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A MAXIMUM PRINCIPLE RELATED TO LEVEL SURFACES OF SOLUTIONS OF PARABOLIC EQUATIONS

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Abstract

Let u be a solution of a parabolic equation $u_t = F(u, Du, D^2u)$. Under convenient hypotheses it is proved that the angle between a given direction and the normal to the level surfaces of $u(\cdot, t)$ satisfies a maximum principle.

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1. Introduction

Let Ω be an open, connected, bounded set in $\mathbf{R}^n$, T a positive constant and $H = \Omega \times (0, T]$. Let u be a sufficiently smooth solution in H of a parabolic equation of the form

$$u_t = F(u, DU, D^2u), \tag{1}$$

where $Du = (\partial u/\partial x_1, \dots, \partial u/\partial x_n)$, and D^2u is the hessian matrix of u with respect to the space variables.

Let $|Du| \neq 0$ in $\overline{H}$ and let $w(x,t)$ be the angle between $Du(x,t)$ and a given direction in $\mathbf{R}^n$. We will prove the following strong maximum principle.

If $w \leq \pi/2$ in H, then

$$w(x,t) \leq \max_{\partial_p H} w \quad \text{for } (x,t) \in H, \tag{2}$$

where $\partial_p H = \{\partial\Omega \times [0,T]\} \cup \{(x,0); x \in \Omega\}$ is the parabolic boundary of H; furthermore w is constant in H if equality holds in (2) for some (x,T).

We will also show that for $n > 2$ the hypothesis $w \leq \pi/2$ is essential.

Note that no hypothesis on the sign of the derivative of F with respect to u is assumed.

Analogous results for solutions of elliptic equations have been obtained in [5].

The maximum principle for w gives information on the behaviour of the level sets of $u(\cdot,t)$. Geometric properties of these level sets have been investigated by Brascamp and Lieb [1], Matano [4], Jones [3], Gage [2], Tso [7].

The results obtained in this paper were announced in [6] where references can be found about geometric properties of level sets of solutions of elliptic and parabolic equations.

2. A differential equation

Let Γ be the class of real functions u, $u \in C^1(\overline{H})$, such that $Du \in C^1(H)$, and D^2u is differentiable with respect to the space variables.

In this paper we denote by F a real differentiable function on the set $\mathbf{R} \times \mathbf{R}^n \times M$, M being the space of the real, symmetric, $n \times n$ matrices. Let us suppose that a positive constant α exists such that in H

$$(3) \qquad \sum_{r,s}^{1,n} \frac{\partial F}{\partial u_{rs}}(u, Du, D^2u)\lambda_r\lambda_s \geq \alpha|\lambda|^2 \quad \text{for } \lambda \in \mathbf{R}^n,$$

where u_{rs} is the second derivative of u with respect to x_r and x_s. Furthermore let us assume throughout this paper that

$$|Du| \neq 0 \quad \text{in } \overline{H}.$$

THEOREM I. *Under the stated hypothesis the angle $w(x,t)$, between $Du(x,t)$ and a given direction μ in $\mathbf{R}^n$, is a function of class $C^0(\overline{H})$; in the set $K = \{(x,t); (x,t) \in H, 0 < w(x,t) < \pi\}$ w is of class C^1 and Dw is differentiable with respect to the space variables; moreover w satisfies in K the following parabolic equation*

$$(4) \qquad w_t = \sum_{r,s}^{i,n} \frac{\partial F}{\partial u_{rs}} w_{rs} + \sum_{r}^{1,n} b_r w_r + \operatorname{cotg} w \sum_{r,s}^{1,n} \frac{\partial F}{\partial u_{rs}} w_r w_s - g \operatorname{cotg} w,$$

where b_r, $g \in C^0(H)$,

$$(5) \qquad g \geq 0;$$

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b_r and g have the following expressions

(6)
$$b_r = \frac{\partial F}{\partial u_r}(u, Du, D^2u) + |Du|^{-2} \sum_s^{1,n} \frac{\partial F}{\partial u_{rs}}(u, Du, D^2u) \sum_i^{1,n} u_i u_{is},$$

(7)
$$g = |Du|^{-2} \sum_{r,s}^{1,n} \frac{\partial F}{\partial u_{rs}}(u, Du, D^2u) \left[\sum_i^{1,n} u_{ir} u_{is} - |Du|^{-2} \sum_i^{1,n} u_i u_{ir} \sum_j^{1,n} u_j u_{js} \right].$$

PROOF. We compute the derivatives of w in terms of the derivatives of u. Since we have

(8)
$$w = \arc\cos \frac{u_\mu}{|Du|}, \qquad \left(u_\mu = \frac{\partial u}{\partial \mu} \right),$$

it follows that

(9)
$$w_r = -[|Du|^2 - u_\mu^2]^{-1/2} \left[u_{\mu r} - u_\mu |Du|^{-2} \sum_i^{1,n} u_i u_{ir} \right],$$

(10)
$$w_t = -[|Du|^2 - u_\mu^2]^{-1/2} \left[u_{\mu t} - u_\mu |Du|^{-2} \sum_i^{1,n} u_i u_{it} \right],$$

$$\begin{aligned} w_{rs} = -[|Du|^2 - u_\mu^2]^{-1/2} \Bigg[& u_{\mu rs} - u_\mu |Du|^{-2} \sum_i^{1,n} u_i u_{irs} - u_{\mu s} |Du|^{-2} \sum_i^{1,n} u_i u_{ir} \\ & + 2u_\mu |Du|^{-4} \sum_i^{1,n} u_i u_{ir} \sum_j^{1,n} u_j u_{js} \\ & - u_\mu |Du|^{-2} \sum_i^{1,n} u_{is} u_{ir} \Bigg] \\ + [|Du|^2 - u_\mu^2]^{-3/2} & \left[\sum_i^{1,n} u_i u_{is} - u_\mu u_{\mu s} \right] \left[u_{\mu r} - u_\mu |Du|^{-2} \sum_i^{1,n} u_i u_{ir} \right]. \end{aligned}$$

By (9) it follows

$$u_{\mu s} = u_\mu |Du|^{-2} \sum_i^{1,n} u_i u_{is} - [|Du|^2 - u_\mu^2]^{1/2} w_s,$$

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and with this substitution we obtain

$$(11)\quad w_{rs} = -[|Du|^2 - u_\mu^2]^{-1/2}\left[u_{\mu rs} - u_\mu|Du|^{-2}\sum_i^{1,n} u_i u_{irs} + u_\mu|Du|^{-4}\sum_i^{1,n} u_u u_{ir}\sum_j^{1,n} u_j u_{js} - u_\mu|Du|^{-2}\sum_i^{1,n} u_{is}u_{ir}\right] - |Du|^{-2}\left[\sum_i^{1,n} u_i(u_{ir}w_s + u_{is}w_r)\right] - [|Du|^2 - u_\mu^2]^{-1/2}u_\mu w_r w_s.$$

By (1) we obtain

$$\sum_{r,s}^{1,n}\frac{\partial F}{\partial u_{rs}}u_{\mu rs} = -\sum_j^{1,n}\frac{\partial F}{\partial u_j}u_{\mu j} - \frac{\partial F}{\partial u}u_\mu + u_{t\mu},$$

and

$$\sum_{r,s}^{1,n}\frac{\partial F}{\partial u_{rs}}u_{irs} = -\sum_j^{1,n}\frac{\partial F}{\partial u_j}u_{ij} - \frac{\partial F}{\partial u}u_i + u_{ti}.$$

Hence, by (9) and (10), we have

$$(12)\quad \sum_{r,s}^{1,n}\frac{\partial F}{\partial u_{rs}}\left[u_{\mu rs} - u_\mu|Du|^{-2}\sum_i^{1,n} u_i u_{irs}\right] = [|Du|^2 - u_\mu^2]^{1/2}\left\{\sum_j^{1,n}\frac{\partial F}{\partial u_j}w_j - w_t\right\}.$$

Therefore the equation (4) follows from (11) and (12), taking into account (6) and (7).

Let $\mathscr{F}$ be the matrix $(\partial F/\partial u_{rs})$; by the assumption (3) it follows that the matrix $(D^2u)^*\mathscr{F}(D^2u)$ is symmetric and positive definite. Hence

$$\operatorname{tr}((D^2u)^*\mathscr{F}(D^2u)) - |Du|((D^2u)^*\mathscr{F}(D^2u)Du, Du) \geq 0,$$

that is, (5) holds.

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3. The maximum principle

As a consequence of the previous theorem we obtain

THEOREM II. *Let us suppose*

$$u_\mu \geq 0 \quad \text{in } H \tag{13}$$

where μ is a given direction in $\mathbf{R}^n$. Then the angle $w(x,t)$ between μ and $Du(x,t)$ satisfies the strong maximum principle, that is, (2) *holds and w is constant in $\overline{\Omega} \times [0,\tau]$ if equality holds in* (2) *for some $(\xi,\tau) \in H$. Furthermore, if w is constant and less than $\pi/2$ in $\overline{\Omega} \times [0,\tau]$, then Du has constant direction in this set.*

REMARKS. (1) Changing μ to $-\mu$ yields the analogous statement for the minimum of w.

(2) The hypothesis of smoothness of u can be relaxed. It is sufficient to suppose smoothness of u such that the maximum principle holds for w.

(3) The hypothesis $|Du| \neq 0$ is necessary to define w. In the case $|Du| = 0$ in a subset of H, the theorem gives information on the behaviour of w in the neighbourhood of any point at which $|Du| \neq 0$.

(4) If w is constant in H and equal to $\pi/2$, then

$$\frac{\partial u}{\partial \mu} = 0 \quad \text{in } H.$$

In this case, u can be considered as a function of $n-1$ space variables. In case w is constant in H and less than $\pi/2$, u can be considered as a function of only one space variable.

(5) The hypothesis (13) needs to be justified. We shall show that it is superfluous for $n=2$ (Theorem III) and it is essential for $n>2$.

PROOF. By Theorem I, w satisfies (4) in K. By (13) it follows that $w \leq \pi/2$; hence by (5) we get

$$-w_t + \sum_{r,s}^{1,n} \frac{\partial F}{\partial u_{rs}} w_{rs} + \sum_{r=1}^{n} B_r w_r \geq 0 \quad \text{in } K,$$

with B_r continuous in K.

$$B_r = b_r + \operatorname{cotg} w \sum_{s=1}^{n} \frac{\partial F}{\partial u_{rs}} w_s.$$

Then $\max_{\overline{K}} w = \max_{\partial_p K} w$; where $\partial_p K$ is the parabolic boundary of the open set K, as usually defined. Since $H = K \cup \{w = 0\}$, we get (2). Furthermore the strong parabolic maximum principle holds in H: if there is $(\xi,\tau) \in H$ such that $w(\xi,\tau) = \max_{\partial_p H} w$, then w is constant in $\overline{\Omega} \times [0,\tau]$. Let us consider now this latter case with $w < \pi/2$ to complete the proof of the theorem. If $w = 0$, Du has constant direction μ. Let $0 < w < \pi/2$. Let y be a given point in Ω; it uniquely defines a direction λ in $\mathbf{R}^n$, coplanar with $Du(y,\tau)$ and μ, orthogonal to $Du(y,\tau)$ and such that the angle between λ and μ is $\pi/2 - w$. Let $\gamma(x,t)$ the angle between λ and $Du(x,t)$; by the inequality $\gamma \le w + \widehat{\mu\lambda}$ it follows that $\gamma(x,t) \le \pi/2$ in $\overline{H}$. Thus γ has a maximum at (y,τ) and, by the previous strong maximum principle, γ is constant in $\overline{\Omega} \times [0,\tau]$. Hence, at any point of this set Du is orthogonal to λ and the angle w between μ and Du is constant, then the direction of Du is constant.

THEOREM III. *Let us suppose $n = 2$ and $w < \pi$ in H. Then (2) holds and, if the maximum of w is achieved in a point (ξ,τ) of H, then Du has constant direction in $\overline{\Omega} \times [0,\tau]$.*

PROOF. Because of Theorem II, it is sufficient to prove the theorem under the hypothesis $\max_{\overline{H}} w > \pi/2$.

Let us suppose that there exists (ξ,τ) such that

$$w(\xi,\tau) = \max_{\overline{H}} w, \qquad (\xi,\tau) \in H. \tag{14}$$

By the continuity of Du, a positive constant δ exists such that the angle between $Du(x,t)$ and $Du(\xi,\tau)$ is less than $w(\xi,\tau) - \pi/2$ in

$$M \equiv \{(x,t); |x-\xi| < \delta, \tau - \delta < t \le \tau\} \subset H. \tag{15}$$

A direction λ in $\mathbf{R}^2$, orthogonal to μ, is uniquely defined such that the angle between λ and $Du(\xi,\tau)$ is equal to $w(\xi,\tau) - \pi/2$. Let $\gamma(x,t)$ be the angle between $Du(x,t)$ and λ; we have

$$\gamma(x,t) \le \gamma(\xi,\tau) < \frac{\pi}{2} \quad \text{for } (x,t) \in M.$$

Then $u_\lambda > 0$ in M. By Theorem II it follows that Du has constant direction in M; hence w is constant in M. We have proved that, for any (ξ,τ) for which (14) holds, there is a set M, defined by (15), in which w is constant and Du has constant direction. Hence w is constant and Du has constant direction in $\overline{\Omega} \times [0,\tau]$.

The following example shows that the hypothesis (12) cannot be relaxed in the case $n > 2$.

Let ε be a negative constant,

$$\begin{aligned} u(x,t) = {} & x_1 + x_1^2 - x_3^2 - 6x_2(T-t) - x_3^3 \\ & + \varepsilon\left[\tfrac{1}{2}x_1^2 - \tfrac{1}{2}x_1x_2^2 + \tfrac{1}{2}x_2^3 - 2x_2x_3^3 + \tfrac{1}{6}x_1^3\right], \end{aligned}$$

$$\mu = \left(\frac{2\varepsilon}{\sqrt{1+4\varepsilon^2}}, \frac{-1}{\sqrt{1+4\varepsilon^2}}, 0\right),$$

and let w be the angle between μ and Du. The function u satisfies the heat equation

$$u_t = \Delta u \quad \text{in } H = \mathbf{R}^3 \times [0,T].$$

Let $Q = (0,0,0)$; one may check with elementary calculations

$$\begin{aligned} &w_i(Q,T) = 0, \quad i = 1,2,3, \quad w_t(Q,T) = 6, \\ &w_{11}(Q,T) = \varepsilon, \quad w_{12}(Q,T) = -\varepsilon, \quad w_{13}(Q,T) = 0, \\ &w_{22}(Q,T) = 3\varepsilon - 6, \quad w_{23}(Q,T) = 0, \quad w_{33}(Q,T) = 4\varepsilon. \end{aligned}$$

Hence $w(x,t) < w(1,T) = \arccos(2\varepsilon/\sqrt{1+4\varepsilon^2})$ for (x,t) in a neighbourhood of (Q,T), $t \le T$. We can observe $w(Q,T) > \pi/2$ and $w(Q,T) \to \pi/2$ if $\varepsilon \to 0$.

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