A SIMPLE PROOF OF HERMITE'S THEOREM ON THE ZEROS OF A POLYNOMIAL

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In 1856 Hermite showed how to determine by purely rational operations the number of zeros of a given polynomial lying in a specified half plane [**1**]: one inspects the signature of a certain Hermitian form. This type of result is still of interest for practical applications, and several authors have provided alternatives to Hermite's original, highly computational proof (for example [**2**, **3**]). Recently V. Pták and the author gave a simple matrix-theoretic proof and generalization of a class of Hermite-type theorems [**4**]. This class included the Schur–Cohn test for zeros in a circle, but not, to our regret, the original theorem of Hermite. The purpose of this note is to show that a slight modification of our method does indeed provide a simple proof of Hermite's theorem.

Hermite's test uses the Bezoutian Bez(f, g) of two polynomials f and g of degree n (over $\mathbb{C}$): this is defined to be the $n \times n$ matrix $[c_{ij}]$, where

$$\sum_{i,j=0}^{n-1} c_{ij} z^i w^j = \frac{f(z)g(w) - g(z)f(w)}{z - w}.$$

The present proof uses a matrix identity for the Bezoutian. It is a matter of straightforward computation that, if $f(z) = \sum_0^n a_r z^r$ and $g(z) = \sum_0^n b_r z^r$,

$$\text{Bez}(f, g) = \left[\sum_{k=0}^{i} b_{j-k} a_{i+1+k} - a_{j-k} b_{i+1+k} \right]_{i,j=0}^{n-1}.$$

This can also be written

$$\text{Bez}(f, g) = \begin{bmatrix} a_1 & a_2 & \dots & a_n \\ a_2 & a_3 & \dots & 0 \\ \cdot & \cdot & \dots & \cdot \\ a_n & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} b_0 & b_1 & \dots & b_{n-1} \\ 0 & b_0 & \dots & b_{n-2} \\ \cdot & \cdot & \dots & \cdot \\ 0 & 0 & \dots & b_0 \end{bmatrix}$$

$$- \begin{bmatrix} b_1 & b_2 & \dots & b_n \\ b_2 & b_3 & \cdots & 0 \\ \cdot & \cdot & \dots & \cdot \\ b_n & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} a_0 & a_1 & \dots & a_{n-1} \\ 0 & a_0 & \cdots & a_{n-2} \\ \cdot & \cdot & \dots & \cdot \\ 0 & 0 & \dots & a_0 \end{bmatrix}$$

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or, if we introduce the matrices

$$S=\begin{bmatrix}0 & 1 & 0 & \dots & 0\\ 0 & 0 & 1 & \dots & 0\\ . & . & . & \dots & .\\ 0 & 0 & 0 & \dots & 1\\ 0 & 0 & 0 & \dots & 0\end{bmatrix}, \qquad J=\begin{bmatrix}0 & 0 & 0 & \dots & 1\\ 0 & 0 & 0 & \dots & 0\\ . & . & . & \dots & .\\ 0 & 1 & 0 & \dots & 0\\ 1 & 0 & 0 & \dots & 0\end{bmatrix}$$

of type $n\times n$,

$$\operatorname{Bez}(f, g)=Jf_1(S^*)g(S)-Jg_1(S^*)f(S), \tag{1}$$

where the star denotes the conjugate transpose and

$$f_1(z)=z^n f(1/z).$$

We shall also need the polynomials f^* and f_0 defined by

$$f^*(z)=f(\bar{z})^-, \qquad f_0=(f_1)^*.$$

HERMITE'S THEOREM. *Let f be a polynomial over $\mathbb{C}$. Suppose that the matrix*

$$B=-i\operatorname{Bez}(f, f^*)$$

is non-singular. Then the number of + signs in the canonical form of the Hermitian form x^*Bx is equal to the number of zeros of f with positive imaginary parts.

Proof. We can suppose f monic. Let

$$f(z)=(z-\alpha_1)(z-\alpha_2)\dots(z-\alpha_n).$$

Then

$$\begin{aligned} f^*(z)&=(z-\bar{\alpha}_1)(z-\bar{\alpha}_2)\dots(z-\bar{\alpha}_n),\\ f_1(z)&=(1-\alpha_1 z)(1-\alpha_2 z)\dots(1-\alpha_n z),\\ f_0(z)&=(1-\bar{\alpha}_1 z)(1-\bar{\alpha}_2 z)\dots(1-\bar{\alpha}_n z).\end{aligned}$$

Introduce the $n\times n$ matrices

$$E_i=S-\alpha_i I, \qquad F_i=S-\bar{\alpha}_i I, \qquad G_i=I-\alpha_i S, \qquad H_i=I-\bar{\alpha}_i S.$$

Then

$$\begin{aligned} f(S)&=E_1E_2\dots E_n, & f^*(S)&=F_1F_2\dots F_n,\\ f_1(S^*)&=H_n^*H_{n-1}^*\dots H_1^*, & f_0(S^*)&=G_n^*G_{n-1}^*\dots G_1^*.\end{aligned}$$

From (1),

$$\begin{aligned} B&=-i\operatorname{Bez}(f, f^*)\\ &=-iJ\{f_1(S^*)f^*(S)-f_0(S^*)f(S)\}\\ &=-iJ\{H_n^*\dots H_1^*F_1\dots F_n-G_n^*\dots G_1^*E_1\dots E_n\}. \end{aligned} \tag{2}$$

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We now appeal to an identity for the difference of two products. Consider the sum

$$\sum_{k=1}^{n} H_n^* \dots H_{k+1}^* G_{k-1}^* \dots G_1^* (H_k^* F_k - G_k^* E_k) E_1 \dots E_{k-1} F_{k+1} \dots F_n.$$

(In the case $k=1$ we interpret $H_n^* \dots H_{k+1}^* G_{k-1}^* \dots G_1^*$ as $H_n^* \dots H_2^*$, etc.) On writing out the sum we find that, in view of the fact that H_k^* commutes with G_j^* and E_k commutes with E_j, the second component of the kth term cancels with the first component of the $(k+1)$th term, $1 \leqslant k \leqslant n-1$, and the sum telescopes to

$$H_n^* \dots H_1^* F_1 \dots F_n - G_n^* \dots G_1^* E_1 \dots E_n,$$

so that (2) can be written

$$B = -iJ \sum_{k=1}^{n} H_n^* \dots H_{k+1}^* G_{k-1}^* \dots G_1^* (H_k^* F_k - G_k^* E_k) E_1 \dots E_{k-1} F_{k+1} \dots F_n.$$

Now it is readily checked that

$$\begin{aligned} H_j^* F_j - G_j^* E_j &= (\alpha_j - \bar{\alpha}_j)(I - S^* S) \\ &= 2i(\operatorname{Im} \alpha_j) e e^*, \end{aligned}$$

where $e^* = [1\,0 \dots 0]$. Thus

$$B = -2iJ \sum_{k=1}^{n} i(\operatorname{Im} \alpha_k) H_n^* \dots H_{k+1}^* G_{k-1}^* \dots G_1^* e e^* E_1 \dots E_{k-1} F_{k+1} \dots F_n.$$

I claim that

$$J H_n^* \dots H_{k+1}^* G_{k-1}^* \dots G_1^* e = F_n^* \dots F_{k+1}^* E_{k-1}^* \dots E_1^* e.$$

To see this, note that if h is any polynomial of degree $n-1$ and $h_1(z) = z^{n-1} h(1/z)$ then $Jh_1(S^*)$ and $h(S^*)$ have the same first column, which is to say that

$$Jh_1(S^*)e = h(S^*)e.$$

On applying this observation with

$$h(z) = (z - \alpha_n) \dots (z - \alpha_{k+1})(z - \bar{\alpha}_{k-1}) \dots (z - \bar{\alpha}_1),$$

one establishes the claim.

We now have

$$\begin{aligned} B &= 2 \sum_{k=1}^{n} (\operatorname{Im} \alpha_k) F_n^* \dots F_{k+1}^* E_{k-1}^* \dots E_1^* e e^* E_1 \dots E_{k-1} F_{k+1} \dots F_n \\ &= 2 \sum_{k=1}^{n} (\operatorname{Im} \alpha_k) v_k v_k^*, \end{aligned}$$

where

$$v_k = F_n^* \dots F_{k+1}^* E_{k-1}^* \dots E_1^* e.$$

This can be written

$$B = V \operatorname{diag}\{\operatorname{Im} \alpha_1, \ldots, \operatorname{Im} \alpha_n\} V^*,$$

where

$$V = \sqrt{2}[v_1 \ldots v_n].$$

If B is non-singular then so is V and hence B is congruent to $\operatorname{diag}\{\operatorname{Im} \alpha_1, \ldots, \operatorname{Im} \alpha_n\}$. Hermite's assertion follows at once.

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