A note on centralizers of involutions involving simple groups

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If a finite group $G$ has a 'central' involution $t$ whose centralizer in $G$ is $\langle t \rangle \times H$ then, under certain conditions on $H$, $G$ cannot be simple.

The purpose of this note is to observe the following generalization of the theorem of [1]. Let $i(X)$ denote the number of conjugacy classes of involutions of a group $X$.

**THEOREM.** Let $G$ be a finite group possessing a 'central' involution $t$ such that $C_G(t) = \langle t \rangle \times H$ where $H$ is a non-abelian simple group such that

(i) the centre of a Sylow 2-subgroup $S$ of $H$ is cyclic, and

(ii) the involution $\tau$ of $Z(S)$ is a square in $H$.

Then

(a) if $i(H) = 1$ then $G = O(G).C_G(t)$;

(b) if $i(H) = 2$ then $G$ has a subgroup of index 2.

**Proof.** Let $\chi$ denote the permutation character of the representation of $G$ on the left cosets of $C_G(t)$. Suppose $i(H) = 1$. Since $t$ is a non-square in $G$, (ii) implies that $t \nmid \tau$ in $G$. If $G \neq O(G).C_G(t)$ then $t \sim \tau t$ from [2]. By inducing the identity character of $C_G(t)$ to $G$ one sees that $\chi(t) \equiv 0 \pmod{2}$. Therefore

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\[ [G : C_G(t)] = \chi(1) \equiv 0 \pmod{2}, \]

contradicting the fact that \( t \) is 'central' in \( G \).

Suppose \( i(H) = 2 \) and let \( \tau \in H \) be an involution not conjugate to \( \tau \) in \( H \). Again \( t \perp \tau \) in \( G \). If \( t \sim t\tau \) in \( G \) then, since \( Z(S) \) has a unique involution, one again obtains \( \chi(t) \equiv 0 \pmod{2} \) and \( \chi(1) \equiv 0 \pmod{2} \), a contradiction. Therefore \( t \perp t\tau \) in \( G \). Suppose \( G \) has no subgroup of index 2. Then by a well-known transfer lemma (see, for example, [3], p. 265) \( S \) must contain a representative of each conjugacy class of involutions of \( G \). Therefore \( i(G) = 2 \). Since \( t \) is conjugate in \( G \) to neither of \( \tau, t\tau \), one must have \( \tau \sim t\tau \) in \( G \). Since \( Z(S) \) has a unique involution one sees that \( \chi(\tau) \equiv 0 \pmod{2} \). Therefore \( [G : C_G(t)] = \chi(1) \equiv 0 \pmod{2} \), contradicting the fact that \( t \) is 'central' in \( G \). The theorem is proved.

REMARK. Most of the sporadic simple groups, including the Mathieu groups, satisfy the conditions imposed on \( H \).

References


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