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COMMUTING OF TOEPLITZ OPERATORS ON THE BERGMAN SPACES OF THE BIDISC

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In this paper we describe when two Toeplitz operators T_f and T_g on the Bergman space of the bidisc commute, where $f = f_1 + \overline{f_2}, g = g_1 + \overline{g_2}$, $f_i, g_i \in H^\infty(D^2)(i = 1, 2)$.

1. INTRODUCTION

For a bounded domain Ω in C^n, let dA denote the Lebesgue measure on Ω normalised so that Ω has measure 1. The *Bergman space* $L_a^2(\Omega)$ is the Hilbert space consisting of analytic functions on Ω that are also in $L^2(\Omega, dA)$. The norm $\|.\|_2$ and the inner product $\langle\ ,\ \rangle$ are those of the space $L^2(\Omega, dA)$. In this paper let Ω denote the bidisc D^2. For $z \in \Omega$, the *Bergman reproducing kernel* is the function $K_z \in L_a^2(\Omega)$ such that

$$f(z) = \langle f, K_z \rangle$$

for every $f \in L_a^2(\Omega)$. The *normalised Bergman reproducing kernel* k_z is the function $K_z/\|K_z\|_2$.

Let P denote the orthogonal projection of $L^2(\Omega, dA)$ onto $L_a^2(\Omega)$. For a function $f \in L^\infty(\Omega, dA)$, the *Toeplitz operator* $T_f : L_a^2(\Omega) \to L_a^2(\Omega)$ is defined by

$$T_f(g) = P(fg), g \in L_a^2(\Omega).$$

These are clearly bounded linear operators for every function $f \in L^\infty(\Omega)$. Note that if f is a bounded analytic function on Ω, then T_f is simply the operator of multiplication by f on $L_a^2(\Omega)$.

For $f \in L^2(\Omega, dA)$, we define $\widetilde{f}$, the Berezin symbol of f, by

$$\widetilde{f}(z) = \langle f k_z, k_z \rangle = \int_\Omega f(w)\left|k_z(w)\right|^2 dA(w), z \in \Omega.$$

By making the change of variable $z = \varphi_w(u)$, where $\varphi_z(w)$ is the Möbius transformation on Ω, we have

$$\widetilde{f} = \int_\Omega f \circ \varphi_z(w) dA(w).$$

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For the polydisk D^n, Möbius transformations are described in [**4**, Section 10.1].

The general problem that we are interested in is the following: when two Toeplitz operators commute, what is the relationship between their symbols? On the Hardy space of the circle, this question has been solved in [**3**]. On the Bergman space of the unit disk, partial results were obtained by Axler and Čučković [**1**] and Axler and Čučković and Rao [**2**]. On the Bergman space of several complex variables, the situation is more complicated. In this paper we give a partial result for the general problem on the Bergman space of the bidisc.

2. Commutants of Toeplitz operators on the bidisc

Following [**6**, Definition 2.1.1], we say $f \in C(D^n)$ is n-harmonic if f is harmonic in each variable separately, that is,

$$\Delta_1 f = \Delta_2 f = \cdots = \Delta_n f = 0.$$

Here variable means complex variable. If $z_j = x_j + iy_j$,

$$\Delta_j f = 4\frac{\partial}{\partial \overline{z}}\left(\frac{\partial f}{\partial z}\right) = \frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2}.$$

Since the harmonic functions are those for which $\sum \Delta_j = 0$, every n-harmonic function is harmonic. The two classes coincide if and only if $n = 1$.

In this section, we describe when T_f and T_g defined on the Bergman space of the bidisc commute, where $f = f_1 + \overline{f_2}, g = g_1 + \overline{g_2}$, $f_i, g_i \in H^\infty(D^2)$.

We use $f(z_1)$ $\big($or $f(z_2)\big)$ to denote the function $f(z_1, z_2)$ on D^2 which depends only z_1(or z_2).For a function $f(z_1)$, we set $\dfrac{\partial f}{\partial z_1} = f'$. In this section, c always denotes a constant, and it may be different in different places. Suppose $f = f_1 + \overline{f_2}, g = g_1 + \overline{g_2}$, $f_i, g_i (i = 1, 2) \in H^\infty(D^2)$. We study this problem for the following four types of functions.

TYPE 1. f, g have the following form:

$$f = f_1(z_2) + \overline{f_2},\ g = \overline{g_2(z_1)} + c,$$

or

$$f = f_1(z) + \overline{f_2(z_2)},\ g = g_1(z_1) + c.$$

Interchanging f and g or z_1 and z_2,we can obtain all of Type 1 functions. For example, $f = f_2(z_1) + \overline{f_2}$, $g = \overline{g_2(z_2)} + c$ is Type 1, too.

TYPE 2. f, g have the following form.

$$f = f_1(z_2) + \overline{f_2(z_1)},\ g = g_1(z_2) + \overline{g_2(z_1)},$$

or

$$f = f_1(z_1) + \overline{f_2(z_2)},\ g = g_1(z_1) + \overline{g_2(z_2)}.$$

TYPE 3. f, g satisfy the following condition. There exist constants c, c_1, and an analytic function a such that

$$f_1 = cg_1(z_2) + c_0,\ f_2 = \bar{c}g_2(z) + a(z_1).$$

Interchanging f and g or z_1 and z_2, we can obtain all of Type 3 functions. For example, $f_1 = cg_1(z) + a(z_1),\ f_2 = \bar{c}g_2(z_2) + c_o$.

TYPE 4. f, g satisfy the following condition. There exist analytic functions a_1, a_2, b_1, b_2 and constants c_1, c_2, c_3, such that

$$f_1 = c_1a_1(z_2) + c_2b_1(z_1),\ f_2 = \overline{c_1}a_2(z_2) + \overline{c_2}b_2(z_1)$$

and

$$g_1 = c_3a_1(z_2) + b_1(z_1),\ g_2 = \overline{c_3}a_2(z_2) + b_2(z_1).$$

If both a_1 and a_2 are constant, then exists a constant c such that $f + cg$ is constant.

LEMMA 2.1. *Suppose $f, g \in L^\infty(D^2, dA)$ and $f = f_1 + \overline{f}_2,\ g = g_1 + \overline{g}_2$, where $f_i,\ g_i \in H^\infty(D^2)(i = 1, 2)$. If f and g are one of the above four Types, then $T_fT_g = T_gT_f$.*

PROOF:

(1) If f, g are Type 1. Without loss of generality, we assume

$$f = f_1(z) + \overline{f_2(z_2)},\ g = g_1(z_1).$$

Then $T_f = T_{f_1} + T_{f_2}{}^*,\ T_g = T_{g_1}$. By [7], $T_{f_2}{}^*T_{g_1} = T_{g_1}T_{f_2}{}^*$, so $T_fT_g = T_gT_f$.

(2) If f, g are Type 2. Without loss of generality, we assume

$$f = f_1(z_1) + \overline{f_2(z_2)},\ g = g_1(z_1) + \overline{g_2(z_2)}.$$

By [7], $T_{f_1}T_{g_2}{}^* = T_{g_2}{}^*T_{f_1}$ and $T_{f_2}{}^*T_{g_1} = T_{g_1}T_{f_2}{}^*$, hence $T_fT_g = T_gT_f$.

(3) If f, g are Type 3. Without loss of generality, we assume

$$f_1 = cg_1(z) + a_1(z_2),\ f_2 = \bar{c}g_2(z_1).$$

Then

$$T_{f_2}{}^*T_{g_1} + T_{f_1}T_{g_2}{}^* = cT_{g_2}{}^*T_{g_1} + cT_{g_1}T_{g_2}{}^* + T_{a_1}T_{g_2}{}^*,$$
$$T_{g_2}{}^*T_{f_1} + T_{g_1}T_{f_2}{}^* = cT_{g_2}{}^*T_{g_1} + cT_{g_1}T_{g_2}{}^* + T_{g_2}{}^*T_{a_1}.$$

Using [7] again, we have $T_fT_g = T_gT_f$.

(4) If f, g are Type 4, the proof is analogous. ∎

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THEOREM 2.2. *Suppose $f, g \in L^\infty(D^2, dA)$ and $f = f_1 + \overline{f}_2$, $g = g_1 + \overline{g}_2$, where f_i, $g_i \in H^\infty(D^2)(i = 1, 2)$. Then $T_f T_g = T_g T_f$ if and only if*

(a) *f and g are both analytic on D^2; or*

(b) *f and g are both co-analytic on D^2; or*

(c) *f or g is a constant function; or*

(d) *f and g are one of the above four Types.*

PROOF: By Lemma 2.1, the sufficiency of the Theorem is obvious. Now assume $T_f T_g = T_g T_f$.Then

$$\begin{aligned}\langle T_f T_g k_z, k_z\rangle &= \langle f P(g k_z), k_z\rangle = \Big\langle (f_1 + \overline{f_2})\big(g_1 k_z + \overline{g_2}(z) k_z\big), k_z\Big\rangle \\ &= \langle f_1 g_1 k_z, k_z\rangle + \langle \overline{f_2} g_1 k_z, k_z\rangle + \overline{g_2}(z)\langle f_1 k_z, k_z\rangle + \overline{g_2}(z)\langle \overline{f_2} k_z, k_z\rangle \\ &= f_1(z) g_1(z) + \langle \overline{f_2} g_1 k_z, k_z\rangle + f_1(z)\overline{g_2}(z) + \overline{f_2}(z)\overline{g_2}(z).\end{aligned}$$

Analogously, we obtain

$$\langle T_g T_f k_z, k_z\rangle = f_1(z) g_1(z) + \langle \overline{g_2} f_1 k_z, k_z\rangle + g_1(z)\overline{f_2}(z) + \overline{g_2}(z)\overline{f_2}(z).$$

Combining the above results we conclude

$$\big\langle (T_f T_g - T_g T_f) k_z, k_z\big\rangle = \widetilde{u}(z) - u(z) = 0,$$

where $u = \overline{f_2} g_1 - f_1 \overline{g_2}$. By [5, Theorem 3.1] we get u is a 2-harmonic function. Thus

$$\Delta_1 u = \Delta_2 u = 0.$$

If f is analytic on D^2, then for $i = 1, 2$,

$$\frac{\partial f}{\partial \overline{z_i}} = 0, \quad \frac{\partial \overline{f}}{\partial z_i} = 0, \quad \text{and} \quad \frac{\partial \overline{f}}{\partial \overline{z_i}} = \overline{\frac{\partial f}{\partial z_i}}.$$

So for $i = 1, 2$

$$0 = 4\frac{\partial}{\partial \overline{z_i}}\Big(\frac{\partial u}{\partial z_i}\Big) = 4\Big(\overline{\frac{\partial f_2}{\partial z_i}}\frac{\partial g_1}{\partial z_i} - \frac{\partial f_1}{\partial z_i}\overline{\frac{\partial g_2}{\partial z_i}}\Big) \qquad \text{on } D^2.$$

Thus we have

$$\overline{\frac{\partial f_2}{\partial z_1}}\frac{\partial g_1}{\partial z_1} = \frac{\partial f_1}{\partial z_1}\overline{\frac{\partial g_2}{\partial z_1}} \qquad \text{on } D^2, \tag{2.3}$$

$$\overline{\frac{\partial f_2}{\partial z_2}}\frac{\partial g_1}{\partial z_2} = \frac{\partial f_1}{\partial z_2}\overline{\frac{\partial g_2}{\partial z_2}} \qquad \text{on } D^2. \tag{2.4}$$

To finish the proof, we consider three cases. Because there are many detailed subcases in each case, we let (1.1), (1.2) and (1.3) denote three different subcases in (1). Analogously, (1.3.1), (1.3.2), (1.3.3) denote three different subcases in (1.3), and so on.

(1) Suppose $\frac{\partial g_1}{\partial z_1} = 0$, then $g_1 = g_1(z_2)$. In this case, either $\frac{\partial f_1}{\partial z_1} = 0$ or $\frac{\partial g_2}{\partial z_1} = 0$. (1.1) If $\frac{\partial f_1}{\partial z_1} = 0$, then $f_1 = f_1(z_2)$. By equation (2.4), we get

$$f_1' \overline{\frac{\partial g_2}{\partial z_2}} = g_1' \overline{\frac{\partial f_2}{\partial z_2}}. \tag{2.5}$$

We analysis equation (2.5). (1.1.1) If $g_1'(z_2) = 0$. Then g_1 is a constant. (1.1.1.1) When $f_1'(z_2) = 0$, then f and g are coanalytic, (b) would hold. (1.1.1.2) When $\frac{\partial g_2}{\partial z_2} = 0$, then $g = c + \overline{g_2(z_1)}$, $f = f_1(z_2) + \overline{f_2}$. So f, g are Type 1, (d) would hold. (1.1.2) If $\frac{\partial g_2}{\partial z_2} = 0$, then $g = g_1(z_2) + \overline{g_2(z_1)}$. (1.1.2.1) When $g_1'(z_2) = 0$, this case is (1.1.1.2) and (d) would hold. (1.1.2.2) When $\frac{\partial f_2}{\partial z_2} = 0$, then $f = f_1(z_2) + \overline{f_2(z_1)}$, and $g = g_1(z_2) + \overline{g_2(z_1)}$. So f, g are Type 2, (d) would hold. Thus we may assume (1.1.3) that neither g_1' or $\frac{\partial g_2}{\partial z_2}$ is identically 0 on D^2. By (2.5) we have

$$\overline{\frac{\partial f_2}{\partial z_2} \Big/ \frac{\partial g_2}{\partial z_2}} = \frac{f_1'}{g_1'}.$$

Therefore $f_1(z_2) = cg_1(z_2) + c_0$, $f_2(z) = \bar{c} g_2(z) + a_2(z_1)$, f, g are Type 3 and (d) holds. (1.2) If $\frac{\partial g_2}{\partial z_1} = 0$, then $g = g_1(z_2) + \overline{g_2(z_2)}$. By (2.4), we have

$$\frac{\partial f_1}{\partial z_2} \overline{g_2'} = \overline{\frac{\partial f_2}{\partial z_2}} g_1'. \tag{2.6}$$

We analysis equation (2.6). (1.2.1) Suppose $g_1' = 0$, then $g = \overline{g_2(z_2)}$. (1.2.1.1) When $g_2{\prime} = 0$, then g is a constant and (c) holds. (1.2.1.2) When $\frac{\partial f_1}{\partial z_2} = 0$, then $f = f_1(z_1) + \overline{f_2}$, $g = \overline{g_2(z_2)}$, hence f, g are Type 1 and (d) holds. (1.2.2) Suppose $g_2{\prime} = 0$, then $g = g_1(z_2) + c$. (1.2.2.1) If $g_1{\prime} = 0$, then g is a constant. (1.2.2.2) If $\frac{\partial f_2}{\partial z_2} = 0$, then $f = f_1 + \overline{f_2(z_1)}$, $g = g_1(z_2)$. So f, g are Type 1 and (d) would hold. We assume (1.2.3) both g_1' and g_2' are not zero. Then by (2.6),

$$\frac{\partial f_1/\partial z_2}{g_1{}'} = \frac{\overline{\partial f_2/\partial z_2}}{g_2{}'}.$$

Thus there are a constant c and two analytic functions $a_1(z_1)$, $a_2(z_1)$, such that

$$f_1 = cg_1(z_2) + a_1(z_1), \quad f_2 = \bar{c} g_2(z_2) + a_2(z_1).$$

Thus f, g are Type 4 and (d) holds.

(2) Suppose $\frac{\partial g_2}{\partial z_1} = 0$, then $g_2 = g_2(z_2)$. (2.1) If $\frac{\partial g_1}{\partial z_1} = 0$, this case is (1.2) and the theorem are proved. (2.2) If $\frac{\partial f_2}{\partial z_1} = 0$, then $f_2 = f_2(z_2)$. By (2.4) we have

$$\overline{f_2'} \frac{\partial g_1}{\partial z_2} = \overline{g_2'} \frac{\partial f_1}{\partial z_2}. \tag{2.7}$$

We analysis equation (2.7). (2.2.1) Suppose $g_2' = 0$. If (2.2.1.1) $\dfrac{\partial g_1}{\partial z_2} = 0$, then

$$f = f_1 + \overline{f_2(z_2)},\ g = g_1(z_1).$$

Hence f, g are Type 1 and (d) holds. (2.2.1.2) If $f_2' = 0$, then f, g are analytic and (a) would hold. (2.2.2) Suppose $\dfrac{\partial g_1}{\partial z_2} = 0$. (2.2.2.1) If $g_2' = 0$, then this case is (2.2.1.1) and is proved. (2.2.2.2) If $\dfrac{\partial f_1}{\partial z_2}$=0, then

$$f = f_1(z_1) + \overline{f_2(z_2)},\ g = g_1(z_1) + \overline{g_2(z_2)}.$$

Thus f, g are Type 2 and (d) holds. (2.2.3) Suppose both g_2' and $\dfrac{\partial g_1}{\partial z_2}$ are not zero. By (2.7) we get

$$\frac{\partial f_1/\partial z_2}{\partial g_1/\partial z_2} = \frac{\overline{f_2'}}{g_2'}.$$

Then there exist constant c, c_0 and a analytic function $a_1(z_1)$, such that

$$f_1 = cg_1(z) + a_1(z_1),\ f_2 = \bar{c}g_2(z_2) + c_0.$$

Thus f, g are Type 3 and (d) holds.

(3) Suppose both $\dfrac{\partial g_1}{\partial z_1}$ and $\dfrac{\partial g_2}{\partial z_1}$ are not zero. By (2.3),

$$\overline{\frac{\partial f_2/\partial z_1}{\partial g_2/\partial z_1}} = \frac{\partial f_1/\partial z_1}{\partial g_1/\partial z_1}.$$

So there exist a constant c and analytic functions $a_1(z_2)$, $a_2(z_2)$ such that

$$f_1(z) = cg_1(z) + a_1(z_2),\ f_2(z) = \bar{c}g_2(z) + a_2(z_2). \tag{2.8}$$

Combining (2.4), (2.8), we obtain

$$\overline{a_2'}\frac{\partial g_1}{\partial z_2} = a_1'\overline{\frac{\partial g_2}{\partial z_2}}. \tag{2.9}$$

(3.1) Suppose $a_1' = 0$. (3.1.1) If $a_2' = 0$, then there exist constant a, b such that $f = ag + b$. So f, g are Type 4 and (d) holds. (3.1.2) If $\dfrac{\partial g_1}{\partial z_2} = 0$, then

$$f_1 = cg_1(z_1) + c_1,\ f_2 = \bar{c}g_2(z) + a_2(z_2).$$

So f, g are Type 3 and (d) holds. Similiarly, if (3.2) $a_2' = 0$, then (d) holds. Suppose (3.3) both a_1' and a_2' are not zero. By (2.9) we have

$$\frac{\partial g_1/\partial z_2}{a_1'} = \frac{\overline{\partial g_2/\partial z_2}}{a_2'}.$$

Thence there are a constant c_1 and two analytic functions $b_1(z_1)$ and $b_2(z_1)$ such that

$$g_1 = c_1 a_1(z_2) + b_1(z_1),\ g_2 = \overline{c_1}a_2(z_2) + b_2(z_1). \tag{2.10}$$

By (2.8) and (2,10), f, g are Type 4 and (d) would hold. ∎

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