THE CONVOLUTION \( x^{-r} \cdot x^s \)

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1. Introduction. In a recent paper [1], Jones extended the definition of the convolution of distributions so that further convolutions could be defined. The convolution \( w_1 \ast w_2 \) of two distributions \( w_1 \) and \( w_2 \) was defined as the limit of the sequence \( \{w_{1n} \ast w_{2n}\} \), provided the limit \( w \) exists in the sense that

\[
\lim_{n \to \infty} \int_{-\infty}^{\infty} \phi(x)w_{1n}\ast w_{2n}dx = \int_{-\infty}^{\infty} \phi(x)w(x)dx
\]

for all fine functions \( \phi \) in the terminology of Jones [2], where

\[
w_{1n}(x) = w_1(x)\tau(x/n), \quad w_{2n}(x) = w_2(x)\tau(x/n)
\]

and \( \tau \) is an infinitely differentiable function satisfying the following conditions:

(i) \( \tau(x) = \tau(-x) \),
(ii) \( 0 \leq \tau(x) \leq 1 \),
(iii) \( \tau(x) = 1 \) for \( |x| \leq 1/2 \),
(iv) \( \tau(x) = 0 \) for \( |x| \geq 1 \).

2. The convolution \( x^{-r} \cdot x^s \). We prove that in the sense of Jones' definition of convolution for distributions that

\[
x^{-r} \cdot x^s = 0
\]

for \( s = 0, 1, 2, \ldots, r-1 \) and \( r = 1, 2, \ldots \).

We write

\[
(x^{-r})_n = x^{-r}\tau(x/n), \quad (x^s)_n = x^s\tau(x/n).
\]

Formally we have

\[
(x^{-r})_n \ast (x^s)_n = \int_{-\infty}^{\infty} t^{-r}\tau(t/n)(x-t)^{s}\tau\left(\frac{x-t}{n}\right)dt
\]

but since \( t^{-r} \) is not a summable function we must interpret the integral in the distributional sense. Putting

\[
\phi_s(t) = \tau(t/n)(x-t)^{s}\tau\left(\frac{x-t}{n}\right),
\]

we note that \( \phi_s(t) \) is a fine function so that we can write

\[
(x^{-r})_n \ast (x^s)_n = (t^{-r}, \phi_s(t)).
\]
We must now distinguish between odd and even \( r \). First of all we have

\[
(x^{-2r-1})_n \ast (x^n)_n = (t^{-2r-1}, \phi_s(t))
\]

\[
= \int_0^\infty t^{-2r-1} \left\{ \phi_s(t) - \phi_s(-t) - 2 \sum_{i=1}^r \frac{t^{2i-1}}{(2i-1)!} \phi_s^{(2i-1)}(0) \right\} dt
\]

\[
= \int_0^n t^{-2r-1} \left\{ \phi_s(t) - \phi_s(-t) - 2 \sum_{i=1}^r \frac{t^{2i-1}}{(2i-1)!} \phi_s^{(2i-1)}(0) \right\} dt,
\]

since \( \phi_s(t) = 0 \) for \( |t| \geq n \).

Now for arbitrary fixed \( c > 0 \) choose \( N \) such that \( 4c < N \). Then if

\[
0 \leq t \leq n/4, \quad |x| \leq c, \quad N \leq n
\]

it follows that

\[
\left| \frac{x \pm t}{n} \right| \leq 1/2.
\]

From Taylor's Theorem we have

\[
\phi_s(t) - \phi_s(-t) - 2 \sum_{i=1}^r \frac{t^{2i-1}}{(2i-1)!} \phi_s^{(2i-1)}(0) = \frac{2\xi^2 r + 1}{(2r)!} \phi_s^{(2r+1)}(t_0),
\]

where \( 0 \leq \xi \leq 1 \) and \( -\xi t \leq t_0 \leq \xi t \). But since

\[
\phi_s(t) = \tau(t/n)(x-t)^s \left( \frac{x-t}{n} \right)
\]

and

\[
\tau(t/n) = 1 \quad \text{for} \quad |t| \leq n/2,
\]

it follows that if \( t, x \) and \( n \) are subject to the above inequalities (2), then

\[
\phi_s^{(2r+1)}(t_0) = 0.
\]

Thus

\[
(x^{-2r-1})_n \ast (x^n)_n = \int_{n/4}^{n/4} \left\{ \phi_s(t) - \phi_s(-t) - 2 \sum_{i=1}^r \frac{t^{2i-1}}{(2i-1)!} \phi_s^{(2i-1)}(0) \right\} dt
\]

\[
= n^{-2r} \int_{n/4}^{n/4} u^{-2r-1} \left\{ \phi_s(nu) - \phi_s(-nu) - 2 \sum_{i=1}^r \frac{(nu)^{2i-1}}{(2i-1)!} \phi_s^{(2i-1)}(0) \right\} du,
\]

where \( t = nu \). Obviously

\[
\lim_{n \to \infty} n^{-2r} \int_{n/4}^{n/4} u^{-2r-1} \sum_{i=1}^r \frac{(nu)^{2i-1}}{(2i-1)!} \phi_s^{(2i-1)}(0) du = 0.
\]

Since

\[
\phi_s(nu) - \phi_s(-nu) = \tau(u)(x-nu)^s \left( \frac{x-nu}{n} \right) - \tau(u)(x+nu)^s \left( \frac{x+nu}{n} \right)
\]
it is obvious that, if \( s < 2r \),
\[
\lim_{n \to \infty} n^{-2r} \int_{\frac{1}{n}}^{1} u^{-2r-1} \{ \phi_s(nu) - \phi_s(-nu) \} \, du = 0
\]
and if \( s = 2r \)
\[
\lim_{n \to \infty} n^{-2r} \int_{\frac{1}{n}}^{1} u^{-2r-1} \{ \phi_{2r}(nu) - \phi_{2r}(-nu) \} \, du \\
= \lim_{n \to \infty} \int_{\frac{1}{n}}^{1} u^{-2r-1} \tau(u) \left\{ \tau \left( \frac{x}{n} - u \right) - \tau \left( \frac{x}{n} + u \right) \right\} \, du \\
= 0
\]
since
\[
\lim_{n \to \infty} \tau \left( \frac{x}{n} - u \right) = \lim_{n \to \infty} \tau \left( \frac{x}{n} + u \right) = \tau(u).
\]

We have thus proved that
\[
\lim_{n \to \infty} (x^{-2r-1})_a(x^a)_n = 0
\]
for each \( x \), the convergence obviously being uniform on every finite interval.

Thus for arbitrary fine function \( \phi \) we have
\[
\lim_{n \to \infty} \int_{-\infty}^{\infty} \phi(x)(x^{-2r-1})_a(x^a)_n \, dx = 0
\]
since a fine function vanishes identically outside a finite interval. This completes the proof of equation (1) for odd \( \tau \).

Next we have
\[
(x^{-2r})_a(x^a)_n = (t^{-2r}, \phi_x(t))
\]
\[
= \int_{0}^{\infty} t^{-2r} \left\{ \phi_x(t) + \phi_x(-t) - 2 \sum_{i=0}^{r-1} \frac{t^{2i}}{(2i)!} \phi_x^{(2i)}(0) \right\} \, dt \\
= \int_{0}^{\infty} t^{-2r} \left\{ \phi_x(t) + \phi_x(-t) - 2 \sum_{i=0}^{r-1} \frac{t^{2i}}{(2i)!} \phi_x^{(2i)}(0) \right\} \, dt.
\]

From Taylor's Theorem we have
\[
\phi_x(t) + \phi_x(-t) - 2 \sum_{i=0}^{r-1} \frac{t^{2i}}{(2i)!} \phi_x^{(2i)}(0) = \frac{2\xi t^{2r}}{(2r-1)!} \phi_x^{(2r)}(t_0),
\]
where \( 0 \leq \xi \leq 1 \) and \( -\xi t \leq t_0 \leq \xi t \). It follows that if \( t, x \) and \( n \) again satisfy inequalities (2), then
\[
\phi_x^{(2r)}(t_0) = 0.
\]
Thus

\[ (x^{-2r})_n \ast (x^z)_n = \int_{-\frac{1}{4}}^{\frac{1}{4}} t^{-2r} \left\{ \phi_x(t) + \phi_x(-t) - 2 \sum_{i=0}^{r-1} \frac{t^{2i}}{(2i)!} \phi_x^{(2i)}(0) \right\} dt \]

and it follows, as in the previous case, that if \( s \leq 2r - 1 \)

\[ \lim_{n \to \infty} (x^{-2r})_n \ast (x^z)_n = 0 \]

for each \( x \), the convergence being uniform on every finite interval.

Again, for arbitrary fine function \( \phi \), we have

\[ \lim_{n \to \infty} \int_{-\infty}^{\infty} \phi(x)(x^{-2r})_n \ast (x^z)_n dx = 0, \]

which completes the proof of equation (1).

REFERENCES


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