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Sufficient conditions for a continuous linear operator to be weakly compact

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A locally convex topological vector (LCTV) space E is said to have property V (Dieudonné property) if for every complete separated LCTV space F, every unconditionally converging (weakly completely continuous) operator $T : E \to F$ is weakly compact. First, an investigation of the permanence of property V is given. The permanence of the Dieudonné is analogous. Relationships between property V and the Dieudonné property are then given.

1. Preliminaries

In the following definitions (E, τ) and (F, τ') will denote separated locally convex topological vector spaces (LCTVS) with topologies τ and τ' respectively. All linear operators are to be continuous. We use this fact without making further reference to it. A series $\sum_{i=1}^{\infty} x_i$ in (E, τ) is unconditionally convergent (uc) if it is subseries convergent relative to τ. Equivalent conditions for uc series are given in [4]. A series $\sum_{i=1}^{\infty} x_i$ in (E, τ) is said to be weakly unconditionally convergent (wuc) if $\sum_{i=1}^{\infty} |f(x_i)| < \infty$ for every f in E'. An equivalent condition for wuc is that $S = \left\{\sum_{i\in\sigma} x_i : \sigma \text{ finite}\right\}$ be bounded relative to τ.

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A linear operator $T : E \to F$ (F complete) is said to be unconditionally converging (uc operator) if it sends every wuc series in E into uc series in F. T is said to be weakly compact if T maps bounded sets of E into weakly relatively compact sets of F. This definition, which can be found in Lemma 1 of [2], differs slightly from the more common one given in [1]. It is easy to show using condition (E) of [4] that every weakly compact operator is a uc operator.

$N(E)$ is to denote JE (J is the natural map) plus all $\sigma(E'', E')$-limits of wuc series in E. $N(E)$ is a subset of E''. The following theorem is a consequence of Proposition 9.4.9 of [1], and therefore a proof will not be given.

THEOREM 1.1. *The following conditions on* E *are equivalent.*

(1) *For every complete separated LCTVS* F, *every uc operator* $T : E \to F$ *is weakly compact.*

(1') *As* (1), F *being restricted to a Banach space.*

(2) *Any continuous linear map* $T : E \to F$ (F *as in* (1)) *for which* $T''(N(E)) \subset F$ *satisfies* $T''(E'') \subset F$.

(2') *As* (2), F *being restricted to a Banach space.*

(3) *Any equicontinuous, convex, balanced, and* $\sigma(E', N(E))$*-compact set in* E' *is also* $\sigma(E', E'')$*-compact.*

That (1) and (2) are equivalent can be seen from the facts that $T : E \to F$ is uc if and only if $T''(N(E)) \subset F$ and $T : E \to F$ is weakly compact if and only if $T''(E'') \subset F$.

DEFINITION 1.2. Any LCTVS E which has one of the above equivalent properties is said to have property V.

This definition is a generalization of property V for Banach spaces studied by Pełczyński in [5].

Let $K(E)$ denote JE plus all $\sigma(E'', E')$-limits of weak Cauchy sequences in E. By using $K(E)$ instead of $N(E)$ in Proposition 9.4.9 of [1], we have a theorem similar to Theorem 1.1 above. We state this as a definition.

DEFINITION 1.3. E is said to have the Dieudonné property if one of the following equivalent properties is satisfied.

(1) For every complete separated LCTVS F, every operator $T : E \to F$ which transforms weak Cauchy sequences into weakly convergent sequences is weakly compact.

(1') As (1), F being restricted to a Banach space.

(2) Any continuous linear map $T : E \to F$ (F as in (1)), for which $T''(K(E)) \subset F$ satisfies $T''(E'') \subset F$.

(2') As (2), F being restricted to a Banach space.

(3) Any equicontinuous, convex, balanced and $\sigma(E', K(E))$-compact set in E' is also $\sigma(E', E'')$-compact.

A complete discussion of the Dieudonné property is found in [1] and [2].

Another property which is somewhat related to both property V and the Dieudonné property is the following property.

DEFINITION 1.4. A LCTV space E is said to have property (u) if for every weak Cauchy sequence $\{x_n\}$ in E there exists a wuc series $\sum u_k$ such that the sequence $\left\{x_n - \sum_{k=1}^{n} u_k\right\}$ converges weakly to 0.

2. Permanence of property V

Since condition (3) of Theorem 1.1 is a condition on the dual space E', compatible topologies for E must all agree on E having (or not having) property V. Hence, if E is a Banach space having (not having) property V, then E with the weak topology has (does not have) property V. In particular, l_1 with the weak topology does not have property V since l_1 with the norm topology does not [5]. An example of a LCTV space which does have property V is a reflexive space.

PROPOSITION 2.1. *Suppose E is the regular inductive limit* [3] *of the LCTVS E_n. If each E_n has property V, then E has property V.*

Proof. Let $T : E \to F$ be a uc operator, F complete, and B a

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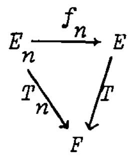
bounded subset of E . Then for some n , $f_n^{-1}(B)$ is a bounded set in E_n where f_n is the continuous linear mapping from E_n to E such that $\cup f_n(E_n)$ spans E . Now define T_n such that the diagram



is commutative. Then $T_n = Tf_n$ is a uc operator and since E_n has property V , T_n is a weakly compact operator. Hence the weak closure of $T_n\left(f_n^{-1}(B)\right) = T(B)$ is compact in the weak topology of F . Therefore T is weakly compact, and E has property V .

EXAMPLE 2.2. Projective limits do not necessarily preserve property V .

Proof. Let R denote the reals and define the map $h : E \to E_f = R$ by $h : e \to f(e)$ where f belongs to E' . Now if we take $E = l_1$ with the weak topology, then l_1 is the projective limit of $|l_\infty|$ copies of R , where $|l_\infty|$ denotes the cardinality of l_∞ . $R = E_f$ has property V since it is reflexive, but l_1 with the weak topology does not.

REMARK. Suppose $E = E_1 \times E_2$. Then $T : E \to F$ is weakly compact if and only if $T|E_1$ and $T|E_2$ (the restriction of T to E_1 and E_2 , respectively) is weakly compact. This is also true for uc operators. Hence, if $E_1, E_2, \ldots, E_n$ are LCTVS with property V , then $E_1 \times E_2 \times \ldots \times E_n$ has property V . The following proposition shows that this is also true for infinite products.

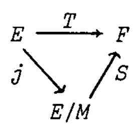
PROPOSITION 2.3. *Suppose E is the infinite direct product of the LCTVS E_n . If each E_n has property V , then E has property V .*

Proof. Let $T : E \to F$ be a continuous operator, F a Banach space, and h_n the natural map of E_n into E . Then $T_n = T \circ h_n$ is continuous from E_n into F , so $T_n = 0$ for all but a finite set of indices. Therefore, it suffices to prove the case for a finite product. But this is contained in the above remark.

REMARK. The direct sum of spaces with property V has property V . The proof is analogous to that for the direct products.

PROPOSITION 2.4. *If* E *is a normed linear space having property* V *, then every quotient space* E *has property* V *.*

Proof. Let M be a subspace of E . Define T such that the diagram



is commutative where j is the canonical map and F is complete. Assume S is a uc operator and let $\sum x_n$ be a wuc series in E . Since j is continuous, $\sum j(x_n)$ is a wuc series in E/M and therefore $\sum T(x_n) = \sum Sj(x_n)$ is a uc series in F . So T is a uc operator and since E has property V , T is a weakly compact operator.

Let $B \neq \{0\}$ be a bounded set in E/M . Since E is a normed space, $j^{-1}(B)$ is a bounded set in E and hence the weak closure of $Tj^{-1}(B) = S(B)$ is compact in the weak topology of F . Therefore S is a weakly compact operator, so E/M has property V .

REMARKS. (1) For LCTVS, Property V is not necessarily preserved for quotient spaces. In Problem 20, page 195 of [7], there is given a Montel space E which has a quotient space isomorphic to l_1 . Since E is a Montel space, E is reflexive and hence has property V . However, l_1 does not have property V . Since an inductive limit topology can be considered as a quotient topology, this example also shows that property V is not preserved by inductive limits.

(2) Property V is not preserved for subspaces since l_1 is linearly isometric to a subspace of $C(S)$, S a compact Hausdorff space, and $C(S)$ has property V [5] while l_1 with the norm topology does not. However a space E has property V if and only if every complemented subspace has property V .

(3) It is an open question whether property V is preserved under tensor products. Swartz has partially answered this question in [8].

3. The Dieudonné property and property V

Permanence properties for the Dieudonné property are analogous to those for property V ; hence they are omitted.

If E has property V then E has the Dieudonné property since every wcc operator (a wcc operator transforms weak Cauchy sequences into weakly convergent sequences) is a uc operator. In general the converse is not true (Example 3.3), however a space having property (u) (Definition 1.4) is a sufficient condition for the converse to hold.

LEMMA 3.1. *E has property (u) if and only if $N(E) = K(E)$.*

Proof. Since $N(E) \subset K(E)$ it will suffice to show $K(E) \subset N(E)$. If $G \in K(E)$, then there exists a weak Cauchy sequence $\{x_n\}$ in E such that $w^* - \lim_n Jx_n = G$. Since E has property (u) , there exists a wuc series $\sum u_i$ in E such that $\left\{x_n - \sum_{i=1}^{n} u_i\right\}$ converges weakly to 0 , thus $w^* - \lim_n \sum_{i=1}^{n} Ju_i = G$ and $G \in N(E)$.

Conversely, assume $K(E) = N(E)$ and let $\{x_n\}$ be a weak Cauchy sequence in E . Since $\{x_n\}$ is weak Cauchy there exists a $G \in K(E)$ such that $w^* - \lim_n Jx_n = G$, and since $K(E) = N(E)$ we have $G \in N(E)$, which implies that there exists a wuc series $\sum u_i$ such that

$w^* - \lim_n \sum_{i=1}^{n} Ju_i = G$, thus $\left\{x_n - \sum_{i=1}^{n} u_i\right\}$ converges weakly to 0 , so E has property (u) .

THEOREM 3.2. *If* E *has property* (u) *, then* E *has property* V *if and only if* E *has the Dieudonné property.*

Proof. It suffices to show that every uc operator is a wcc operator. Let $T : E \to F$ be a uc operator. Then $T''(N(E)) \subset JE$, but since E has property (u) , $N(E) = K(E)$, so $T''(K(E)) \subset JF$. Since T is a wcc operator if and only if $T''(K(E)) \subset JF$, T is a wcc operator.

REMARK. It is not possible to refine Theorem 3.2 to: E has property V if and only if E has the Dieudonné property and property (u) . For example $C[0, 1]$ has both property V and the Dieudonné property, but not property (u) .

EXAMPLE 3.3. James defined a Banach space B_3 such that B_3, B_3' , and B_3'' are separable but B_3''' is non-separable and $B_3'' = B_3 \oplus l_1$. B_3'' is separable, so every bounded sequence in B_3 will have a Cauchy subsequence, and thus every wcc operator will be weakly compact. Hence B_3' has the Dieudonné property.

However, the identity map $i : B_3' \to B_3'$ is a uc operator, since if B_3' contained a subspace isomorphic to c_0 , it would contain a subspace isomorphic to m and B_3' would not be separable, a contradiction. If i were weakly compact, then the unit disk of B_3' would be weakly compact, hence B_3' would be reflexive, which it is not, so B_3' does not have property V . Notice i is an example of a uc operator that is not a wcc operator.

REMARK. Several conclusions can easily be seen by considering the sets $N(X)$ and $K(X)$. Here are a few.

(1) If $N(X) = X''$, then X has property V .

(2) If $K(X) = X''$, then X has the Dieudonné property.

(3) $N(X) = JX$ if and only if X has no subspace isomorphic to c_0

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(equivalent to every wuc series is uc).

(4) X is weakly complete if and only if X has property (u) and $N(X) = JX$ if and only if $K(X) = JX$.

It can be shown that if Y is a subspace of X, then $N(Y) = Y'' \cap N(X)$ and $K(Y) = Y'' \cap K(X)$. This gives the inheritance properties (such as those given in this paper) for subspaces.

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