We use Ramsey’s theorem to generalise a result of L. Babai and T.S. Sós on Sidon subsets and then use this to prove that for an integer \( n > 1 \) the class of groups in which every infinite subset contains a rewritable \( n \)-subset coincides with the class of groups in which every infinite subset contains \( n \) mutually disjoint non-empty subsets \( X_1, \ldots, X_n \) such that \( X_1 \cdots X_n \cap X_{\sigma(1)} \cdots X_{\sigma(n)} \neq \emptyset \) for some non-identity permutation \( \sigma \) on the set \( \{1, \ldots, n\} \).

**Introduction and results**

In [4], Babai and Sós called a subset \( S \) of a group \( G \) a Sidon subset of the first (second) kind, if for any \( x, y, z, w \in S \) of which at least 3 are different, \( xy \neq zw \) (\( xy^{-1} \neq zw^{-1} \), respectively). Among other things, they proved [4, Proposition 8.1] that an infinite subset of a group contains an infinite subset which is a Sidon subset of both kinds simultaneously.

We generalise the above definition as follows:

Let \( n \) be a positive integer greater than 1 and \( \alpha_1, \ldots, \alpha_{2n} \) be non-zero integers. We say that a subset \( S \) of a group is a Sidon subset of kind \((\alpha_1, \ldots, \alpha_{2n})\) if and only if for any \( x_1, \ldots, x_{2n} \in S \) of which at least \( n + 1 \) are different, \( x_1^{\alpha_1} \cdots x_{2n}^{\alpha_{2n}} \neq x_1^{\alpha_{n+1}} \cdots x_{2n}^{\alpha_{2n}} \). Thus in our terminology every Sidon set of kind \((1,1,1,1)\) (\((1,-1,1,-1)\)) is a Sidon set of the first (respectively, second) kind.

Using Ramsey’s Theorem [14], we prove the following which generalises [4, Proposition 8.1].

**Theorem A.** Let \( n > 1 \) be an integer, \( \alpha_1, \ldots, \alpha_{2n} \) be non-zero integers and \( X \) be an infinite subset of a group such that for all \( i \in \{1, \ldots, 2n\} \) and for all distinct elements \( x, y \in X \), \( x^{\alpha_i} \neq y^{\alpha_i} \). Then \( X \) contains an infinite Sidon subset of kind \((\alpha_{f(1)}, \ldots, \alpha_{f(2k)})\) for all \( k \in \{2, \ldots, n\} \) and for all functions \( f : \{1, \ldots, 2k\} \to \{1, \ldots, 2n\} \) simultaneously.

B.H. Neumann proved in [13] that a group is centre-by-finite if and only if every infinite subset of the group contains a pair of commuting elements. Extensions of problems of this type are to be found in [10] and [12]. The notion of commutativity was extended

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to rewritable products in [7] with a complete description obtained in [9]. Detailed study of rewritable groups may be found in [5] and [6]. In [8], the authors introduced a class \( R_n \) of groups, where \( n \) is an integer greater than 1, as follows: a group \( G \) is called an \( R_n \) group if every infinite subset \( X \) of \( G \) contains a subset \( \{x_1, \ldots, x_n\} \) of \( n \) elements such that \( x_1, \ldots, x_n = x_{\sigma(1)}, \ldots, x_{\sigma(n)} \) for some non-identity permutation \( \sigma \) on the set \( \{1, \ldots, n\} \). They proved there that a group \( G \) is an \( R_n \) group for some integer \( n \) if and only if \( G \) has a normal subgroup \( F \) such that \( G/F \) is finite, \( F \) is an FC-group and the exponent of \( F/Z(F) \) is finite.

In [1] and [2] we considered the following condition on a group. Let \( n > 1 \) be an integer. A group \( G \) is called restricted \((\infty, n)\)-permutable if and only if for all \( n \) infinite subsets \( X_1, \ldots, X_n \) of \( G \) there exists a non-identity permutation \( \sigma \) on the set \( \{1, \ldots, n\} \) such that \( X_1 \cdots X_n \cap X_{\sigma(1)} \cdots X_{\sigma(n)} \neq \emptyset \), where, as usual, for any non-empty subsets \( Y_1, \ldots, Y_k \) of a group, \( Y_1 \cdots Y_k \) equals the set of products \( y_1 \cdots y_k \) where \( y_1 \in Y_1, \ldots, y_k \in Y_k \).

In [1] we showed that every infinite restricted \((\infty, 2)\)-permutable group is Abelian and in [2] we extended this result by proving that every restricted \((\infty, n)\)-permutable group is \( n \)-permutable for all integers \( n > 1 \). Also in [3] we considered a class \( Q_n \) of groups which is defined as follows: a group \( G \) is a \( Q_n \)-group if for all \( n \) infinite subsets \( X_1, \ldots, X_n \) of \( G \) there exists two distinct permutations \( \sigma, \tau \) on the set \( \{1, \ldots, n\} \) such that \( X_{\sigma(1)} \cdots X_{\tau(1)} \cap X_{\sigma(2)} \cdots X_{\tau(2)} \cap \cdots \neq \emptyset \). We proved in [3] that every infinite \( Q_n \)-group is \( n \)-rewritable.

Now, for every integer \( n \) greater than 1, we consider another class of groups called \( \overline{R}_n \)-groups and defined as follows: a group \( G \) is called an \( \overline{R}_n \)-group if every infinite subset \( X \) of \( G \) contains \( n \) mutually disjoint non-empty subsets \( X_1, \ldots, X_n \) such that \( X_1 \cdots X_n \cap X_{\sigma(1)} \cdots X_{\sigma(n)} \neq \emptyset \) for some non-identity permutation \( \sigma \) on the set \( \{1, \ldots, n\} \). Clearly, every \( R_n \)-group is an \( \overline{R}_n \)-group and we use Theorem A to prove that the converse also holds; namely we have:

**Theorem B.** For every integer \( n \) greater than 1, the class \( R_n \) coincides with the class \( \overline{R}_n \).

**Proofs**

To prove Theorem A we need Ramsey's Theorem [14].

**Proof of Theorem A:** Let \( k \in \{2, \ldots, n\} \), let \( r \) be an arbitrary element of the set \( \{1, \ldots, k\} \) and let \( Y \) be an infinite subset of \( X \). Suppose that \( f \) is a function from the set \( \{1, \ldots, 2k\} \) to the set \( \{1, \ldots, 2n\} \) and put \( e_i = \alpha_{f(i)} \) for all \( i \in \{1, \ldots, 2k\} \). List the elements of \( Y \) as \( x_1, x_2, \ldots \) under some well order \( \preceq \) so that \( x_i < x_j \) if \( i < j \). Consider the set \( Y^{(k+r)} \) of all \( (k+r) \)-element subsets of \( Y \). For each \( s \in Y^{(k+r)} \), list the elements \( x_{i_1}, \ldots, x_{i_{k+r}} \) of \( s \) in the ascending order given by \( \preceq \) and write \( s = (x_{i_1}, \ldots, x_{i_{k+r}}) \). Now let \( (t_1, t_2, \ldots, t_{2k}) \) be a \( 2k \)-tuple of elements of \( \{1, 2, \ldots, k+r\} \) such that \( \{t_1, \ldots, t_{2k}\} = k+r \),
and let \( T_r \) be the set of all such \( 2k \)-tuples. Define \(|T_r| + 1\) sets, one \( U_t(Y) \) for each element \( t \) of \( T_r \) and \( V_r(Y) \), as follows. For each \( s \in Y^{(k+r)} \), \( \bar{s} = (x_{i_1}, \ldots, x_{i_{k+r}}) \), put \( s \in U_t(Y) \) if \( x_{i_1} \cdot \cdots \cdot x_{i_{k+r}} = x_{i_{k+1}} \cdot \cdots \cdot x_{i_{2k}} \) and put \( s \in V_r(Y) \) if \( s \notin U_t(Y) \) for any \( t \). By Ramsey's Theorem, there exists an infinite subset \( Z \subset Y \) such that \( Z^{(k+r)} \subset U_t(Y) \) for some \( t \) or \( Z^{(k+r)} \subset V_r(Y) \). By restricting the order \( \leq \) to \( Z \), we may assume that \( Z = \{x_1, x_2, \ldots\} \) and \( x_i < x_j \) if \( i < j \). Suppose, if possible, that \( Z^{(k+r)} \subset U_t(Y) \) for some \( t = (t_1, \ldots, t_{2k}) \in T_r \). Hence for any \( i_1 < i_2 < \cdots < i_{k+r} \),
\[
x_{i_1}^{t_1} \cdot \cdots \cdot x_{i_{k+r}}^{t_{k+r}} = x_{i_{k+1}}^{t_{k+1}} \cdot \cdots \cdot x_{i_{2k}}^{t_{2k}}.
\]
Since \( \{t_1, \ldots, t_{2k}\} = k + r > k \), there exists \( l \in \{1, \ldots, 2k\} \) such that \( t_l \neq t_h \) for any \( h \in \{1, \ldots, 2k\} \setminus \{l\} \). Thus we may write \( x_{i_1}^{t_1} \) as a product of \( x_{i_h}^{t_h} \)'s where \( t_h \in \{t_1, \ldots, t_{2k}\} \setminus \{l\} \). Now, since \( Z \) is infinite there exist sequences \( j_1 < j_2 < \cdots < j_{k+r} \) and \( p_1 < p_2 < \cdots < p_{k+r} \) such that \( j_l \neq p_l \) and \( j_h = p_h \) when \( h \in \{1, \ldots, 2k\} \setminus \{l\} \). Therefore \( x_{j_1}^{j_1} = x_{p_1}^{p_1} \), contrary to the hypothesis on the set \( X \). Hence \( Z^{(k+r)} \subset V_r(Y) \).

Now, list the elements of \( X \) as \( x_1, x_2, \ldots \) under some well order \( < \) so that \( x_i < x_j \) if \( i < j \). Replace \( Y \) by \( X \) and put \( r = 1 \) in the above argument, then there is an infinite subset \( X_1 \) of \( X \) such that \( X_1^{(k+1)} \subset V_1(X) \). By restricting the order \( \leq \) to each infinite subset of \( X \), the above process yields a chain of infinite subsets \( X_1, X_2, \ldots, X_k \) such that \( X_i^{(k+i)} \subset V_i(X_{i-1}) \) for all \( i = 1, 2, \ldots, k \). Thus \( X_k \) is a Sidon subset of kind \((e_1, \ldots, e_{2k})\). Since the set of such \( 2k \)-tuples is finite, where \( k \) ranges over the set \{2, \ldots, n\}, the proof is complete.

As we mentioned before, Theorem A generalises [4, Proposition 8.1]. In fact we have:

**Corollary 1.** Let \( G \) be an infinite group, \( n > 1 \) an integer and \( e_1, \ldots, e_{2n} \in \{-1,1\} \). Then every infinite subset of \( G \) contains a Sidon subset of kind \((e_{f(1)}, \ldots, e_{f(2k)})\) for all \( k \in \{2, \ldots, n\} \) and for all functions \( f : \{1, \ldots, 2k\} \rightarrow \{1, \ldots, 2n\} \) simultaneously.

Using the fact that in a torsion-free nilpotent group no two distinct elements can have the same non-zero power (for example see [11, Theorem 16.2.8]), we also have the following corollary of Theorem A on nilpotent groups.

**Corollary 2.** Let \( G \) be a nilpotent group and let \( T \) be the torsion subgroup of \( G \). If \( n > 1 \) and \( a_1, \ldots, a_{2n} \) are non-zero integers, then every infinite subset \( X \) of \( G \) such that \( xT \neq yT \) for all distinct elements \( x, y \in X \), contains a Sidon set of kind \((a_{f(1)}, \ldots, a_{f(2k)})\) for all \( k \in \{2, \ldots, n\} \) and for all functions \( f : \{1, \ldots, 2k\} \rightarrow \{1, \ldots, 2n\} \) simultaneously.

To prove Theorem B we need only to prove that every infinite \( \widehat{R}_n \)-group is an \( \widehat{R}_n \)-group for which we use Theorem A.

**Proof of Theorem B:** Let \( G \) be an infinite \( \widehat{R}_n \)-group. Let \( X \) be an infinite subset of \( G \). By Corollary 1, \( X \) contains an infinite Sidon subset \( X_0 \) of kind \((1, \ldots, 1)\). Now since \( G \in \widehat{R}_n \), \( X_0 \) contains mutually disjoint non-empty subsets \( X_1, \ldots, X_n \) such that...
Therefore there exist elements \( x_1, \ldots, x_{2n} \in X \) of which at least \( n \) are distinct, such that \( x_1 \cdots x_n = x_{n+1} \cdots x_{2n} \). Thus by the property of the set \( X \), we must have \( \{ x_1, \ldots, x_n \} = \{ x_{n+1}, \ldots, x_{2n} \} \). Therefore there exists a non-identity permutation \( \tau \) on the set \( \{ 1, \ldots, n \} \) such that \( x_{n+i} = x_{\tau(i)} \) for all \( i \in \{ 1, \ldots, n \} \), and so \( x_1 \cdots x_n = x_{\tau(1)} \cdots x_{\tau(n)} \). This completes the proof. \( \square \)

By [8, Theorem A] and Theorem B, we obtain the following corollary which generalises the key lemmas of [2] and [3] ([2, Lemma 2.3] and [3, Lemma 4]).

**Corollary 3.** A group \( G \) is an \( R_n \)-group for some integer \( n > 1 \) if and only if \( G \) has a normal subgroup \( F \) such that \( G/F \) is finite, \( F \) is an FC-group and the exponent of \( F/Z(F) \) is finite.

Lastly in this paper we consider another class of groups. Let \( n > 1 \) be an integer and \( m > 0 \) be a cardinality of a countable set (finite or infinite). We say that a group \( G \) is an \( R^*(n, m) \)-group if and only if every infinite set of \( m \)-sets (a set with cardinality \( m \) is said to be an \( m \)-set) in \( G \) contains \( n \) distinct members \( X_1, \ldots, X_n \) such that

\[
X_1 \cdots X_n \cap X_{\sigma(1)} \cdots X_{\sigma(n)} \neq \emptyset
\]

for some non-identity permutation \( \sigma \) on the set \( \{ 1, \ldots, n \} \). It is natural to ask what are the relations between this class of groups and \( R_n \)-groups. In fact we have:

**Proposition 4.** \( R^*(n, m) = R_n \) for all \( n \) and \( m \).

**Proof:** By Theorem B, it is enough to prove \( R^*(n, m) = R_n \). Suppose, for a contradiction, that \( G \) is an infinite \( R^*(n, m) \)-group which is not in \( R_n \). Thus there exists an infinite subset \( X \) of \( G \) such that for every \( n \) mutually disjoint non-empty subsets \( X_1, \ldots, X_n \) and for all non-identity permutation \( \sigma \) on the set \( \{ 1, 2, \ldots, n \} \), we have \( X_1 \cdots X_n \cap X_{\sigma(1)} \cdots X_{\sigma(n)} = \emptyset \). Since \( X \) is infinite and \( m \) is the cardinality of a countable set, there exists an infinite set of mutually disjoint \( m \)-sets of \( X \). Now the existence of the latter set contradicts the property \( R^*(n, m) \). Conversely, suppose that \( G \in R_n = R_n \). Let \( X \) be an infinite set of \( m \)-sets of \( G \). If two distinct members \( X_1, X_2 \) of \( X \) intersect in an element \( x \) then by considering \( n - 2 \) other arbitrary different members \( X_3, \ldots, X_n \) of \( X \), we have \( X_1 X_2 \cdots X_n \cap X_2 X_1 X_3 \cdots X_n \neq \emptyset \), so we may assume that the members of \( X \) are mutually disjoint. Thus by choosing one element from each member of \( X \), we obtain an infinite set \( X \) such that each element of \( X \) belongs to one and only one member of \( X \). Now since \( G \in R_n \), there exist \( n \) elements \( x_1, \ldots, x_n \) and a permutation \( \sigma \) on the set \( \{ 1, 2, \ldots, n \} \) such that \( x_1 \cdots x_n = x_{\sigma(1)} \cdots x_{\sigma(n)} \). Now by the choice of \( X \) for each \( i \in \{ 1, 2, \ldots, n \} \), there exists an element \( x_i \in X \) such that \( x_i = x_i \); that is, there exist \( X_1, X_2, \ldots, X_n \in X \) such that

\[
x_1 \cdots x_n \in X_1 X_2 \cdots X_n \cap X_{\sigma(1)} \cdots X_{\sigma(n)}
\]

so that \( G \in R^*(n, m) \). This completes the proof. \( \square \)
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