J. Austral. Math. Soc. (*Series A*) **45** (1988), 66–77

PRIMITIVE PERMUTATION GROUPS WITH A DOUBLY TRANSITIVE SUBCONSTITUENT

CHERYL E. PRAEGER

(Received 5 June 1986)

Communicated by H. Lausch

Dedicated to Robert Edwards in recognition of 25 years' distinguished contribution to mathematics in Australia, on the occasion of his retirement

Abstract

Let G be a primitive permutation group on a finite set Ω. We investigate the subconstitutents of G, that is the permutation groups induced by a point stabilizer on its orbits in Ω, in the cases where G has a diagonal action or a product action on Ω. In particular we show in these cases that no subconstituent is doubly transitive. Thus if G has a doubly transitive subconstituent we show that G has a unique minimal normal subgroup N and either N is a nonabelian simple group or N acts regularly on Ω: we investigate further the case where N is regular on Ω.

1980 *Mathematics subject classification* (*Amer. Math. Soc.*): 20 B 15, 05 C 25.

Finite primitive permutation groups with a doubly transitive subconstituent were first studied by W. A. Manning (see [12, 17.7]). His results were generalized by P. J. Cameron ([2] and see also [4, 8, 9]). The analogues of these groups in the area of symmetric graphs, namely, 2-arc transitive graphs, have also received a great deal of attention in the literature. In this paper we show that these groups have a unique minimal normal subgroup which either is a nonabelian simple group or is regular. We begin by studying the nature of the subconstituents of a primitive group G with a diagonal or a product action: we show in particular

The work for this paper was supported by a special CTEC grant at the University of Western Australia.

https://doi.org/10.1017/S1446788700032274 Published online by Cambridge University Press

that no subconstituent is doubly transitive for these groups. Using the O'Nan Scott Theorem (see [1, 7, or 11]) we can then deduce immediately

THEOREM A. *Let G be a primitive permutation group on a finite set Ω such that the stabilizer of $\alpha \in \Omega$ is doubly transitive on one of its orbits in $\Omega \backslash \{\alpha\}$. Then either*

(a) $T \leq G \leq \operatorname{Aut} T$ *for a nonabelian simple group T, or*

(b) G *has a unique minimal normal subgroup which is regular on Ω.*

We continue the investigation of those groups with a regular normal subgroup.

1. Primitive groups with a simple diagonal action

Here we assume that $G \leq X = N(\operatorname{Aut} T \times S_k)$ is a primitive permutation group on Ω where we assume the following.

(i) The group N is the socle of G and $N = T_1 \times \cdots \times T_k$ is a direct product of $k \geq 2$ groups T_i, each isomorphic to a fixed nonabelian simple group T.

(ii) The group $\operatorname{Aut} T$ acts naturally on each factor T_i of N and each inner automorphism $\sigma_t \colon x \to x^t$, for $x, t \in T$, is identified with a diagonal element $(t, t, \ldots, t)$ of N, that is $\operatorname{Aut} T \cap N = D = \{(t, t, \ldots, t) | t \in T\}$.

(iii) The group S_k permutes the set $\{T_1, \ldots, T_k\}$ naturally and the subgroup P of S_k induced by G is either primitive of degree k, or $k = 2$ and $P = 1$.

(iv) We may assume that for some $\alpha \in \Omega$, $X_\alpha = \operatorname{Aut} T \times S_k$ with the group of inner automorphisms of T identified with the subgroup D of N; then $D \leq G_\alpha \leq A(T) \times P$ where $A(T)$ is the projection of G_α to $\operatorname{Aut} T$ and $D \leq A(T)$. The points of Ω can be identified with the set of right cosets of D in N so that

$$\alpha = D(1, 1, \ldots, 1)$$

and for $\beta = D(t_1, \ldots, t_k) \in \Omega$, $\mathbf{s} = (s_1, \ldots, s_k) \in N$, $\sigma \in \operatorname{Aut} T$ and $\tau \in S_k$ we have

$$\begin{aligned} \beta^{\mathbf{s}} &= D(t_1 s_1, \ldots, t_k s_k), \\ \beta^{\sigma} &= D\left(t_1^{\sigma}, \ldots, t_k^{\sigma}\right), \\ \beta^{\tau} &= D\left(t_{1\tau^{-1}}, \ldots, t_{k\tau^{-1}}\right). \end{aligned}$$

Here we are taking τ to act on the set $\{1, 2, \ldots, k\}$ in the natural way. We note that $|\Omega| = |T|^{k-1}$, and that each coset of D has a unique member with kth entry equal to 1. Let Π_A, Π_P denote the projections of G_α onto $A(T)$ and P respectively. We shall investigate G_α-orbits Δ which are quasiprimitive (that is all normal subgroups of G_α either fix Δ pointwise or are transitive on Δ). We begin by showing that the only fixed point of D in Ω is α.

https://doi.org/10.1017/S1446788700032274 Published online by Cambridge University Press

LEMMA 1.1. *The diagonal subgroup D fixes only the point $\alpha = D(1,\dots,1)$.*

PROOF. Suppose that D fixes $\delta = D(t_1,\dots,t_{k-1},1) \neq \alpha$. Since $\delta \neq \alpha$ at least one entry say t_i is not the identity element of T. Now for $\mathbf{t} = (t,\dots,t) \in D$ we have $\delta = \delta^{\mathbf{t}} = D(t^t, t^t, \dots, t^t_{k-1}, 1)$ and as each D-coset contains a unique element with kth entry 1 we have $t_i = t_i^t$: since this is true for all $t \in T$ it follows that t_i is in the centre of T. This contradiction completes the proof.

Thus if G_α is quasiprimitive on one of its orbits Δ then D is transitive on Δ:

LEMMA 1.2. *Suppose that D is transitive on a G_α-orbit $\Delta \neq \{\alpha\}$, and that $\delta = D(t_1,t_2,\dots,t_k) \in \Delta$. Then $t_1,t_2,\dots,t_k$ are distinct elements of T.*

PROOF. This is true if $k = 2$ so suppose that $k \geq 3$: then P is primitive on $\{T_1,\dots,T_k\}$. Suppose that exactly x entries t_i in $(t_1,t_2,\dots,t_k)$ are equal to t_k; then $1 \leq x < k$ since $\delta \neq \alpha$. Let i be an index for which $t_i \neq t_k$. Then for some $\tau \in P$ we have $i^\tau = k$, and for some $\sigma \in A(T)$ we have $\sigma\tau \in G_\alpha$: thus Δ contains $\delta^{\sigma\tau} = D(t^\sigma_{1\tau^{-1}},\dots,t^\sigma_{k\tau^{-1}})$ with kth entry $t^\sigma_{k\tau^{-1}} = t^\sigma_i$. However as D is transitive on Δ, $\delta^{\sigma\tau} = \delta^{\mathbf{t}}$ for some $\mathbf{t} = (t,\dots,t) \in D$. It follows that exactly x entries in $(t^\sigma_{1\tau^{-1}},\dots,t^\sigma_{k\tau^{-1}})$ are equal to $t^\sigma_{k\tau^{-1}} = t^\sigma_i$ and hence that exactly x entries in $(t_1,\dots,t_k)$ are equal to t_i. Thus the partition of $\{1,2,\dots,k\}$ determined by equality of the elements $t_1,t_2,\dots,t_k$ has k/x blocks of size x. Further this partition is independent of the coset representative $(t_1,\dots,t_k)$ chosen for δ, and as D is transitive on Δ this partition is independent of the point δ chosen from Δ. Thus this partition must be preserved by P. Since P is primitive and $x < k$ we must have $x = 1$.

Next we investigate the case where G_α is unfaithful on Δ.

LEMMA 1.3. *Suppose that $\Delta \neq \{\alpha\}$ is a G_α-orbit such that D^Δ is transitive and the kernel K of G_α on Δ is nontrivial. The following hold.*

(a) *The group $K \leq G_\alpha \cap P$ and K is regular on $\mathcal{T} = \{T_1,\dots,T_k\}$. Moreover if $\delta = D(t_1,t_2,\dots,t_k) \in \Delta$ with $t_k = 1$ then $S = \{t_1,t_2,\dots,t_k = 1\}$ is a subgroup of T isomorphic to K: the map $\varphi\colon K \to S$ defined by $\varphi(\tau) = t_{k\tau^{-1}}$ for $\tau \in K$ is an isomorphism.*

(b) *If $K = G_\alpha \cap P$ then $K \simeq S$ is elementary abelian, and P is soluble. Further if G_α^Δ is primitive then $G_{\alpha\delta} = G_\alpha \cap (N_{A(T)}(S) \times P)$ and $N(S) \cap (G_\alpha \cap A(T)) = C(S) \cap (G_\alpha \cap A(T))$: in particular S is not a Sylow subgroup of $G_\alpha \cap A(T)$. The orbit Δ is self-paired if and only if some element of T inverts (each element of) S, and if Δ is self-paired and primitive then S is an elementary abelian 2-group.*

(c) *If $G_\alpha \cap P$ is transitive on Δ then G_α^Δ is primitive with two regular normal subgroups $(G_\alpha \cap P)^\Delta$ and $(G_\alpha \cap A(T))^\Delta$ each isomorphic to T (so $G_\alpha \cap A(T) = D$). Moreover $S = T \simeq K$ and $G_\alpha \cap P \simeq T \times T$ so that $k = |T|$. (We can*

https://doi.org/10.1017/S1446788700032274 Published online by Cambridge University Press

change the subscripts $1, 2, \ldots, k$ *of the* T_i *to elements of* T *and choose a coset representative for* δ *with* x*-entry* t_x *and* $t_1 = 1$ *so that* K *identified with* T *acts on the subscripts by left multiplication. Then for some* $\sigma = \sigma(\delta) \in \mathrm{Aut}\, T$, $t_x = x^\sigma$ *for all* $x \in T$: *and the stabilizer of* δ *in* $(D \times (G_\alpha \cap P))^\Delta$ *is* $\{(\mathbf{x}, \sigma x \sigma^{-1}) | x \in T\} \simeq T$, *where* $\mathbf{x} = (x, x, \ldots, x)$.)

REMARKS 1.4. (a) In part (a), K is a regular normal subgroup of a primitive group P and hence is elementary abelian or a direct power of a nonabelian simple group. We show in (b) that K is elementary abelian when $G_\alpha \cap P = K$.

(b) If G_α^Δ is quasiprimitive then $(G_\alpha \cap P)^\Delta$ is transitive or trivial: the latter case corresponds to K being elementary abelian. In the former case we show in (c) that $K \simeq T$ and G_α^Δ is in fact primitive with two regular normal subgroups isomorphic to T.

(c) If G_α^Δ were 2-transitive then G_α^Δ would have a unique minimal normal subgroup, namely D^Δ, so case (c) could not hold. Further in none of the almost simple 2-transitive groups does D_δ^Δ have a nontrivial centre, and hence case (b) does not hold either. Thus in Lemma 1.3, G_α^Δ is not 2-transitive.

PROOF. (a) Since D is faithful on Δ, $K \cap A(T) = K \cap D = 1$ and it follows that K centralizes D. Hence $\Pi_A(K)$ centralizes D, and as D has trivial centralizer in $A(T)$ we conclude that $\Pi_A(K) = 1$, that is $K \subseteq P$. Thus we have $K \leq G_\alpha \cap P$, (so $P \neq 1$).

Now K is normal in $\Pi_P(G_\alpha) = P$ and as P is primitive on $\mathcal{T} = \{T_1, \ldots, T_k\}$ it follows that$K$ is transitive on $\mathcal{T}$. If $\tau \in K$ fixes k then since $\delta = \delta^\tau = D(t_{1\tau^{-1}}, \ldots, t_{k\tau^{-1}})$ with $t_{k\tau^{-1}} = t_k = 1$ we must have $t_{i\tau^{-1}} = t_i$ for all i: since all the t_i are distinct it follows that $\tau = 1$. Thus K is regular on $\mathcal{T}$.

Now for each $i \in \{1, \ldots, k\}$ there is a unique $\tau = \tau_i \in K$ such that $i^\tau = k$: then $\delta^\tau = \delta$ yields $t_{j\tau^{-1}} = t_i t_j$ for each $j \in \{1, \ldots, k\}$. Thus $S = \{t_1, t_2, , \ldots, t_k = 1\}$ is closed under multiplication and hence is a subgroup of T. Further $\tau = \tau_i$ induces a permutation of S by $t_j^\tau = t_{j\tau^{-1}}$ and we have seen that this permutation is simply left multiplication by t_i. Moreover by definition of τ_i we have $\tau_i \tau_j = \tau_x$ where $x = j\tau_i^{-1}$ and we also have $t_i t_j = t_{j\tau_i^{-1}} = t_x$. It follows that the map $\varphi(\tau_i) = t_i$, is an isomorphism.

(b) Assume next that $G_\alpha \cap P = K$. Then $A(T) = \Pi_A(G_\alpha) \simeq G_\alpha/(G_\alpha \cap P) = G_\alpha/K$, $P/K = \Pi_P(G_\alpha)/\Pi_P(K) \simeq G_\alpha/K(G_\alpha \cap A(T))$ and $K(G_\alpha \cap A(T))/K \simeq (G_\alpha \cap A(T))/(K \cap (G_\alpha \cap A(T))) \simeq G_\alpha \cap A(T)$. It follows that P/K is isomorphic to a section of Out T and hence P/K is soluble. From [1] the only primitive groups P with a regular normal subgroup K such that P/K is soluble are those of affine type, that is those with K elementary abelian.

Now Δ is self-paired if and only if Δ contains $\delta^{-1} = D(t_1^{-1}, t_2^{-1}, \ldots, t_k^{-1})$, and as D is transitive on Δ, this is true if and only if some $t \in T$ inverts each of the t_i.

For $\gamma = D(s_1, s_2, \ldots, s_k) \in \Delta$, with $s_k = 1$, let $S(\gamma) = \{s_1, s_2, \ldots, s_k = 1\}$. Then as D^Δ is transitive $S(\gamma)$ is a subgroup of T conjugate to S. The subset $\Delta_S = \{\gamma \in \Delta \mid S(\gamma) = S\}$ is clearly a block of imprimitivity for G_α in Δ, and Δ_S contains δ^{-1} if Δ is self-paired. If G_α^Δ is primitive then $\Delta_S = \{\delta\}$ (so that $\delta = \delta^{-1}$, that is S is a 2-group, whenever Δ is self-paired). Now an element $\sigma\tau \in G_\alpha$, where $\sigma \in A(T)$, $\tau \in P$, fixes Δ_S setwise if and only if $\sigma \in N_{A(T)}(S)$, that is the setwise stabilizer of Δ_S in G_α is $G_\alpha \cap (N_{A(T)}(S) \times P)$. Thus if G_α^Δ is primitive then $G_{\alpha\delta}$ is $G_\alpha \cap (N_{A(T)}(S) \times P)$; moreover $\sigma \in G_\alpha \cap N_{A(T)}(S)$ fixes δ if and only if σ centralizes S, so $G_\alpha \cap N_{A(T)}(S) = G_\alpha \cap C_{A(T)}(S)$: by [6, 7.4.3] S is not a Sylow subgroup of $G_\alpha \cap A(T)$.

(c) Finally assume that $G_\alpha \cap P$ is transitive on Δ. Then $G_\alpha^\Delta = G_\alpha/K$ has normal subgroups $(G_\alpha \cap A(T))^\Delta = ((G_\alpha \cap A(T)) \times K)/K$ and $(G_\alpha \cap P)^\Delta = (G_\alpha \cap P)/K$ with trivial intersection. Thus $(G_\alpha \cap A(T))^\Delta$ centralizes $(G_\alpha \cap P)^\Delta$ and as both are transitive it follows that both are regular and are isomorphic to each other: thus as D^Δ is transitive we have $G_\alpha \cap A(T) = D \simeq (G_\alpha \cap P)/K$. Now the stabilizer of δ in $D^\Delta \times (G_\alpha \cap P)^\Delta$ is a diagonal subgroup and hence for each $\mathbf{t} = (t, t, \ldots, t) \in D$ there is a $\tau \in G_\alpha \cap P$ such that $\delta^{\mathbf{t}\tau} = \delta$. If we define $S(\gamma)$ as above for $\gamma \in \Delta$, we have $S(\delta^{\mathbf{t}\tau}) = S^t$ and hence t normalizes S. Since this holds for all $t \in T$ we must have $S = T$. It follows that $G_\alpha \cap P = K \times L$ where $L \simeq (G_\alpha \cap P)^\Delta \simeq T$. Since K is regular on T we may replace the set of labels $\{1, 2, \ldots, k\}$ by the set T so that K identified with T acts by left multiplication: we choose a coset representative for δ with x-entry t_x and $t_1 = 1$. Then we have, for each $y \in K = T$, $\delta = \delta^y$ and hence for each $x \in T$, $t_x = t_y^{-1} t_{yx}$ that is $t_y t_x = t_{yx}$. Thus there is an isomorphism $\sigma \in \operatorname{Aut} T$ such that $t_x = x^\sigma$. Now L must act on the labels by right multiplication and we find that for $y \in L$, with L identified with T, and for $\mathbf{x} = (x, \ldots, x) \in D$, $\mathbf{x}y$ fixes δ if and only if $(yzy^{-1})^{\sigma x} = z^\sigma$ for all $z \in T$, and this holds if and only if $\sigma = y^{-1}\sigma x$, that is $y = \sigma x \sigma^{-1}$. Thus $(D \times (G_\alpha \cap P))_\delta^\Delta = \{(\mathbf{x}, \sigma x \sigma^{-1}) | x \in T\}$.

Finally we consider the case where G_α is faithful and quasiprimitive on Δ.

LEMMA 1.5. *Suppose that $\Delta \neq \{\alpha\}$ is a faithful G_α-orbit such that D^Δ is transitive.*

(a) *If $G_\alpha \cap P = 1$ then $G_\alpha \simeq A(T)$ and $P \simeq G_\alpha/(G_\alpha \cap A(T))$ which is isomorphic to a section of $\operatorname{Out} T$ and hence is soluble.*

(b) *If $(G_\alpha \cap P)^\Delta$ is transitive then $G_\alpha \cap A(T) = D \simeq G_\alpha \cap P$; both D and $G_\alpha \cap P$ are regular on Δ. In this case G_α^Δ is primitive. (If we identify $G_\alpha \cap P$ with T then with $\delta = D(t_1, \ldots, t_k) \in \Delta$ and $t_k = 1$, there is an automorphism σ*

of T such that $(D \times (G_\alpha \cap P))_\delta = \{(\mathbf{x}^\sigma, x) | x \in T\}$ where $\mathbf{x}^\sigma = (x^\sigma, x^\sigma, \dots, x^\sigma)$ and for all $1 \le i \le k$ we have $x^{-\sigma} t_i x^\sigma = t_{kx^{-1}}^{-1} t_{ix^{-1}}$.)

REMARKS 1.6. (a) In part (a) the stabilizer of δ in D is $\{\mathbf{t} | t \in \bigcap_{1 \le i \le k} C_T(t_i)\}$ and as in Remark 1.4(c) we see that G_α^Δ cannot be 2-transitive. In part (b), G_α^Δ has two minimal normal subgroups and again cannot be 2-transitive. Thus we can conclude

COROLLARY 1.7. *If G has a simple diagonal action on Ω then G has no 2-transitive subconstituents.*

(b) In case (a) if we identify D with T then $D_\delta = \bigcap_{1 \le i \le k} C_T(t_i)$ that is D_δ is the centralizer in T of the set $S(\delta) = \{t_1, t_2, \dots, t_k = 1\}$. Since $G_\alpha = DG_{\alpha\delta}$ we have $P = \Pi_P(G_{\alpha\delta})$ and so for each $\tau \in P$ there is a $\sigma \in A(T)$ such that $\sigma\tau \in G_{\alpha\delta}$: moreover $\sigma\tau \in G_{\alpha\delta}$ if and only if $t_i = (t_{k\tau^{-1}}^{-1} t_{i\tau^{-1}})^\sigma$ for all $1 \le i \le k$. Now $\sigma\tau \in G_\alpha$ induces a map from $S(\delta)$ to $S(\delta^{\sigma\tau})$ and $G_{\alpha\delta}$ fixes $S(\delta)$ setwise. If G_α^Δ is primitive then $G_{\alpha\delta}$ must be the full setwise stabilizer of $S(\delta)$ in G_α (since the other possibility that $S(\delta) = S(\gamma)$ for all $\gamma \in \Delta$ is not allowed: for $S(\delta)$ would contain the conjugacy class in D of each t_i and hence $D_\delta = \bigcap C_T(t_i) = 1$. This is not possible for an almost simple primitive group).

(c) In case (b) since D is regular on δ we have

$$\bigcap_{1 \le i \le k} C_T(t_i) = 1.$$

Also since $G_\alpha \cap P$ is transitive on Δ, each $\gamma = D(x_1, \dots, x_k) \in \Delta$ with $x_k = 1$ has its entries x_i of the form $t_j^{-1} t_l$ for some j, l. It follows that the set S of entries in points of Δ on the one hand is the union of the T-conjugacy classes of the t_i (since D^Δ is transitive) and on the other hand is $\{t_j^{-1} t_l | j, l \in [1, k]\}$: that is S is a normal set with a kind of closure property. We make also the following observations.

(i) If $t \in G_\alpha \cap P = T$ fixes positions k and i, for some $1 \le i < k$ then t^σ centralizes t_i.

(ii) If Δ is self-paired then $\delta^{-1} = D(t_1^{-1}, t_2^{-1}, \dots, t_{k-1}^{-1}, 1) \in \Delta$ so, for some $\mathbf{t} = (t, \dots, t) \in D$, $\delta^{\mathbf{t}} = \delta^{-1}$, that is t inverts each t_i. Thus for any two such elements $t = x$ and $t = y$ say the product xy^{-1} centralizes all the t_i and we deduce $xy^{-1} = 1$; so t is unique and has order at most 2.

PROOF OF LEMMA 1.5. (a) If $G_\alpha \cap P = 1$ then $A(T) = \Pi_A(G_\alpha) \simeq G_\alpha$ and $P = \Pi_P(G_\alpha) \simeq G_\alpha / (G_\alpha \cap A(T))$ which is isomorphic to a quotient of $A(T)/D$: thus (a) holds.

(b) Suppose that $(G_\alpha \cap P)^\Delta$ is transitive. As in the proof of Lemma 1.3(c), $(G_\alpha \cap A(T))^\Delta$ and $(G_\alpha \cap P)^\Delta$ centralise each other, so each is regular on Δ and

$G_\alpha \cap A(T) = D \simeq G_\alpha \cap P$. Thus $G_{\alpha\delta} \cap (D \times (G_\alpha \cap P)) = \{(\mathbf{x}^\sigma, x) | x \in T\}$ for some $\sigma \in \operatorname{Aut} T$ where $\mathbf{x}^\sigma = (x^\sigma, x^\sigma, \ldots, x^\sigma)$ and so $x^{-\sigma} t_i x^\sigma = t_{kx^{-1}} t_{ix^{-1}}$ for all i.

2. Primitive groups with a product action

Here we assume that $G \leq X = H \operatorname{wr} S_x$ acts primitively on $\Omega = \Gamma^x = \Gamma_1 \times \cdots \times \Gamma_x$ for some $x \geq 2$ where the following hold.

(i) The group H is a primitive permutation group on Γ and $T \leq H \leq \operatorname{Aut} T$ for a nonabelian simple group T, or H has a simple diagonal action on Γ as described in the previous section, say H has socle T^y with T a nonabelian simple group and $y \geq 2$.

(ii) The group induced by G on Γ_i is H_i, a copy of H: the base group of X is $H^x = H_1 \times \cdots \times H_x$, and G and X have the same socle $N = T^{xy} = T_1 \times \cdots \times T_{xy}$ (where $y = 1$ if H is almost simple and the socle of H_i is

$$\operatorname{soc} H_i = T_{(i-1)y+1} \times \cdots \times T_{iy}).$$

(iii) The top group S_x of X permutes the sets $\mathcal{G} = \{\Gamma_1, \ldots, \Gamma_x\}$ and $\mathcal{H} = \{H_1, \ldots, H_x\}$ naturally, and G induces a transitive subgroup P of S_x.

(iv) For $\alpha = (\gamma, \gamma, \ldots, \gamma) \in \Omega = \Gamma^x$ the stabilizer $G_\alpha = G \cap (H_\gamma \operatorname{wr} S_x)$, G_α contains $N_\alpha = (\operatorname{soc} H)_\gamma^x$, and as $G = NG_\alpha$, G_α induces P on $\mathcal{G}$ and $\mathcal{H}$.

THEOREM 2.1. *Suppose that G is a primitive permutation group on Ω with the product action as described above, and suppose that Δ is a G_α-orbit in $\Omega \backslash \{\alpha\}$.*

(a) *If G_α is quasiprimitive on Δ then $\Delta = \Delta(\gamma)^x$ where $\Delta(\gamma)$ is an orbit of H_γ in Γ.*

(b) *If $(H \operatorname{wr} P)_\alpha$ is quasiprimitive on Δ then H_γ is quasiprimitive on $\Delta(\gamma)$.*

(c) *Also $(H \operatorname{wr} P)_\alpha$ is primitive on Δ if and only if H_γ is primitive on $\Delta(\gamma)$. Thus if G_α is primitive on Δ then H_γ is primitive on $\Delta(\gamma)$.*

(d) *The orbit Δ is self-paired for G_α if and only if $\Delta(\gamma)$ is a self-paired oribt of H_γ.*

REMARKS 2.2. (a) We have not quite shown that the action of H_γ on $\Delta(\gamma)$ must be quasiprimitive; the problem is that some $M \lhd H_\gamma$ with $M^{\Delta(\gamma)} \neq 1$ may be such that $(M^x \cap G_\alpha)^\Delta = 1$, so that the quasiprimitivity of G_α^Δ yields no information about the action of M on $\Delta(\gamma)$. Can this situation really occur? (Note that $(M^x \cap G_\alpha)^\Delta = 1$ implies that $(M \cap (\operatorname{soc} H)_\gamma)^{\Delta(\gamma)} = 1$.)

(b) If $\gamma' \in \Delta(\gamma)\backslash\{\gamma\}$ then $\delta' = (\gamma', \gamma', \gamma', \dots, \gamma') \in \Delta$ and $\delta'' = (\gamma', \gamma, \dots, \gamma) \in \Delta$ and it is not possible for an element of $G_{\alpha\delta}$ to map δ' to δ''. Thus

COROLLARY 2.3. *A primitive group with product action has no 2-transitive subconstitutents.*

In fact similar considerations show that G_α must have rank at least $x+1$ on Δ and that if G_α^Δ has rank $x+1$ then $P \supseteq A_x$ and $H_\gamma^{\Delta(\gamma)}$ is 2-transitive.

PROOF OF THEOREM 2.1. Let $\Delta_0(\gamma) = \{\gamma\}$, $\Delta_1(\gamma), \dots, \Delta_r(\gamma)$ be the orbits of H_γ in Γ where $r \geq 1$. Let $S = \{\mathbf{j} = (j_1, \dots, j_x) | 0 \leq j_i \leq r$ for each $i = 1, \dots, x\}$. Then P acts on S by permuting coordinates in the same way as it acts on the set $\mathcal{G} = \{\Gamma_1, \dots, \Gamma_x\}$. Moreover the orbits of $H_\gamma \operatorname{wr} P$ in Ω are in one-to-one correspondence with the orbits of P in S, namely the orbit $\Delta_J(\alpha)$ of $(H \operatorname{wr} P)_\alpha$ in Ω corresponding to the orbit J of P in S is the union of $\Delta_{\mathbf{j}}(\alpha) = \Delta_{j_1}(\gamma) \times \cdots \times \Delta_{j_x}(\gamma)$ over all $\mathbf{j} \in J$. Thus $\Delta \subseteq \Delta_J(\alpha)$ for some orbit $J \neq \{(0, 0, \dots, 0)\}$ of P in S. By [10, Lemma 2.3] $(\operatorname{soc} H)_\gamma$ acts nontrivially on $\Delta_j(\gamma)$ for all $j \neq 0$ and it follows that $(\operatorname{soc} H)_\gamma^x (\lhd G_\alpha \cap H^x)$ is nontrivial and $\frac{1}{2}$-transitive on $\Delta_{\mathbf{j}}(\alpha)$ for each $\mathbf{j} \in J$. Now G_α and P induce the same action on $\mathcal{G}$ and hence on S and hence $\Delta \cap \Delta_{\mathbf{j}}(\alpha) \neq \varnothing$ for each $\mathbf{j} \in J$. Thus $(\operatorname{soc} H)_\gamma^x$ is nontrivial on Δ, is normal in G_α, and as G_α is quasiprimitive on Δ it follows that $(\operatorname{soc} H)_\gamma^x$ is transitive on Δ. Then as $(\operatorname{soc} H)_\gamma^x$ fixes $\Delta \cap \Delta_{\mathbf{j}}(\alpha)$ setwise for each $\mathbf{j} \in J$ it follows that $J = \{\mathbf{j} = (j, j, \dots, j)\}$ for some $j > 0$, and so $\Delta_J(\alpha) = \Delta_j(\gamma)^x$.

Now $(\operatorname{soc} H)_\gamma^x$ is transitive on Δ and hence $\Delta = A_1 \times \cdots \times A_x$ where each A_i is an orbit of $(\operatorname{soc} H)_\gamma$ in $\Delta_j(\gamma)$. However by [10, Lemma 2.2(b)] the setwise stabilizer of Γ_i in G_α induces H_γ of Γ_i and hence $A_i = \Delta_j(\gamma)$ for each i, that is, $\Delta = \Delta_j(\gamma)^x$. If $(H \operatorname{wr} P)_\alpha^\Delta$ is quasiprimitive then clearly $H_\gamma^{\Delta_j(\gamma)}$ is quasiprimitive, and if $(H \operatorname{wr} P)_\alpha^\Delta$ is primitive then also $H_\gamma^{\Delta_j(\gamma)}$ is primitive. Conversely if H_γ is primitive on $\Delta_j(\gamma)$ then $(H \operatorname{wr} P)_\alpha^\Delta$ is primitive by [3]. Finally Δ is clearly self paired if and only if $\Delta_j(\gamma)$ is self paired.

3. Primitive groups with a doubly transitive subconstituent

Suppose that G is a primitive permutation group on Ω and that for $\alpha \in \Omega$, G_α is 2-transitive on one of its orbits $\Gamma(\alpha) \subseteq \Omega\backslash\{\alpha\}$. By Corollaries 1.7 and 2.3 G has neither a simple diagonal action nor a product action on Ω. It follows from the O'Nan Scott Theorem [1] that Theorem A is true. In this section we shall discuss the case where G has a unique minimal normal subgroup N which is regular on Ω. We consider first the case where N is abelian.

https://doi.org/10.1017/S1446788700032274 Published online by Cambridge University Press

PROPOSITION 3.1. *Suppose that G is a simply transitive primitive permutation group on Ω with an elementary abelian regular normal subgroup N, and suppose that G_α, $\alpha \in \Omega$, is doubly transitive on one of its orbits $\Gamma(\alpha) \subseteq \Omega\backslash\{\alpha\}$. We identify Ω with N in such a way that α is identified with the zero of N (written additively).*

(a) *Then the orbit $\Gamma^*(\alpha)$ paired with $\Gamma(\alpha)$ is $-\Gamma(\alpha) = \{-\beta | \beta \in \Gamma(\alpha)\}$.*

(b) *The orbit $\Gamma(\alpha)$ is self-paired if and only if N is a 2-group.*

(c) *The orbit $\Gamma \circ \Gamma^*(\alpha) = \{\gamma|$ there is a β such that $\beta \in \Gamma(\alpha) \cap \Gamma(\gamma)$, and $\gamma \neq \alpha\}$ is self-paired.*

PROOF. Identifying Ω with N such that $\alpha = 0$, for each $\beta \in \Gamma(\alpha)$ we can translate by $-\beta \in N$ to obtain $-\beta \in \Gamma^*(\alpha)$. Hence $\Gamma^*(\alpha) = -\Gamma(\alpha) = \{-\beta|\beta \in \Gamma(\alpha)\}$. Thus if N is a 2-group then $\beta = -\beta$ and so $\Gamma(\alpha)$ is self-paired. Conversely if $\Gamma(\alpha)$ is self-paired then $\{\beta, -\beta\}$ is a block of imprimitivity in $\Gamma(\alpha)$ for G_α and so $\beta = -\beta$, that is N is a 2-group.

Finally if β and γ are distinct points of $\Gamma(\alpha)$ then translating $(\alpha, -\beta)$ and $(\alpha, -\gamma)$ by $\gamma \in N$ and $\beta \in N$ respectively we find that $\gamma - \beta \in \Gamma^*(\gamma)\backslash\{\alpha\} \subseteq \Gamma \circ \Gamma^*(\alpha)$ and $\beta - \gamma \in \Gamma^*(\beta)\backslash\{\alpha\} \subseteq \Gamma \circ \Gamma^*(\alpha)$. It follows as above that $\Gamma \circ \Gamma^*(\alpha)$ is self-paired. ($\Gamma \circ \Gamma^*(\alpha)$ was shown to be a G_α-orbit in [2].)

REMARKS 3.2. (a) A similar argument shows that for $\beta \in \Gamma(\alpha)$ we have $i\beta$ in $\Gamma(\alpha)$ where $i \in \mathbf{Z}$, if and only if $i\beta = \beta$ (since the set of $i\beta$ in $\Gamma(\alpha)$, $i \in \mathbf{Z}$, is a block of imprimitivity for G_α in $\Gamma(\alpha)$). In particular if $N = \mathbf{Z}_p^d$ and we regard G_α as a subgroup of $GL(d,p)$ then G_α contains no nontrivial scalar transformations. Similarly if in fact $G_\alpha \leq \Gamma L(d/a, p^a)$ then G_α contains no nontrivial $GF(p^a)$-scalar transformations.

(b) The stabilizer G_α regarded as a subgroup of $GL(d,p)$ is irreducible since G is primitive, and so we could choose a basis so that $\Gamma(\alpha)$ contains all the standard basis vectors $e_i = (o^{i-1}10^{d-i})$, $1 \leq i \leq d$. Suppose that $|\Gamma(\alpha)| = d$: then G_α permutes the standard basis vectors amongst themselves and so fixes the point $e_1 + e_2 + \cdots + e_d \neq \alpha$. This contradicts the fact that G is primitive, so always $|\Gamma(\alpha)| > d$.

Now we consider the case where G has a unique minimal normal subgroup N which is nonabelian and regular on Ω. Here (see [1] and [7]) $N = T_1 \times \cdots \times T_k$ where each T_i is isomorphic to a fixed nonabelian simple group T and $k \geq 2$: the group G is a twisted wreath product $T \operatorname{twr}_\varphi P$, where P is a transitive subgroup of S_k permuting the T_i naturally, and the twisting homomorphism $\varphi\colon P_1 \to \operatorname{Aut} T$ is such that $\varphi(P_1)$ contains the group of inner automorphisms of T. We can identify Ω with N so that α is identified with the identity $(1, 1, \ldots, 1)$ of N, and then $G_\alpha = P$.

https://doi.org/10.1017/S1446788700032274 Published online by Cambridge University Press

PROPOSITION 3.3. *Suppose that $G = T \operatorname{twr}_{\varphi} P$ is primitive on Ω with regular normal subgroup $N = T^k$ as described above. Suppose that G_α is doubly transitive on an orbit $\Gamma(\alpha) \subseteq \Omega \backslash \{\alpha\}$, where Ω is identified with N and $\alpha = (1, 1, \ldots, 1)$. Then G_α has an orbit $\hat{\Gamma}(\alpha)$ in $\Omega \backslash \{\alpha\}$, possibly equal to $\Gamma(\alpha)$, such that the actions of G_α on $\Gamma(\alpha)$ and $\hat{\Gamma}(\alpha)$ are permutationally equivalent, and for $\beta = (\beta_1, \ldots, \beta_k) \in \hat{\Gamma}(\alpha)$ all the nontrivial β_i lie in a single $\varphi(P_1)$-conjugacy class $\mathcal{C}$ and $G_{\alpha\beta} (\leq P)$ is transitive on the support $\Delta(\beta) = \{i | \beta_i \neq 1\}$ of β.*

If $\Gamma(\alpha)$ is self-paired then $\hat{\Gamma}(\alpha)$ is also self-paired and β has order 2 as an element of N.

PROOF. Choose a transversal $\{\rho_1 = 1, \rho_2, \ldots, \rho_k\}$ for P_1 in P such that $1\rho_i = i$ for all i. Then each $\sigma \in G_\alpha = P$ is such that the jth entry of β^σ is

$$\beta_i^{\rho_i \sigma \rho_j^{-1}}$$

where $i = j\sigma^{-1}$ and we write t^τ for the image of $t \in T$ under $\tau \in P$. Let $\Delta_1, \ldots, \Delta_r$ be the $G_{\alpha\beta}$-orbits in $\mathbf{k} = \{1, 2, \ldots, k\}$; we note that all β-entries in a fixed Δ_i are conjugate under $\varphi(P_1)$. Call Δ_i nontrivial if the entries in Δ_i-positions in β are not the identity. As $\beta \neq \alpha$, there is at least one nontrivial orbit, say Δ. Define $\beta(\Delta) \in N$ to have entries equal to β-entries at positions in Δ, and entries equal to 1 otherwise. Then $G_{\alpha\beta}$ fixes $\beta(\Delta)$, and as $G_{\alpha\beta}$ is a maximal subgroup of G_α and G_α does not fix the point $\beta(\Delta) \neq \alpha$, it follows that $G_{\alpha\beta} = G_{\alpha\beta(\Delta)}$. If $\hat{\Gamma}(\alpha)$ is the G_α-orbit containing $\beta(\Delta)$ then the G_α-actions on $\Gamma(\alpha)$ and $\hat{\Gamma}(\alpha)$ are equivalent and $G_{\alpha\beta(\Delta)}$ is transitive on Δ. Finally $\beta^{-1} \in \Gamma^*(\alpha)$ and arguing as in Proposition 3.1 we see that $\Gamma(\alpha)$ is self-paired if and only if $\beta = \beta^{-1}$ that is if and only if β has order 2 in N. In this case $\beta(\Delta)$ also has order 2 so that $\hat{\Gamma}(\alpha)$ is also self-paired.

3.4. *Further discussion of the twisted wreath product case.* Let us assume that $\beta = \beta(\Delta)$ (that is replace $\Gamma(\alpha)$ by $\hat{\Gamma}(\alpha)$), $G_{\alpha\beta}$ is transitive on $\Delta(\beta)$, and all β-entries in $\Delta(\beta)$-positions lie in the $\varphi(P_1)$-conjugacy class $\mathcal{C}$. We define a design $\mathcal{D}$ as follows: the set of points is $\Gamma(\alpha)$, the set of blocks is $\mathbf{k} = \{1, 2, \ldots, k\}$ with β incident with i whenever $i \in \Delta(\beta)$, that is $\beta_i \neq 1$. Then G_α acts faithfully (since N is regular) as an automorphism group of this design; G_α is 2-transitive on points, transitive on blocks, and the stabilizer of a point is transitive on the blocks incident with the point. A counting argument shows that each pair of points is incident with

$$\lambda = l(vl - k)/k(v - 1)$$

blocks where $l = |\Delta(\beta)|$. Of course $\mathcal{D}$ is a degenerate design when $l = k$ but if $l < k$ then $k \geq v$. We note that the parameter λ is $|\Delta(\beta) \cap \Delta(\gamma)|$ for distinct β, γ in $\Gamma(\alpha)$.

Further if β, γ are distinct points of $\Gamma(\alpha)$ then the points $\beta^{-1}\gamma$ and $\gamma^{-1}\beta$ lie in $\Gamma \circ \Gamma^*(\alpha)$. Thus points of $\Gamma \circ \Gamma^*(\alpha)$ have $l_2 = k - 2l + \lambda + L$ entries equal to 1 since there are $k = 2l + \lambda$ entries i with $\beta_i = \gamma_i = 1$ and $L \geq 0$ entries i with $\beta_i = \gamma_i \neq 1$. The parameters L, l_2 can be determined in terms of v, l, k and $|\mathcal{D}|$ as follows:

(i) Given $i \in \mathbf{k}$ and $t \in \mathcal{C}$ the number $n(t, i)$ of $\beta \in \Gamma(\alpha)$ with $\beta_i = t$ is independent of i and t.

PROOF. The subset of P consisting of those elements which map i to 1 is $\rho_i^{-1} P_1$; for each $\sigma \in \rho_i^{-1} P_1$ and each $\beta \in \Gamma(\alpha)$ with $\beta_i = t$ we have

$$(\beta^\sigma)_1 = \beta_i^{\rho_i \sigma} = t^{\rho_i \sigma}.$$

Thus $n(t, i) = n(t^{\rho_i \sigma}, 1)$ for all $\rho_i \sigma \in P_1$, that is $n(t, i) = n(t', 1)$ for all $t' \in \mathcal{C}$.

(ii) Counting the nonidentity entries in points of $\Gamma(\alpha)$ we obtain $vl = kn|\mathcal{C}|$, where $n = n(t, i)$ above.

(iii) Counting triples (β, γ, i), where β, γ are in $\Gamma(\alpha)$ and the entry i is such that $\beta_i = \gamma_i \neq 1$ we obtain $v(v-1)L = k|\mathcal{C}|n(n-1)$, and subsituting for n

$$L = (vl - k|\mathcal{C}|)l / k|\mathcal{C}|(v-1).$$

One final remark about the self-paired case (where $\beta^2 = 1$): by a theorem of Baer and Suzuki, (see [6, 3.8.2]), there is some entry in $\beta\gamma$ whose order as an element of T is not a power of 2. In particular $\beta\gamma \neq \gamma\beta$, that is no pair of distinct elements of $\Gamma(\alpha)$ commutes. Further if $\beta\gamma$ had odd order then the elements of $\mathcal{C}$ would be isolated in any Sylow 2-subgroup of T which would give a contradiction by Glauberman's Z^*-theorem [5]. Thus the order of $\beta\gamma$ is an even integer, not a power of 2.

Now $\beta\gamma\beta$ has order 2 (as it is conjugate in N to γ) and is joined to $\gamma\beta$ so it lies in $\Gamma(\alpha) \cup \Gamma_3(\alpha)$ where $\Gamma_3(\alpha) = \Gamma \circ \Gamma \circ \Gamma(\alpha) \backslash \Gamma(\alpha)$. If $\beta\gamma\beta \in \Gamma(\alpha)$, then as it is joined to $(\gamma\beta)^2$, and $(\gamma\beta)^2$ does not have order 2 we would have $(\gamma\beta)^2 \in \Gamma_2(\alpha)$. Then $\gamma\beta$ and $(\gamma\beta)^2$ would have the same order, and hence would have odd order, a contradiction. Thus $\beta\gamma\beta \in \Gamma_3(\alpha)$. If G_α were transitive on $\Gamma_3(\alpha)$ also, then all entries in points of $\Gamma_3(\alpha)$ would lie in $\mathcal{C} \cup \{1\}$. It follows that $T = \{1\} \cup \mathcal{C} \cup \mathcal{C}^2$. This seems to be a strong restriction on the group T, and hence on Γ.

3.5. FINAL REMARKS. Suppose that G is primitive on Ω and that G_α is 2-transitive and unfaithful on an orbit $\Gamma(\alpha) \subseteq \Omega \backslash \{\alpha\}$. Then it follows from Theorem A that $T \leq G \leq \operatorname{Aut} T$ for some nonabelian simple group T. Can such groups be classified?

References

[1] M Aschbacher and L. L. Scott, 'Maximal subgroups of finite groups', *J. Algebra* **92** (1985), 44–80.

[2] P. J. Cameron, 'Permutation groups with multiply transitive suborbits I', *Proc. London Math. Soc.* (3) **25** (1972), 427–440; II *Bull. London Math. Soc.* **6** (1974), 1–5.

[3] P. J. Cameron, 'Finite permutation groups and finite simple groups', *Bull. London Math. Soc.* **13** (1981), 1–22.

[4] P. J. Cameron and C. E. Praeger, 'Graphs and permutation groups with projective subconstitutents', *J. London Math. Soc.* (2) **25** (1982), 62–74.

[5] G. Glauberman, 'Central elements in core-free groups', *J. Algebra* **4** (1966), 403–420.

[6] D. Gorenstein, *Finite groups* (Harper and Row, New York, Evanston and London, 1968).

[7] L. G. Kovacs, 'Maximal subgroups in composite groups', *J. Algebra*, **99** (1986), 114–131.

[8] C. E. Praeger, 'On primitive permutation groups with a doubly transitive suborbit', *J. London Math. Soc.* (2) **17** (1978), 67–73.

[9] C. E. Praeger, 'Primitive permutation groups and a characterization of the odd graphs', *J. Combin. Theory Ser. B* **31** (1981), 117–142.

[10] C. E. Praeger, J. Saxl, and K. Yokoyama, 'Distance transitive graphs and finite simple groups', *Proc. London Math. Soc.* (3) **55** (1987), 1–21.

[11] L. L. Scott, 'Representations in characteristic p', *Proc. Sympos. Pure Math.* **37** (1980), 319–331.

[12] H. Wielandt, *Finite permutation groups* (Academic Press, New York, 1964).

Department of Mathematics
The University of Western Australia
Nedlands, Western Australia 6009
Australia

https://doi.org/10.1017/S1446788700032274 Published online by Cambridge University Press