{\Lambda}$ as a disjoint union of locally closed subschemes.

6.5.2 Similarly,

$$\mathrm{RZ}^{\mathrm{red}} = \bigcup_{\text{all } \Lambda} \mathrm{BT}_{\Lambda}$$

defines a stratification of $\mathrm{RZ}^{\mathrm{red}}$ as a disjoint union of locally closed subschemes. This is the GSpin analogue of the Bruhat–Tits stratification for unitary Rapoport–Zink spaces found in [VW11, RTW14]. However, this terminology should be taken with a grain of salt: unlike in [VW11], the strata here are not in bijection with the vertices in the Bruhat–Tits building of the group J_b^{der}. See [HP14, § 2.7] for more details in the special case $n = 6$.

6.5.3 For a special lattice $L \subset V_K$, recall from Proposition 5.2.2 the vertex lattice $\Lambda(L)$ characterized by

$$\Lambda(L)^{\vee} = \{x \in L : \Phi(x) = x\}.$$

If we rewrite the bijections of Theorem 6.3.1 and § 5.3.1 as

$$\begin{aligned} p^{\mathbb{Z}} \backslash \mathrm{RZ}^{\mathrm{red}}_{\Lambda}(k) &\xrightarrow{\sim} S_{\Lambda}(k) \\ &\xrightarrow{\sim} \{\text{special lattices } L \subset V_K : \Lambda^{\vee} \subset L\} \\ &= \{\text{special lattices } L \subset V_K : \Lambda(L) \subset \Lambda\}, \end{aligned}$$

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the inclusion $\mathrm{BT}_\Lambda \subset \mathrm{RZ}_\Lambda^{\mathrm{red}}$ identifies

$$\begin{aligned} p^{\mathbb{Z}}\backslash \mathrm{BT}_\Lambda(k) &\xrightarrow{\sim} \{\text{special lattices } L \subset V_K : \Lambda(L) = \Lambda\} \\ &= \{\text{special lattices } L \subset V_K : L + \Phi(L) + \cdots + \Phi^d(L) = \Lambda_W\}. \end{aligned}$$

6.5.4 Fix a vertex lattice Λ of type $t_\Lambda = 2d$, and recall from § 5.3.1 the $2d$-dimensional $\mathbb{F}_p$-quadratic space $\Omega_0 = \Lambda/\Lambda^\vee$. Set

$$\Omega = \Omega_0 \otimes_{\mathbb{F}_p} k \xrightarrow{\sim} \Lambda_W/\Lambda_W^\vee,$$

and let $\Phi = \mathrm{id} \otimes \sigma$ be the Frobenius on Ω.

We recall the set-up of [HP14, § 3.2]. Fix a basis $\{e_1, \ldots, e_d, f_1, \ldots, f_d\}$ of Ω in such a way that $\mathrm{Span}_k\{e_1, \ldots, e_d\}$ and $\mathrm{Span}_k\{f_1, \ldots, f_d\}$ are totally isotropic, $[e_i, f_j] = \delta_{i,j}$, and the Frobenius Φ fixes $e_1, \ldots, e_{d-1}, f_1, \ldots, f_{d-1}$ but interchanges $e_d \leftrightarrow f_d$. This choice of basis determines a maximal Φ-stable torus $T \subset \mathrm{SO}(\Omega)$.

The isotropic flags $\mathscr{F}_\bullet^+$ and $\mathscr{F}_\bullet^-$ in Ω, defined by

$$\begin{aligned} \mathscr{F}_i^\pm &= \mathrm{Span}_k\{e_1, \ldots, e_i\}, \quad \text{for } 1 \leqslant i \leqslant d-1, \\ \mathscr{F}_d^+ &= \mathrm{Span}_k\{e_1, \ldots, e_{d-1}, e_d\}, \\ \mathscr{F}_d^- &= \mathrm{Span}_k\{e_1, \ldots, e_{d-1}, f_d\}, \end{aligned}$$

satisfy $\mathscr{F}_\bullet^\pm = \Phi(\mathscr{F}_\bullet^\mp)$, and have the same stabilizer $B \subset \mathrm{SO}(\Omega)$. It is a Φ-stable Borel subgroup containing T. The corresponding set of simple reflections in the Weyl group $W = N(T)/T$ is $\{s_1, \ldots, s_{d-2}, t^+, t^-\}$, where

- s_i interchanges $e_i \leftrightarrow e_{i+1}$ and $f_i \leftrightarrow f_{i+1}$, and fixes the other basis elements;
- t^+ interchanges $e_{d-1} \leftrightarrow e_d$ and $f_{d-1} \leftrightarrow f_d$, and fixes the other basis elements;
- t^- interchanges $e_{d-1} \leftrightarrow f_d$ and $f_{d-1} \leftrightarrow e_d$, and fixes the other basis elements.

Notice that $\Phi(s_i) = s_i$, and $\Phi(t^\pm) = t^\mp$, and so the products

$$w^\pm = t^\mp s_{d-2} \cdots s_2 s_1 \in W$$

are *Coxeter elements*: products of exactly one representative from each Φ-orbit in the set of simple reflections above.

6.5.5 The *Deligne–Lusztig variety*

$$X_B(w^\pm) = \{g \in \mathrm{SO}(\Omega)/B : \mathrm{inv}(g, \Phi(g)) = w^\pm\}$$

is a smooth quasi-projective k-variety of dimension $d-1$. Here inv is the relative position invariant

$$\mathrm{SO}(\Omega)/B \times \mathrm{SO}(\Omega)/B \xrightarrow{(g_1,g_2)\mapsto g_1^{-1}g_2} B\backslash \mathrm{SO}(\Omega)/B \xrightarrow{\sim} W.$$

Theorem 6.5.6. *There are isomorphisms $X_B(w^+) \xrightarrow{\sim} X_B(w^-)$, and*

$$p^{\mathbb{Z}}\backslash \mathrm{BT}_\Lambda \xrightarrow{\sim} X_B(w^+) \sqcup X_B(w^-).$$

Proof. Recall that the k-variety

$$S_\Lambda(k) = \{\text{Lagrangians } \mathscr{L} \subset \Omega : \dim_k(\mathscr{L} + \Phi(\mathscr{L})) = d+1\}$$

has two connected components X_Λ^+ and X_Λ^-, interchanged by the action of any $g \in \mathrm{O}(\Omega_0)(\mathbb{F}_p)$ with $\det(g) = -1$.

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After possibly relabeling $\mathscr{F}_\bullet^+$ and $\mathscr{F}_\bullet^-$, [HP14, Proposition 3.8] gives an open immersion $X_B(w^\pm) \to S_\Lambda^\pm$ defined by $g \mapsto g\mathscr{F}_d^\pm$. Thus

$$X_B(w^+) \sqcup X_B(w^-) \subset S_\Lambda$$

as an open subset with k-points

$$\begin{aligned} X_B(w^\pm)(k) &= \{\mathscr{L} \in S_\Lambda^\pm(k) : \mathscr{L} \cap \Phi(\mathscr{L}) \cap \Phi^2(\mathscr{L}) \cap \cdots \cap \Phi^d(\mathscr{L}) = 0\} \\ &= \{\mathscr{L} \in S_\Lambda^\pm(k) : \mathscr{L} + \Phi(\mathscr{L}) + \Phi^2(\mathscr{L}) + \cdots + \Phi^d(\mathscr{L}) = \Omega\}. \end{aligned}$$

The action of any g as above interchanges $X_B(w^+)$ with $X_B(w^-)$.

By § 6.5.3, we have bijections

$$\begin{aligned} p^{\mathbb{Z}}\backslash \mathrm{BT}_\Lambda(k) &\xrightarrow{\sim} \{\mathscr{L} \in S_\Lambda(k) : \mathscr{L} + \Phi(\mathscr{L}) + \Phi^2(\mathscr{L}) + \cdots + \Phi^d(\mathscr{L}) = \Omega\} \\ &\xrightarrow{\sim} X_B(w^+)(k) \sqcup X_B(w^-)(k). \end{aligned}$$

This is nothing more than the restriction of the isomorphism $p^{\mathbb{Z}}\backslash \mathrm{RZ}_\Lambda^{\mathrm{red}} \xrightarrow{\sim} S_\Lambda$ of Theorem 6.3.1, and hence arises from an isomorphism of varieties

$$p^{\mathbb{Z}}\backslash \mathrm{BT}_\Lambda \xrightarrow{\sim} X_B(w^+) \sqcup X_B(w^-). \qquad \square$$

Remark 6.5.7. The quotient $p^{\mathbb{Z}}\backslash \mathrm{RZ}_\Lambda^{\mathrm{red}}$ is itself isomorphic to a disjoint union of two Deligne–Lusztig varieties. Indeed, if $P^\pm \subset \mathrm{SO}(\Omega)$ denotes the maximal parabolic subgroup stabilizing $\mathscr{F}_d^\pm$, then [HP14, Proposition 3.6] shows that

$$p^{\mathbb{Z}}\backslash \mathrm{RZ}_\Lambda \xrightarrow{\sim} S_\Lambda \xrightarrow{\sim} X_{P^+}(1) \sqcup X_{P^-}(1).$$

7. Shimura varieties for spinor similitude groups

Finally, we apply our results to study the supersingular loci of Shimura varieties of type GSpin. Throughout § 7 we fix a quadratic space (V, Q) of signature $(n-2, 2)$ over $\mathbb{Z}_{(p)}$. We always assume that $n \geqslant 3$, and that the corresponding bilinear form $[x, y]$ induces an isomorphism from V to its $\mathbb{Z}_{(p)}$-linear dual.

7.1 The GSpin Shimura variety

First, we attach to the quadratic space V a Shimura variety of Hodge type.

7.1.1 As in the local set-up of § 4.1.2, the Clifford algebra $C(V)$ is endowed with a $\mathbb{Z}/2\mathbb{Z}$-grading $C(V) = C^+(V) \oplus C^-(V)$ and a canonical involution $c \mapsto c^*$. The group of spinor similitudes $G = \mathrm{GSpin}(V)$ is the reductive group over $\mathbb{Z}_{(p)}$ defined by

$$G(R) = \{g \in C^+(V_R)^\times : gV_Rg^{-1} = V_R, g^*g \in R^\times\}$$

for any $\mathbb{Z}_{(p)}$-algebra R. As before, the spinor similitude $\eta_G : G \to \mathbb{G}_m$ is defined by $\eta_G(g) = g^*g$, and there is a representation $G \to \mathrm{SO}(V)$ defined by $g \bullet v = gvg^{-1}$. By a slight abuse of notation, we denote again by G the generic fiber of the $\mathbb{Z}_{(p)}$-group scheme G just defined.

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7.1.2 As in (1.2.1.2), define a hermitian symmetric domain

$$\mathcal{H} = \{z \in V_{\mathbb{C}} : [z, z] = 0,\ [z, \bar{z}] < 0\}/\mathbb{C}^{\times}$$

of dimension $n-2$. The group $G(\mathbb{R})$ acts on $\mathcal{H}$ through the representation $G \to \mathrm{SO}(V)$, and the action of any $g \in G(\mathbb{R})$ with $\eta_G(g) < 0$ interchanges the two connected components of $\mathcal{H}$.

Writing $z \in \mathcal{H}$ as $z = u + iv$ with $u, v \in V_{\mathbb{R}}$, the subspace $\mathrm{Span}_{\mathbb{R}}\{u, v\}$ is a negative definite plane in $V_{\mathbb{R}}$, oriented by the ordered orthogonal basis u, v. There are natural $\mathbb{R}$-algebra maps

$$\mathbb{C} \xrightarrow{\sim} C^+(\mathrm{Span}_{\mathbb{R}}\{u, v\}) \to C^+(V_{\mathbb{R}}).$$

The first is determined by

$$i \mapsto \frac{uv}{\sqrt{Q(u)Q(v)}},$$

and the second is induced by the inclusion $\mathrm{Span}_{\mathbb{R}}\{u, v\} \subset V_{\mathbb{R}}$. The above composition restricts to an injection $\boldsymbol{h}_z : \mathbb{C}^{\times} \to G(\mathbb{R})$, which arises from a morphism $\boldsymbol{h}_z : \mathbb{S} \to G_{\mathbb{R}}$ of real algebraic groups. Here $\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m$ is Deligne's torus. The construction $z \mapsto \boldsymbol{h}_z$ realizes $\mathcal{H} \subset \mathrm{Hom}(\mathbb{S}, G_{\mathbb{R}})$ as a $G(\mathbb{R})$-conjugacy class.

Using the conventions of [Del79], the Hodge structure on V determined by $\boldsymbol{h}_z$ is

$$V_{\mathbb{C}}^{(1,-1)} = \mathbb{C}z, \quad V_{\mathbb{C}}^{(0,0)} = (\mathbb{C}z + \mathbb{C}\bar{z})^{\perp}, \quad V_{\mathbb{C}}^{(-1,1)} = \mathbb{C}\bar{z}. \tag{7.1.2.1}$$

7.1.3 The $\mathbb{Z}_{(p)}$-quadratic space V admits an orthogonal basis, and so one can choose orthogonal vectors $e, f \in V$ of negative length with $Q(e), Q(f) \in \mathbb{Z}_{(p)}^{\times}$. If we set $\delta = ef \in C(V)^{\times}$ then, exactly as in § 4.1.4, δ determines a perfect G-equivariant symplectic form

$$\psi_{\delta} : C(V) \otimes_{\mathbb{Z}_{(p)}} C(V) \to \mathbb{Z}_{(p)}(\eta_G),$$

where G acts on $C(V)$ via left multiplication. For any $z \in \mathcal{H}$ the bilinear form $\psi_{\delta}(\boldsymbol{h}_z(i)c_1, c_2)$ on $C(V_{\mathbb{R}})$ is either positive definite or negative definite, depending on the connected component of $\mathcal{H}$ containing z.

The Hodge structure on $C(V_{\mathbb{Q}})$ determined by $\boldsymbol{h}_z \in \mathrm{Hom}(\mathbb{S}, G_{\mathbb{R}})$ is

$$C(V_{\mathbb{C}})^{(0,-1)} = zC(V_{\mathbb{C}}), \quad C(V_{\mathbb{C}})^{(-1,0)} = \bar{z}C(V_{\mathbb{C}}).$$

From this it follows that the faithful representation

$$G \to \mathrm{GSp}(C(V), \psi_{\delta})$$

defines a morphism of Shimura data from $(G, \mathcal{H})$ to the Siegel Shimura datum determined by the symplectic space $(C(V), \psi_{\delta})$.

7.1.4 Define a hyperspecial subgroup $U_p = G(\mathbb{Z}_p)$ of $G(\mathbb{Q}_p)$, and choose any sufficiently small compact open subgroup $U^p \subset G(\mathbb{A}_f^p)$. Setting $U = U_pU^p$, there is an associated Shimura variety $\mathrm{Sh}_U(G, \mathcal{H})$ over $\mathbb{Q}$ with complex points

$$\mathrm{Sh}_U(G, \mathcal{H})(\mathbb{C}) = G(\mathbb{Q})\backslash\mathcal{H} \times G(\mathbb{A}_f)/U.$$

Let $2g = \dim C(V_{\mathbb{Q}}) = 2^n$ so that, as in § 3.1, the morphism of Shimura data $(G, \mathcal{H}) \to (\mathrm{GSp}_{2g}, \mathcal{H}_{2g})$ constructed in § 7.1.3 determines a morphism from $\mathrm{Sh}_U(G, \mathcal{H})$ to a moduli space of

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polarized abelian varieties up to prime-to-p-isogeny. Pulling back the universal object over this moduli space yields an abelian scheme up to prime-to-p-isogeny

$$A \to \mathrm{Sh}_U(G, \mathcal{H}),$$

often called the *Kuga–Satake abelian scheme*; see [Mad16] for more information.

The fiber of the Kuga–Satake abelian scheme at a point $(z, g) \in \mathcal{H} \times G(\mathbb{A}_f)$ can be made very explicit: it is the abelian variety up to prime-to-p-isogeny $A_{(z,g)}$ whose Betti homology is the $\mathbb{Z}_{(p)}$-module

$$\mathrm{H}_1(A_{(z,g)}(\mathbb{C}), \mathbb{Z}_{(p)}) = g \cdot C(V) \subset C(V_{\mathbb{Q}})$$

with the Hodge structure $\boldsymbol{h}_z$ defined above. Note that $A_{(z,g)}$ carries a prime-to-p polarization λ inherited from the symplectic form ψ_δ, and an action of $C(V)^{op}$ induced by the right multiplication action of $C(V)$ on itself.

7.2 Uniformization of the supersingular locus

As in § 3.1, let $\mathscr{S} = \mathscr{S}_U(G, \mathcal{H})$ be Kisin's [Kis10] smooth integral model of $\mathrm{Sh}_U(G, \mathcal{H})$ over $\mathbb{Z}_{(p)}$, and let

$$\mathscr{S}_{U_p} = \varprojlim\nolimits_{U^p} \mathscr{S}_{U^p U_p}(G, \mathcal{H}).$$

By the very construction of the integral model, the Kuga–Satake abelian scheme extends to an abelian scheme up to prime-to-p-isogeny $A \to \mathscr{S}$.

7.2.1 We denote by

$$\mathscr{S}_{\mathrm{ss}} \subset \mathscr{S} \otimes_{\mathbb{Z}_{(p)}} k$$

the supersingular locus: the largest reduced closed subscheme over which the Kuga–Satake abelian scheme is supersingular. The fiber of A at any point of $\mathscr{S} \otimes_{\mathbb{Z}_{(p)}} k$ is supersingular if and only if its p-divisible group is isoclinic. Thus Lemma 4.2.4 implies that the supersingular locus is precisely the basic locus. Moreover, along the supersingular locus the slope of the universal p-divisible group must by $1/2$, and so the classification of basic elements in Proposition 4.2.5 tells us that $\mathscr{S}_{\mathrm{ss}}$ must be the Newton stratum $\mathscr{S}_b$ for the basic b appearing in Proposition 4.2.6.

Denote by $(\widehat{\mathscr{S}_W})_{/\mathscr{S}_{\mathrm{ss}}}$ the formal completion of $\mathscr{S}_W$ along $\mathscr{S}_{\mathrm{ss}}$.

Lemma 7.2.2. *The supersingular locus $\mathscr{S}_{\mathrm{ss}}$ is non-empty.*

Proof. This can be understood as a special case of recent results on the non-emptiness of the basic locus in Hodge type Shimura varieties; see Remark 1.1.3. Here we give a direct argument.

Let $V_{0\mathbb{Q}} \subset V_{\mathbb{Q}}$ be a rational 2-plane on which Q is negative definite. The $\mathbb{Z}/2\mathbb{Z}$-grading on the Clifford algebra of $V_{0\mathbb{Q}}$ has the simple form $C(V_{0\mathbb{Q}}) = F \oplus V_{0\mathbb{Q}}$, where the even part is the quadratic imaginary field

$$F = \mathbb{Q}\big(\sqrt{-\det(V_{0\mathbb{Q}})}\big).$$

We leave it as an exercise to the reader to check that one may choose $V_{0\mathbb{Q}}$ so that p is inert in F (reduce to the case $n = 3$, and use the classification of quadratic forms from [Ser73]).

The action of F by left multiplication makes $V_{0\mathbb{Q}}$ into an F-vector space of dimension 1. The $\mathbb{C}$-quadratic space $V_{0\mathbb{C}}$ is a hyperbolic plane, and its two isotropic lines are distinguished by the two embeddings $F \to \mathbb{C}$: on one line F acts through one embedding, and on the other F acts through the conjugate embedding. These two lines determine two points of $\mathcal{H}$, and we pick one of them, $z_0 \in \mathcal{H}$. For any $g \in G(\mathbb{A}_f)$, an exercise in linear algebra shows that $A_{(z_0,g)}$ is isogenous to a product of elliptic curves with complex multiplication by F.

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Let $x_{\mathbb{C}} \in \mathscr{S}(\mathbb{C})$ be the point defined by (z_0, g). This is a special point in the sense of Deligne, and so the underlying point $x \in \mathscr{S}$ has residue field a finite extension of $\mathbb{Q}$. By completing the residue field at a prime above p and passing to its maximal unramified extension, we obtain a finite extension $\Phi/\mathbb{Q}_p^{\mathrm{unr}}$ and a point $x_\Phi \in \mathscr{S}(\Phi)$ above x. As the Kuga–Satake abelian scheme A_{x_Φ} has complex multiplication, the Néron–Ogg–Shafarevich criterion guarantees that we may replace Φ by a finite extension so that the ℓ-adic Tate module of A_{x_Φ} is unramified for all $\ell \neq p$. The extension property of Kisin's integral models now gives an extension of x_Φ to a point of $\mathscr{S}(\mathcal{O}_\Phi)$, whose reduction to $\mathscr{S}(k)$ is necessarily supersingular (as p is inert in the CM field F). $\square$

Proposition 7.2.3. *There exists a point $x \in \mathscr{S}_{U_p}(k)$ such that the local Hodge–Shimura datum $(G_{\mathbb{Z}_p}, b_x, \mu_x, C(V_{\mathbb{Z}_p}))$ obtained from x (by the procedure of § 3.1.4) agrees with the local Hodge–Shimura datum of Proposition 4.2.6.*

Proof. Let b and μ be as in Proposition 4.2.6. Using Lemma 7.2.2, we can find a point

$$x_0 \in \mathscr{S}_{\mathrm{ss}}(k) = \mathscr{S}_b(k),$$

which determines a local Shimura–Hodge datum $(G_{\mathbb{Z}_p}, b_{x_0}, \mu_{x_0}, C(V_{\mathbb{Z}_p}))$ as in § 3.1.4.

The cocharacters μ_{x_0} and μ are $G(W)$-conjugate. Indeed, using (7.1.2.1), one can see that the conjugacy class of both μ_{x_0} and μ is characterized as the set of all characters $\mathbb{G}_{mW} \to G_W$ such that the composition

$$\mathbb{G}_{mW} \to G_W \xrightarrow{\nu_G} \mathbb{G}_{mW}$$

is $z \mapsto z^{-1}$, and such that the induced grading on V_W has the form $V_W = F_1 \oplus F_0 \oplus F_{-1}$, in which F_1 and F_{-1} are isotropic lines orthogonal to F_0.

The results of § 4.2 now show that there is a unique σ-conjugacy class of basic elements in $G(K)$ making $D_K = \mathrm{Hom}(C(V_K), K)$ into an isocrystal of slope 1/2, and hence the basic elements b_{x_0} and b are σ-conjugate. Thus the claim follows from Remark 3.2.15. $\square$

Let $V'_{\mathbb{Q}}$ be the unique positive definite quadratic space over $\mathbb{Q}$ with the same dimension and determinant as $V_{\mathbb{Q}}$, but with Hasse invariant

$$\epsilon(V'_{\mathbb{Q}_\ell}) = \begin{cases} \epsilon(V_{\mathbb{Q}_\ell}) & \text{if } \ell \neq p, \\ -\epsilon(V_{\mathbb{Q}_\ell}) & \text{if } \ell = p, \end{cases}$$

for all finite primes ℓ. Let $I' = \mathrm{GSpin}(V'_{\mathbb{Q}})$ be the corresponding spinor similitude group over $\mathbb{Q}$, and let $\eta_{I'} : I' \to \mathbb{G}_m$ be the spinor similitude.

The uniformization theorem (Theorem 3.3.2) now gives the following result.

Theorem 7.2.4. *There is an isomorphism of formal W-schemes*

$$I'(\mathbb{Q}) \backslash \mathrm{RZ} \times G(\mathbb{A}^p)/U^p \xrightarrow{\sim} (\widehat{\mathscr{S}_W})_{/\mathscr{S}_{\mathrm{ss}}}$$

for suitable isomorphisms $I'(\mathbb{Q}_p) \xrightarrow{\sim} J_b(\mathbb{Q}_p)$, and $I'(\mathbb{Q}_\ell) \xrightarrow{\sim} G(\mathbb{Q}_\ell)$ for $\ell \neq p$.

Proof. Since $\mathscr{S}_{\mathrm{ss}} = \mathscr{S}_b$, this will follow from Theorem 3.3.2 after we show that the group I in the statement of Theorem 3.3.2 can be identified with the group $I' = \mathrm{GSpin}(V'_{\mathbb{Q}})$ above.

The group $I' = \mathrm{GSpin}(V'_{\mathbb{Q}})$ is an inner form of G: since $V_{\mathbb{Q}}$ and $V'_{\mathbb{Q}}$ have the same dimension and determinant, we can find an isomorphism of quadratic spaces

$$\psi : V'_{\mathbb{Q}} \otimes_{\mathbb{Q}} \bar{\mathbb{Q}} \xrightarrow{\sim} V_{\mathbb{Q}} \otimes_{\mathbb{Q}} \bar{\mathbb{Q}}$$

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which produces a Galois cocycle $\sigma \mapsto \psi\sigma(\psi)^{-1}$ with values in $\mathrm{SO}(V_{\mathbb{Q}})(\bar{\mathbb{Q}})$. Composing this with $\mathrm{SO}(V_{\mathbb{Q}}) \to G^{\mathrm{ad}}$ gives a class $c' \in \mathrm{H}^1(\mathbb{Q}, G^{\mathrm{ad}})$ which defines $I' = \mathrm{GSpin}(V'_{\mathbb{Q}})$. Notice that $V'_{\mathbb{Q}_\ell}$ is isomorphic to $V_{\mathbb{Q}_\ell}$, for all $\ell \neq p$, and $V'_{\mathbb{Q}_p}$ is isomorphic to V_K^{Φ}.

The group I is also an inner form of G. In fact, by the remarks that follow part (iv) of the definition of a Kottwitz triple in [Kis13, (4.3)], I is uniquely determined *as an inner form* (or more correctly *an inner twist*) of G by the local inner twisting isomorphisms at finite places, and the fact that $I_{\mathbb{R}}$ is anisotropic modulo center. (This uses the Hasse principle for adjoint groups (see [PR94, § 6.5, Theorem 6.22]) and the fact that there is a unique element of $\mathrm{H}^1(\mathbb{R}, G^{\mathrm{ad}})$ which corresponds to the compact modulo center form of $G_{\mathbb{R}}$ (see [Kot92, p. 423]).) Therefore, I is given by a well-defined cohomology class $c \in \mathrm{H}^1(\mathbb{Q}, G^{\mathrm{ad}})$ with prescribed localizations c_v in $\mathrm{H}^1(\mathbb{Q}_v, G^{\mathrm{ad}})$, for all places v of $\mathbb{Q}$.

By the definition of the classes c_v, as provided by the inner twists coming from the Kottwitz triple given by x_0, c_ℓ is trivial for $\ell \neq p$, while c_p corresponds to the inner twist $\mathrm{GSpin}(V_K^{\Phi})$; as $V_K^{\Phi} \xrightarrow{\sim} V'_{\mathbb{Q}_p}$, we have $c_p = c'_p$. Hence, $c_v = c'_v$ for all finite places v of $\mathbb{Q}$. Also, since $V'_{\mathbb{R}}$ is positive definite, we have $c_\infty = c'_\infty$ as above. The result then follows as above, by the Hasse principle for adjoint groups. □

From here on we identify

$$I = I' = \mathrm{GSpin}(V'_{\mathbb{Q}}).$$

Remark 7.2.5. As in the proof of Theorem 3.3.2, the group $I = \mathrm{GSpin}(V'_{\mathbb{Q}})$ acts as quasi-endomorphisms of the fiber A_{x_0} of the Kuga–Satake abelian scheme at the base point $x_0 \in \mathscr{S}_{U_p}(k)$. The action $I \subset \mathrm{End}(A_{x_0})^{\times}_{\mathbb{Q}}$ can be explained as follows. The fiber A_{x_0}, like every fiber of the Kuga–Satake abelian scheme, comes endowed with a collection of *special quasi-endomorphisms* $V(A_{x_0}) \subset \mathrm{End}(A_{x_0})_{\mathbb{Q}}$ as in [Mad16, § 5]. This is a quadratic space over $\mathbb{Q}$, with quadratic form determined by $v \circ v = Q(v) \cdot \mathrm{id}$. For any fiber the space of special endomorphisms has dimension less than or equal to $\dim(V_{\mathbb{Q}})$, and equality holds precisely at supersingular points. In fact, using [Mad15, Theorem 6.4], the supersingularity of A_{x_0} implies that $V(A_{x_0}) \xrightarrow{\sim} V'_{\mathbb{Q}}$. After fixing such an isomorphism, we obtain an injection $V'_{\mathbb{Q}} \to \mathrm{End}(A_{x_0})_{\mathbb{Q}}$, which, by the universal property of Clifford algebras, extends to the ring homomorphism $C(V'_{\mathbb{Q}}) \to \mathrm{End}(A_{x_0})_{\mathbb{Q}}$. This homomorphism then restricts to a homomorphism of groups

$$\mathrm{GSpin}(V'_{\mathbb{Q}}) \to \mathrm{End}(A_{x_0})^{\times}_{\mathbb{Q}}.$$

7.2.6 Recalling the decomposition $\mathrm{RZ} = \bigsqcup_{\ell \in \mathbb{Z}} \mathrm{RZ}^{(\ell)}$ of RZ into its connected components, we may rewrite the uniformization of Theorem 7.2.4 as an isomorphism

$$(\widehat{\mathscr{S}_W})_{/\mathscr{S}_{\mathrm{ss}}} \xrightarrow{\sim} I(\mathbb{Q})_0 \backslash \mathrm{RZ}^{(0)} \times G(\mathbb{A}^p)/U^p,$$

where

$$I(\mathbb{Q})_0 = \ker(I(\mathbb{Q}) \xrightarrow{\eta_I} \mathbb{Q}^{\times} \xrightarrow{\mathrm{ord}_p} \mathbb{Z})$$

is the common stabilizer in $I(\mathbb{Q})$ of the connected components of RZ. This may be further rewritten as

$$(\widehat{\mathscr{S}_W})_{/\mathscr{S}_{\mathrm{ss}}} \xrightarrow{\sim} \bigsqcup_{g \in I(\mathbb{Q})_0 \backslash G(\mathbb{A}_f^p)/U^p} \Gamma_g \backslash \mathrm{RZ}^{(0)}, \tag{7.2.6.1}$$

where $\Gamma_g = I(\mathbb{Q})_0 \cap gU^p g^{-1}$.

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7.3 Structure of the supersingular locus

As in the proof of Theorem 7.2.4, we may identify

$$V'_{\mathbb{Q}_p} \xrightarrow{\sim} V_K^{\Phi}$$

as $\mathbb{Q}_p$-quadratic spaces. In particular, we obtain from Definition 5.1.1 the notion of a *vertex lattice* $\Lambda \subset V'_{\mathbb{Q}_p}$, whose *type* $t_\Lambda = \dim_{\mathbb{F}_p}(\Lambda/\Lambda^\vee)$ is a positive even integer less than or equal to the integer $t_{\max}$ of (1.2.3.1).

7.3.1 Fix a vertex lattice $\Lambda \subset V'_{\mathbb{Q}_p}$. Exactly as in § 5.3.1, endow $\Omega_0 = \Lambda/\Lambda^\vee$ with the rescaled $\mathbb{F}_p$-valued quadratic form pQ. Set $\Omega = \Omega_0 \otimes_{\mathbb{F}_p} k$, and let S_Λ be the reduced k-scheme with k-points

$$S_\Lambda(k) = \left\{ \text{Lagrangians } \mathscr{L} \subset \Omega : \dim_k(\mathscr{L} + \Phi(\mathscr{L})) = \frac{t_\Lambda}{2} + 1 \right\},$$

where $\Phi = \mathrm{id} \otimes \sigma$ is the absolute Frobenius on Ω. Recall from Proposition 5.3.2 that $S_\Lambda = S_\Lambda^+ \sqcup S_\Lambda^-$ has two connected components. The components are isomorphic, and each is projective and smooth of dimension $(t_\Lambda/2) - 1$. Up to isomorphism, S_Λ depends only on the type t_Λ.

7.3.2 Taking the reduced scheme underlying both sides of (7.2.6.1) yields an isomorphism

$$\mathscr{S}_{\mathrm{ss}} \xrightarrow{\sim} \bigsqcup_{g \in I(\mathbb{Q})_0 \backslash G(\mathbb{A}_f^p)/U^p} \Gamma_g \backslash \mathrm{RZ}^{(0),\mathrm{red}}.$$

From this, the description of $\mathrm{RZ}^{(0),\mathrm{red}}$ of Theorem 6.4.1, and an argument as in the proof of [Vol10, Theorem 6.1] we deduce the following result.

THEOREM 7.3.3. *For all $U^p \subset G(\mathbb{A}_f^p)$ sufficiently small, the following hold.*

(i) *Each of the k-schemes $\Gamma_g \backslash \mathrm{RZ}^{(0),\mathrm{red}}$ is connected.*

(ii) *The irreducible components of $\Gamma_g \backslash \mathrm{RZ}^{(0),\mathrm{red}}$ are in bijection with the set of orbits*

$$\Gamma_g \backslash \{\text{vertex lattices of type } t_{\max}\},$$

and the irreducible component indexed by a vertex lattice Λ is isomorphic to $S_\Lambda^\pm$. In particular, all irreducible components are isomorphic to one another, and are projective and smooth of dimension

$$\dim(\mathscr{S}_{\mathrm{ss}}) = \frac{t_{\max}}{2} - 1. \qquad \square$$

ACKNOWLEDGEMENTS

We would like to thank W. Kim, M. Kisin, and M. Rapoport for helpful discussions and comments, and the referees for some useful suggestions.

Appendix A. Maximal lattices

For the reader's convenience we recall two results on maximal lattices in quadratic spaces, used in several places in the body of the paper.

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A.1 Eichler's theorem and the elementary divisor theorem

Let F be a field, complete with respect to a discrete valuation. Denote by $\mathcal{O}$ the valuation ring of F. Suppose V is a finite-dimensional F-vector space endowed with a non-degenerate quadratic form $q : V \to F$, and let

$$[x, y] = q(x + y) - q(x) - q(y)$$

be the bilinear form determined by q.

DEFINITION A.1.1. By a *lattice* in V we mean a free $\mathcal{O}$-submodule $M \subset V$ with $\mathrm{rank}_{\mathcal{O}}(M) = \dim_F(V)$. A lattice M is *maximal* (with respect to q) if $q(M) \subset \mathcal{O}$, and if M is not properly contained in any other lattice with this property. The *dual* $M^\vee$ of the lattice M is

$$M^\vee = \{x \in V : [x, m] \in \mathcal{O}, \forall m \in M\}.$$

The lattice M is called *self-dual* if $M = M^\vee$.

For a proof of the following, see [Ger08, Theorem 8.8].

THEOREM A.1.2 (Eichler). *All maximal lattices in V are isomorphic as $\mathcal{O}$-quadratic spaces. If V is anisotropic, then it has a unique maximal lattice*

$$M = \{x \in V : q(x) \in \mathcal{O}\}.$$

THEOREM A.1.3 (Elementary divisor theorem). *Suppose A and B are maximal lattices in V. There is a decomposition*

$$V = Fe_1 \oplus Ff_1 \oplus \cdots \oplus Fe_r \oplus Ff_r \oplus V_0,$$

in which V_0 is anisotropic and orthogonal to all e_i and all f_i,

$$[e_i, e_j] = 0, \quad [f_i, f_j] = 0, \quad [e_i, f_j] = \delta_{i,j},$$

and

$$A = \mathcal{O}e_1 \oplus \mathcal{O}f_1 \oplus \cdots \oplus \mathcal{O}e_r \oplus \mathcal{O}f_r \oplus M_0,$$
$$B = (\beta_1)e_1 \oplus (\beta_1^{-1})f_1 \oplus \cdots \oplus (\beta_r)e_r \oplus (\beta_r^{-1})f_r \oplus M_0,$$

for some $\beta_1, \ldots, \beta_r \in F^\times$. Here $(x) = \mathcal{O}x$, and $M_0 \subset V_0$ is the unique maximal lattice in V_0.

Proof. By applying [Ger08, Lemma 6.36] inductively, there is an orthogonal decomposition

$$V = H_1 \oplus \cdots \oplus H_r \oplus V_0$$

in which each H_i is a hyperbolic plane, V_0 is anisotropic, and

$$A = (A \cap H_1) \oplus \cdots \oplus (A \cap H_r) \oplus (A \cap V_0),$$
$$B = (B \cap H_1) \oplus \cdots \oplus (B \cap H_r) \oplus (B \cap V_0).$$

The maximality of A implies that each $A \cap H_i$ is a maximal lattice of H_i, and that $A \cap V_0$ is a maximal lattice of V_0. Of course similar remarks apply to B, and, in particular,

$$A \cap V_0 = \{x \in V_0 : q(x) \in \mathcal{O}\} = B \cap V_0$$

by Theorem A.1.2.

Choose a basis $e_i, f_i \in H_i$ such that $q(e_i) = q(f_i) = 0$ and $[e_i, f_i] = 1$. Using the fact that Fe_i and Ff_i are the unique isotropic lines in H_i, it follows from [Ger08, Lemma 6.35] that

$$A \cap H_i = (\alpha_i)e_i \oplus (\alpha_i^{-1})f_i,$$
$$B \cap H_i = (\beta_i)e_i \oplus (\beta_i^{-1})f_i,$$

for some $\alpha_i, \beta_i \in F^\times$. The desired decomposition of V is now obtained by rescaling e_i and f_i so that $\alpha_i = 1$. □

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References

Asg02 M. Asgari, *Local L-functions for split spinor groups*, Canad. J. Math. **54** (2002), 673–693; MR 1913914 (2003i:11062).

Bas74 H. Bass, *Clifford algebras and spinor norms over a commutative ring*, Amer. J. Math. **96** (1974), 156–206; MR 0360645 (50 #13092).

Ber96 P. Berthelot, *Cohomologie rigide et cohomologie rigide à supports propres. Première partie.* Preprint (1996), IRMAR 96-03, available at http://perso.univ-rennesl.fr/pierre.berthelot.

BBM82 P. Berthelot, L. Breen and W. Messing, *Théorie de Dieudonné cristalline. II*, Lecture Notes in Mathematics, vol. 930 (Springer, Berlin, 1982); MR 667344 (85k:14023).

BM90 P. Berthelot and W. Messing, *Théorie de Dieudonné cristalline. III. Théorèmes d'équivalence et de pleine fidélité*, in *The Grothendieck Festschrift, Vol. I*, Progress in Mathematics, vol. 86 (Birkhäuser, Boston, 1990), 173–247; MR 1086886 (92h:14012).

BS15 B. Bhatt and P. Scholze, *Projectivity of the Witt vector affine Grassmannian*. Preprint (2015), arXiv:1507.06490.

CKV15 M. Chen, M. Kisin and E. Viehmann, *Connected components of affine Deligne–Lusztig varieties in mixed characteristic*, Compositio Math. **151** (2015), 1697–1762; MR 3406443.

CV15 M. Chen and E. Viehmann, *Affine Deligne–Lusztig varieties and the action of J*. Preprint (2015), arXiv:1507.02806.

DOR10 J.-F. Dat, S. Orlik and M. Rapoport, *Period domains over finite and p-adic fields*, Cambridge Tracts in Mathematics, vol. 183 (Cambridge University Press, Cambridge, 2010); MR 2676072 (2012a:22026).

deJ95 A. J. de Jong, *Crystalline Dieudonné module theory via formal and rigid geometry*, Publ. Math. Inst. Hautes Études Sci. **82** (1995), 5–96; MR 1383213 (97f:14047).

Del79 P. Deligne, *Variétés de Shimura: interprétation modulaire, et techniques de construction de modèles canoniques*, in *Automorphic forms, representations and L-functions* (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, OR, 1977), Part 2, Proceedings of Symposia in Pure Mathematics, vol. XXXIII (American Mathematical Society, Providence, RI, 1979), 247–289; MR 546620 (81i:10032).

Dri76 V. G. Drinfel'd, *Coverings of p-adic symmetric domains*, Funkcional. Anal. i Priložen. **10** (1976), 29–40; MR 0422290 (54 #10281).

EGAIV A. Grothendieck, *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. I*, Publ. Math. Inst. Hautes Études Sci. **20** (1964), 5–251; MR 0173675 (30 #3885).

Fal99 G. Faltings, *Integral crystalline cohomology over very ramified valuation rings*, J. Amer. Math. Soc. **12** (1999), 117–144; MR 1618483 (99e:14022).

Ger08 L. Gerstein, *Basic quadratic forms*, Graduate Studies in Mathematics, vol. 90 (American Mathematical Society, Providence, RI, 2008); MR 2396246 (2009e:11064).

GHKR06 U. Görtz, T. Haines, R. Kottwitz and D. Reuman, *Dimensions of some affine Deligne–Lusztig varieties*, Ann. Sci. Éc. Norm. Supér. (4) **39** (2006), 467–511; MR 2265676 (2008e:14068).

GH15 U. Görtz and X. He, *Basic loci of Coxeter type in Shimura varieties*, Camb. J. Math. **3** (2015), 323–353; MR 3393024.

Ham16 P. Hamacher, *The almost product structure of Newton strata in the Deformation space of a Barsotti–Tate group with crystalline Tate tensors*. Preprint (2016), arXiv:1601.03131.

HP14 B. Howard and G. Pappas, *On the supersingular locus of the* $\mathrm{GU}(2,2)$ *Shimura variety*, Algebra Number Theory **8** (2014), 1659–1699; MR 3272278.

Kim13 W. Kim, *Rapoport–Zink spaces of Hodge type*. Preprint (2013), arXiv:1308.5537.

Kim14 W. Kim, *Rapoport–Zink uniformization of Hodge-type Shimura varieties*. Preprint (2014).

https://doi.org/10.1112/S0010437X17007011 Published online by Cambridge University Press

Kis10 M. Kisin, *Integral models for Shimura varieties of abelian type*, J. Amer. Math. Soc. **23** (2010), 967–1012; MR 2669706.

Kis13 M. Kisin, *Mod p points on Shimura varieties of abelian type*, J. Amer. Math. Soc., to appear. Preprint (2013).

KP15 M. Kisin and G. Pappas, *Integral models for Shimura varieties with parahoric level structure*. Preprint (2015), arXiv:1512.01149.

Kit93 Y. Kitaoka, *Arithmetic of quadratic forms*, Cambridge Tracts in Mathematics, vol. 106 (Cambridge University Press, Cambridge, 1993); MR 1245266 (95c:11044).

Kot84 R. Kottwitz, *Shimura varieties and twisted orbital integrals*, Math. Ann. **269** (1984), 287–300; MR 761308 (87b:11047).

Kot85 R. Kottwitz, *Isocrystals with additional structure*, Compositio Math. **56** (1985), 201–220; MR 809866 (87i:14040).

Kot92 R. Kottwitz, *Points on some Shimura varieties over finite fields*, J. Amer. Math. Soc. **5** (1992), 373–444; MR 1124982 (93a:11053).

Kud04 S. S. Kudla, *Special cycles and derivatives of Eisenstein series*, in *Heegner points and Rankin L-series*, Mathematical Sciences Research Institute Publications, vol. 49 (Cambridge University Press, Cambridge, 2004), 243–270; MR 2083214 (2005g:11108).

KR99 S. S. Kudla and M. Rapoport, *Arithmetic Hirzebruch–Zagier cycles*, J. reine angew. Math. **515** (1999), 155–244; MR 1717613 (2002e:11076a).

KR00 S. S. Kudla and M. Rapoport, *Cycles on Siegel threefolds and derivatives of Eisenstein series*, Ann. Sci. Éc. Norm. Supér. (4) **33** (2000), 695–756; MR 1834500 (2002e:11076b).

Lan02 S. Lang, *Algebra*, Graduate Texts in Mathematics, vol. 211, third edition (Springer, New York, 2002); MR 1878556 (2003e:00003).

Mad15 K. Madapusi Pera, *The Tate conjecture for* K3 *surfaces in odd characteristic*, Invent. Math. **201** (2015), 625–668; MR 3370622.

Mad16 K. Madapusi Pera, *Integral canonical models for spin Shimura varieties*, Compositio Math. **152** (2016), 769–824; MR 3484114.

Mat80 H. Matsumura, *Commutative algebra*, Mathematics Lecture Note Series, vol. 56, second edition (Benjamin/Cummings Publishing Co., Reading, MA, 1980); MR 575344 (82i:13003).

Mes72 W. Messing, *The crystals associated to Barsotti–Tate groups: with applications to abelian schemes*, Lecture Notes in Mathematics, vol. 264 (Springer, Berlin, New York, 1972); MR 0347836 (50 #337).

Moo98 B. Moonen, *Models of Shimura varieties in mixed characteristics*, in *Galois representations in arithmetic algebraic geometry (Durham, 1996)*, London Mathematical Society Lecture Note Series, vol. 254 (Cambridge University Press, Cambridge, 1998), 267–350; MR 1696489 (2000e:11077).

Nis82 Y. A. Nisnevich, *Étale cohomology and arithmetic of semisimple groups*, ProQuest LLC, Ann Arbor, MI, 1982, PhD thesis, Harvard University; MR 2632405.

Ogu84 A. Ogus, *F-isocrystals and de Rham cohomology. II. Convergent isocrystals*, Duke Math. J. **51** (1984), 765–850; MR 771383 (86j:14012).

PR94 V. Platonov and A. Rapinchuk, *Algebraic groups and number theory*, Pure and Applied Mathematics, vol. 139 (Academic Press, Boston, 1994), translated from the 1991 Russian original by R. Rowen; MR 1278263 (95b:11039).

Rap05 M. Rapoport, *A guide to the reduction modulo p of Shimura varieties*, in *Automorphic forms. I*, Astérisque, vol. 298 (Société Mathématique de France, Paris, 2005), 271–318; MR 2141705 (2006c:11071).

RR96 M. Rapoport and M. Richartz, *On the classification and specialization of F-isocrystals with additional structure*, Compositio Math. **103** (1996), 153–181; MR 1411570 (98c:14015).

https://doi.org/10.1112/S0010437X17007011 Published online by Cambridge University Press

RTW14 M. Rapoport, U. Terstiege and S. Wilson, *The supersingular locus of the Shimura variety for* $\mathrm{GU}(1, n-1)$ *over a ramified prime*, Math. Z. **276** (2014), 1165–1188; MR 3175176.

RV14 M. Rapoport and E. Viehmann, *Towards a theory of local Shimura varieties*, Münster J. Math. **7** (2014), 273–326; MR 3271247.

RZ96 M. Rapoport and Th. Zink, *Period spaces for p-divisible groups*, Annals of Mathematics Studies, vol. 141 (Princeton University Press, Princeton, NJ, 1996); MR 1393439 (97f:14023).

SW13 P. Scholze and J. Weinstein, *Moduli of p-divisible groups*, Camb. J. Math. **1** (2013), 145–237; MR 3272049.

Ser73 J.-P. Serre, *A course in arithmetic*, Graduate Texts in Mathematics, vol. 7 (Springer, New York, 1973), translated from the French; MR 0344216 (49 #8956).

Shi10 G. Shimura, *Arithmetic of quadratic forms*, Springer Monographs in Mathematics (Springer, New York, 2010); MR 2665139 (2011m:11003).

Vie14 E. Viehmann, *Truncations of level 1 of elements in the loop group of a reductive group*, Ann. of Math. (2) **179** (2014), 1009–1040; MR 3171757.

Vol10 I. Vollaard, *The supersingular locus of the Shimura variety for* $\mathrm{GU}(1, s)$, Canad. J. Math. **62** (2010), 668–720; MR 2666394.

VW11 I. Vollaard and T. Wedhorn, *The supersingular locus of the Shimura variety of* $\mathrm{GU}(1, n-1)$ *II*, Invent. Math. **184** (2011), 591–627; MR 2800696 (2012j:14035).

Wor13 D. Wortmann, *The μ-ordinary locus for Shimura varieties of Hodge type*. Preprint (2013), arXiv:1310.6444.

Zha13 C. Zhang, *Ekedahl–Oort strata for good reductions of Shimura varieties of Hodge type*. Preprint (2013), arXiv:1312.4869.

Zha15 C. Zhang, *Stratifications and foliations for good reductions of Shimura varieties of Hodge type*. Preprint (2015), arXiv:1512.08102.

Zhu17 X. Zhu, *Affine Grassmannians and the geometric Satake in mixed characteristic*, Ann. of Math. (2) **185** (2017), 403–492.

Zin01 Th. Zink, *Windows for displays of p-divisible groups*, Moduli of abelian varieties (Texel Island, 1999), Progress in Mathematics, vol. 195 (Birkhäuser, Basel, 2001), 491–518; MR 1827031 (2002c:14073).

Zin02 Th. Zink, *The display of a formal p-divisible group*, in *Cohomologies p-adiques et applications arithmétiques, I*, Astérisque, vol. 278 (Société Mathématique de France, Paris, 2002), 127–248; MR 1922825 (2004b:14083).

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