MINIMAL THETA FUNCTIONS

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1. Statement of results. Let $f(\mathbf{u}) = f(u_1, u_2) = au_1^2 + bu_1u_2 + cu_2^2$ be a positive definite binary quadratic form with real coefficients and discriminant $b^2 - 4ac = -1$. Among such forms, let $h(\mathbf{u}) = \frac{1}{\sqrt{3}}(u_1^2 + u_1u_2 + u_2^2)$. The Epstein zeta function of f is defined to be

$$\zeta_f(s) = \sum_{(m,n)\neq\mathbf{0}} f(m, n)^{-s}.$$

Rankin [**7**], Cassels [**1**], Ennola [**5**], and Diananda [**4**] between them proved that for every real $s > 0$,

$$\zeta_f(s) \geq \zeta_h(s). \tag{1}$$

We prove a corresponding result for theta functions. For real $\alpha > 0$, let

$$\theta_f(\alpha) = \sum_{m,n} e^{-2\pi\alpha f(m,n)}.$$

This function satisfies the functional equation

$$\theta_f(1/\alpha) = \alpha\theta_f(\alpha). \tag{2}$$

(This may be proved by using the formula (4) below, and then twice applying the identity (8).)

THEOREM 1. *For any $\alpha > 0$,*

$$\theta_f(\alpha) \geqslant \theta_h(\alpha).$$

If there is an $\alpha > 0$ for which $\theta_f(\alpha) = \theta_h(\alpha)$, then f and h are equivalent forms and $\theta_f \equiv \theta_h$.

Let

$$\xi_f(s) = \zeta_f(s)\Gamma(s)(2\pi)^{-s}.$$

Then

$$\xi_f(s) = \frac{1}{s-1} - \frac{1}{s} + \int_1^\infty (\theta_f(\alpha) - 1)(\alpha^s + \alpha^{1-s})\frac{d\alpha}{\alpha} \tag{3}$$

for all complex numbers s other than 0 and 1. From Theorem 1 it follows that $\xi_f(s) \geq \xi_h(s)$ for all real s, with equality only when $f \sim h$. Hence we obtain the earlier result (1) as a corollary. On differentiating in (3), we see moreover that $\xi_f^{(2k)}(s) \geq \xi_h^{(2k)}(s)$ for all real s, and that $\xi^{(2k+1)}(s) \geq \xi_h^{(2k+1)}(s)$ for all $s \geq 1/2$. (For $s < 1/2$ the inequality is reversed in the case of derivatives of odd order.)

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We may factorize f;

$$f = a(u_1 + zu_2)(u_1 + \bar{z}u_2),$$

where $z = x + iy$, and without loss of generality $y > 0$. Since $b^2 - 4ac = -1$, we deduce that $a = 1/2y$, so that

$$f = \frac{1}{2y}(u_1 + zu_2)(u_1 + \bar{z}u_2) = \frac{1}{2y}(u_1 + xu_2)^2 + \tfrac{1}{2}yu_2^2.$$

Since θ_f and ζ_f are determined by the complex number z in the upper half-plane, we dispense with the notation θ_f, ζ_f, and henceforth write $\theta(\alpha; x, y)$, $\zeta(s; x, y)$ instead. In particular, we see that

$$\theta(\alpha; x, y) = \sum_n e^{-\pi\alpha y n^2} \sum_m e^{-\pi\alpha(m+nx)^2/y}. \tag{4}$$

Two forms are equivalent, $f_1 \sim f_2$, if there is a

$$C = [c_{ij}] \in SL(2, \mathbb{Z})$$

such that $f_2(u) = f_1(C\underline{u})$. In this case

$$z_2 = \frac{c_{12} + c_{22}z_1}{c_{11} + c_{21}z_1}.$$

The form f is *reduced* if $-a < b \le a < c$ or if $0 \le b \le a = c$. Thus f is reduced if and only if z lies in the fundamental domain

$$\mathscr{D} = \{z : -\tfrac{1}{2} < x \le \tfrac{1}{2}, |z| > 1, y > 0\} \cup \{z : 0 \le x \le \tfrac{1}{2}, |z| = 1, y > 0\}$$

of the modular group. We know that each f is equivalent to a unique reduced form, which is to say that any z in the upper half-plane is equivalent to a unique $z' \in \mathscr{D}$. If $f_1 \sim f_2$ then $\theta_{f_1} \equiv \theta_{f_2}$ and $\zeta_{f_1} \equiv \zeta_{f_2}$. Hence we may confine our attention to reduced forms.

The earlier results on ζ_f were derived by combining in a clever way the following two lemmas:

Lemma A. *If $y \ge 3/2$ and $s > 0$, then* $\dfrac{\partial}{\partial y}\zeta(s; x, y) > 0$.

Lemma B. *If $0 \le x \le 1/2$, $y \ge 3/5$, and $0 \le s \le 3$, then* $\dfrac{\partial}{\partial x}\zeta(s; x, y) < 0$.

In Lemmas 4 and 7 below we establish corresponding results, from which Theorem 1 is an obvious consequence.

We now consider positive definite quadratic forms in $n > 2$ variables, but we restrict our attention to diagonal forms. Suppose that $c_1, \ldots, c_n$ are positive real numbers for which

$$c_1, c_2 \ldots c_n = 1, \tag{5}$$

and for $\alpha > 0$ put

$$\theta(\alpha, c) = \sum_k \exp\left(-\pi\alpha \sum_i c_i k_i^2\right) = \prod_{i=1}^n \theta(c_i\alpha), \tag{6}$$

where $\theta(t) = \sum_{-\infty}^{+\infty} e^{-\pi n^2 t}$ for $t > 0$. From the case $b = 0$ of Theorem 1 we see that $\theta(a\alpha)\theta(c\alpha) \geq \theta(\alpha/2)^2$ whenever $ac = 1/4$. Hence it is evident that $\theta(\alpha, c) \geq \theta(\alpha, \mathbf{1})$ for all $\alpha > 0$ where $\mathbf{1} = (1, 1, 1, \ldots, 1)$, but we now show more. We exhibit a curve which starts at $\mathbf{1}$, ends at $\mathbf{c}$, lies in the hypersurface (5), and along which $\theta(\alpha, \cdot)$ is strictly increasing.

THEOREM 2. *Let α and $c_1, \ldots, c_n$ be fixed positive numbers, and suppose that* (5) *holds but that not all the c_i are* 1. *For t real put*

$$U(t) = \prod_{i=1}^u \theta(c_i^t \alpha).$$

Then $U'(0) = 0$, $U'(t) > 0$ for $t > 0$, and $U'(t) < 0$ for $t < 0$.

In particular we see that $U(0) < U(1)$. This implies corresponding results concerning the Epstein zeta function of diagonal positive definite quadratic forms. For related literature see Ennola [**6**], Sandakova [**9**], Delone and Ryškov [**2, 3**], and Ryškov [**8**].

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2. Properties of the classical theta function. We consider the classical one-dimensional theta function

$$\theta(t; \beta) = \sum_{-\infty}^{+\infty} e^{-\pi k^2 t} e(k\beta), \tag{7}$$

where $\operatorname{Re} t > 0$, $e(w) = e^{2\pi i w}$. This function satisfies the functional equation

$$\theta(t; \beta) = t^{-1/2} \sum_{-\infty}^{+\infty} e^{-\pi(m-\beta)^2/t}. \tag{8}$$

The Jacobi triple product formula also gives

$$\theta(t; \beta) = \prod_{r=1}^{\infty} (1 - e^{-2\pi r t})(1 + 2e^{-(2r-1)\pi t} \cos 2\pi\beta + e^{-2(2r-1)\pi t}). \tag{9}$$

LEMMA 1. *Let*

$$Q(t; \beta) = -\frac{\dfrac{\partial}{\partial \beta} \theta(t; \beta)}{\sin 2\pi\beta},$$

and suppose that $t>0$ is fixed. Then $Q(t;\beta)$ is an even function of β, with period 1. The values of $Q(t;\beta)$ are all positive, and $Q(t;\beta)$ is a decreasing function of β in the interval $[0, 1/2]$.

Proof. The first assertion is obvious. By differentiating in (9) we see that

$$Q(t;\beta)=4\pi\sum_{s=1}^{\infty}(1-e^{-2\pi st})e^{-(2s-1)\pi t}\prod_{\substack{r=1\\ r\neq s}}^{\infty}(1-e^{-2\pi rt})$$
$$\times(1+2e^{-(2r-1)\pi t}\cos 2\pi\beta+e^{-2(2r-1)\pi t}).$$

In this formula each term is positive and decreasing in $[0, 1/2]$. Thus the proof is complete.

LEMMA 2. *Let $Q(t;\beta)$ be as above. Put*

$$A(t)=\begin{cases} t^{-3/2}e^{\pi/4t} & (0<t<1),\\ \left(1-\dfrac{1}{3000}\right)4\pi e^{-\pi t} & (t\geq 1),\end{cases}$$

and

$$B(t)=\begin{cases} t^{-3/2} & (0<t<1),\\ \left(1+\dfrac{1}{3000}\right)4\pi e^{-\pi t} & (t\geq 1).\end{cases}$$

Then

$$A(t)\leq Q(t;\beta)\leq B(t)$$

for all β.

Proof. By Lemma 1 it suffices to show that

$$Q(t;1/2)\geq A(t), \tag{10}$$

and that

$$Q(t;0)\leq B(t). \tag{11}$$

By L'Hôpital's rule we see that

$$Q(t;1/2)=\frac{1}{2\pi}\frac{\partial^2}{\partial\beta^2}\theta(t;\beta)\bigg|_{\beta=1/2}.$$

Suppose that $t\geq 1$. By differentiating twice in (7) we see that the above is

$$4\pi\sum_{k=1}^{\infty}(-1)^{k-1}k^2e^{-\pi k^2 t}\geq 4\pi e^{-\pi t}\left(1-\sum_{k=2}^{\infty}k^2e^{-\pi(k^2-1)t}\right).$$

This last sum is a decreasing function of t, and we note that

$$\sum_{k=2}^{\infty} k^2 e^{-\pi(k^2-1)} < 0.0003228 < \frac{1}{3000}. \tag{12}$$

This gives (10) when $t \geq 1$. When $t < 1$ we differemtoate twice in (8) to see that

$$Q(t; 1/2) = 2t^{-3/2} \sum_{m=1}^{\infty} e^{-\pi(m-1/2)^2/t}\left(\frac{2\pi}{t}(m-1/2)^2 - 1\right).$$

Here all terms are non-negative, and the term $m = 1$ contributes an amount

$$t^{-3/2} e^{-\pi/4t}\left(\frac{\pi}{t} - 2\right) > A(t).$$

Thus we have (10) when $t < 1$.

By L'Hôpital's rule we find that

$$Q(t; 0) = -\frac{1}{2\pi} \frac{\partial^2}{\partial \beta^2} \theta(t; \beta)\Big|_{\beta=0}.$$

Suppose that $t \geq 1$. By differentiating twice in (7) we see that

$$Q(t; 0) = 4\pi \sum_{k=1}^{\infty} k^2 e^{-\pi k^2 t} = 4\pi e^{-\pi t}\left(1 + \sum_{k=2}^{\infty} k^2 e^{-\pi(k^2-1)t}\right).$$

This last sum is a decreasing function of t, and so by (12) we see that we have (11) when $t \geq 1$.

Now suppose that $t < 1$. We differentiate twice in (8) to see that

$$Q(t; 0) = t^{-3/2} \sum_{-\infty}^{+\infty} (1 - 2\pi m^2/t) e^{-\pi m^2/t}.$$

Here the term $m = 0$ gives $B(t)$ and the other terms are negative. Thus we have (11) when $t < 1$, and the proof is complete.

In order to establish Theorem 2, we prove the following further result.

Lemma 3. *For $t > 0$ let $\theta(t) = \theta(t; 0) = \sum_{-\infty}^{+\infty} e^{-\pi n^2 t}$. Then*

$$t \frac{\theta'}{\theta}(t) + t^2 \left(\frac{\theta'}{\theta}\right)'(t) > 0$$

for all $t > 0$.

Proof. Let $T(t)$ denote the above expression. We begin by showing that $T(t) = T(1/t)$. To see this, we take $\beta = 0$ in (8) and differentiate logarithmically to find that

$$\frac{\theta'}{\theta}(t) = -\frac{1}{2t} - t^{-2} \frac{\theta'}{\theta}(1/t).$$

We multiply both sides by t and differentiate to see that

$$\frac{\theta'}{\theta}(t)+t\left(\frac{\theta'}{\theta}\right)'(t)=t^{-2}\frac{\theta'}{\theta}(1_{t})+t^{-3}\left(\frac{\theta'}{\theta}\right)'(1/t).$$

On multiplying both sides by t this becomes the desired identity. It now suffices to show that $T(t)>0$ for $t\geq 1$. We set $\beta=0$ in (9) and differentiate logarithmically to see that

$$\frac{\theta'}{\theta}(t)=\sum_{n=1}^{\infty}\left(\frac{2\pi n e^{-2\pi n t}}{1-e^{-2\pi n t}}-\frac{2(2n-1)\pi e^{-(2n-1)\pi t}}{1+e^{-(2n-1)\pi t}}\right).$$

On differentiating this we see also that

$$\left(\frac{\theta'}{\theta}\right)'(t)=\sum_{n=1}^{\infty}\left(-\frac{(2\pi n)^2 e^{-2\pi n t}}{(1-e^{-2\pi n t})^2}+\frac{2(2n-1)^2\pi^2 e^{-(2n-1)\pi t}}{(1+e^{-(2n-1)\pi t})^2}\right).$$

Thus on one hand

$$\frac{\theta'}{\theta}(t)\geq -2\pi\sum_{n=1}^{\infty}(2n-1)e^{-(2n-1)\pi t}=-2\pi\frac{e^{-\pi t}(1+e^{-2\pi t})}{(1-e^{-\pi t})^2}.$$

Since $e^{-\pi t}\leq 0{\cdot}04322$ for $t\geq 1$, this is $\geq -7e^{-\pi t}$. On the other hand $(1-e^{-2\pi n t})^{-2}\leq(1-e^{-2\pi})^{-2}<1{\cdot}004$ for $n\geq 1$, $t\geq 1$, so that

$$\left(\frac{\theta'}{\theta}\right)(t)>\frac{2\pi^2 e^{-\pi t}}{(1+e^{-\pi t})^2}-4\pi^2(1.004)\sum_{n=1}^{\infty}n^2 e^{-2\pi n t}.$$

As $(1+e^{-\pi t})^{-2}\geq(1+e^{-\pi})^{-2}>0.918$, the first term is $>18\,e^{-\pi t}$. The sum in the second term is

$$\frac{e^{-2\pi t}(1+e^{-2\pi t})}{(1-e^{-2\pi t})^3}\leq\frac{e^{-\pi}(1+e^{-2\pi})}{(1-e^{-2\pi})^3}e^{-\pi t}<(0.044)e^{-\pi t}.$$

Hence the second term is $>-2e^{-\pi t}$. On combining these estimates we see that

$$T(t)>(-7+16t)te^{-\pi t}\geq 9t^2e^{-\pi t}>0$$

for $t\geq 1$, and the proof is complete.

3. The first main lemma.

LEMMA 4. *If $\alpha>0$, $0<x<1/2$, and $y\geq 1/2$, then*

$$\frac{\partial}{\partial x}\theta(\alpha;x,y)<0.$$

We note that in particular, this holds when $z=x+iy$ is in the interior of the right hand side of the fundamental domain $\mathcal{D}$. Numerical experiments suggest that the constraint $y\geq 1/2$ can be weakened to $y\geq\sqrt{3}/6$. This would be best possible in view of the behaviour of $\theta(\alpha;x,y)$ near $(x,y)=(\frac{1}{2},\sqrt{\frac{3}{6}})$ when α is large.

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Proof. In view of (2), we may suppose that $\alpha \geq 1$. From (4), (7), and (8) we see that

$$\theta(\alpha; x, y) = (y/\alpha)^{1/2} \sum_n e^{-\pi\alpha y n^2} \theta(y/\alpha; nx). \tag{13}$$

Hence

$$\frac{\partial}{\partial x} \theta(\alpha; x, y) = 2(y/\alpha)^{1/2} \sum_{n=1}^{\infty} n e^{-\pi\alpha y n^2} \frac{\partial}{\partial \beta} \theta(y/\alpha; \beta)\bigg|_{\beta = nx}.$$

In the notation of Lemma 2 this sum is at most

$$-A(y/\alpha)(\sin 2\pi x) e^{-\pi\alpha y} + \sum_{n=2}^{\infty} n e^{-\pi\alpha y n^2} B(y/\alpha)\, |\sin 2\pi n x|.$$

Since $\left|\dfrac{\sin 2\pi n x}{\sin 2\pi x}\right| \leq n$ for all x, the above is negative if

$$\sum_{n=2}^{\infty} n^2 e^{-\pi\alpha y (n^2 - 1)} < A(y/\alpha)/B(y/\alpha). \tag{14}$$

Since $y \geq 1/2$, we see that $\alpha y = \dfrac{\alpha}{y} \cdot y^2 \geq \frac{1}{4}\alpha/y$. Hence the sum above is at most

$$\sum_{n=2}^{\infty} n^2 e^{-\pi(n^2-1)\alpha/4y} = e^{-\pi\alpha/4y} \sum_{n=2}^{\infty} n^2 e^{-\pi(n^2-2)\alpha/4y}.$$

Suppose that $\alpha > y$. In this case the right hand side of (14) is $e^{-1/4\pi\alpha/y}$, while the last sum above is

$$< \sum_{n=2}^{\infty} n^2 e^{-\pi(n^2-2)/4} = 0{\cdot}868649\ldots < 1.$$

This gives the result in this case.

Suppose that $\alpha \leq y$. Then the right hand side of (14) is $\left(1 + \dfrac{1}{3000}\right)\Big/\left(1 - \dfrac{1}{3000}\right)$. On the other hand, we see that $\alpha y \geq \alpha^2 \geq 1$, so that the left hand side in (14) does not exceed the sum (12). Thus we again have (14), and the proof is complete.

4. Differential inequalities.

We wish to show that the inequality

$$\frac{\partial}{\partial y} \theta(\alpha; x, y) \geq 0$$

holds when $z = x + iy$ lies in the fundamental domain $\mathcal{D}$. To this end, we first prove the following subsidiary result.

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LEMMA 5. *If* $\alpha > 0$, $0 \le x \le \frac{1}{2}$, *and* $x^2 + y^2 \ge 1$, *then*

$$\frac{\partial^2}{\partial y^2}\theta(\alpha; x, y) + \frac{2}{y}\frac{\partial}{\partial y}\theta(\alpha; x, y) > 0.$$

Numerical experiments suggest that this inequality holds if $\alpha > 0$ and $y \ge 0.71$. However, it fails when $\alpha = 9/4$, $x = 1/2$, $y = 0{\cdot}70$. For any $Y > 1/2$ there is a $C(Y)$ such that the inequality holds when $\alpha \ge C(Y)$, $y \ge Y$. On the other hand, at the saddle point $(x, y) = (1/2, 1/2)$, we have

$$\frac{\partial \theta}{\partial y} = 0, \; \frac{\partial^2 \theta}{\partial y^2} < 0$$

for all $\alpha > 0$. By differentiating in (13) it is easy to see that

$$\theta - y^{1/2}\alpha^{-1/2}, \; \frac{\partial \theta}{\partial y} \sim \frac{1}{2} y^{-1/2}\alpha^{1/2}, \; \frac{\partial^2 \theta}{\partial y^2} \sim -\frac{1}{4} y^{-3/2}\alpha^{-1/2}$$

as $y/\alpha \to \infty$, $\alpha \ge 1$. Thus it is obvious that the stated inequality holds when $\alpha \ge 1$ and y/α is large.

Proof. In view of (2), we may assume that $\alpha \ge 1$. By direct calculation in (4) we see that the quantity in question is

$$(\pi\alpha)^2 \sum_{m,n} \left(n^2 - \frac{(m-nx)^2}{y^2}\right)^2 e^{-2\pi\alpha f} - \frac{2\pi\alpha}{y} \sum_{m,n} n^2 e^{-2\pi\alpha f} = \sum_1 - \sum_2,$$

say. In $\sum_1$, the terms $(m, n) = (\pm 1, 0)$ contribute an amount

$$2(\pi\alpha)^2 y^{-4} e^{-\pi\alpha/y} = (2\pi\alpha e^{-\pi\alpha(x^2+y^2)/y})(\pi\alpha y^{-4} e^{\pi\alpha(x^2+y^2-1)/y}) = KP_1,$$

where K denotes the first factor on the above right, and P_1 the second. The terms $(m, n) = (0, \pm 1)$ contribute to $\sum_1$ an amount

$$2(\pi\alpha)^2(1 - x^2/y^2)^2 e^{-\pi\alpha(x^2+y^2)/y} = K(\pi\alpha(1 - x^2/y^2)^2) = KP_2,$$

say. Thus

$$\sum_1 \ge K(P_1 + P_2).$$

On the other hand,

$$\sum_2 = K \cdot \left(\frac{2}{y}\sum_m e^{\pi\alpha(x^2-(x-m)^2)/y} + \frac{2}{y} e^{\pi\alpha(x^2+y^2)/y} \sum_{n=2}^{\infty} n^2 e^{-\pi\alpha n^2 y} \sum_m e^{-\pi\alpha(m-nx)^2/y}\right). \tag{15}$$

By pairing m and $-m$ we see that the first of these sums is

$$1 + 2\sum_{m=1}^{\infty} e^{-\pi\alpha m^2/y} \cosh(2\pi\alpha m x/y).$$

This is an increasing function of x. We take $x = 1/2$ and pair m with $1 - m$ to see that the sum is

$$\leq 2\sum_{m=1}^{\infty} e^{-\pi\alpha m(m-1)/y} \leq 2\sum_{m=1}^{\infty} e^{-\pi\alpha(m-1)^2/y} = 1 + \sum_{-\infty}^{+\infty} e^{-\pi\alpha m^2/y}.$$

From (7) it is evident that $\max_\beta \theta(t;\beta) = \theta(t;0)$. Hence the sum in (8) is also maximized when $\beta = 0$. Thus in the second term in (15), the sum over m is at most $\sum_m e^{-\pi\alpha m^2/y}$. Since $0 \leq x \leq \frac{1}{\sqrt{3}} y$ in the domain under consideration, we see that

$$e^{\pi\alpha(x^2+y^2)/y} \leq e^{\pi\alpha(4/3)y}.$$

On combining these estimates we see that

$$\sum_2 \leq K\left(\frac{2}{y} + \frac{2}{y}\left(\sum_{-\infty}^{+\infty} e^{-\pi\alpha m^2/y}\right)\left(1 + \sum_{n=2}^{\infty} n^2 e^{-\pi\alpha(n^2-4/3)y}\right)\right) = KR,$$

say. The sums in R are decreasing functions of α, while P_1 and P_2 are increasing. Thus in order to prove the inequality it suffices to show that $R < P_1 + P_2$ when $\alpha = 1$. Since $y \geq \sqrt{3}/2$, the second sum in R is

$$\leq \sum_{n=2}^{\infty} n^2 e^{-\pi(n^2-4/3)\sqrt{3}/2} < 0{\cdot}002826.$$

By (8) the first sum in R is

$$= y^{1/2} \sum_k e^{-\pi k^2 y} \leq y^{1/2} \sum_k e^{-\pi k^2\sqrt{3}/2} < y^{1/2}(1{\cdot}1317).$$

Hence

$$R \leq \frac{2}{y} + (2{\cdot}27)y^{-1/2} = S,$$

say.

We show that $S < P_1 + P_2$. By logarithmic differentiation it is easy to see that P_1 is an increasing function of y. It is also clear that P_2 is an increasing function of y, while S is decreasing. Hence it suffices to consider $y = \sqrt{1-x^2}$. That is, it remains to verify that

$$2(1-x^2)^{-1/2} + (2{\cdot}27)(1-x^2)^{-1/4} < \pi(1-x^2)^{-2} + \pi\left(1 - \frac{x^2}{1-x^2}\right)^2 \tag{16}$$

for $0 \leq x \leq 1/2$. Since the right hand side is

$$2\pi\left(1 + \frac{x^4}{(1-x^2)^2}\right),$$

we see that the expression is increasing. Hence its least value is $2\pi = 6{\cdot}28\ldots$, at $x = 0$. The left hand side is obviously increasing. Its maximum value is $4.75\ldots$, at $x = 1/2$. Hence (16) holds, and the proof is complete.

5. The second main lemma. We begin with an elementary observation.

LEMMA 6. *Suppose that*

$$f'(y)+\frac{2}{y}f(y)>0$$

for all $y \geq y_0 > 0$, *and that* $f(y_0) \geq 0$. *Then* $f(y) > 0$ *for all* $y > y_0$.

Proof. On multiplying the given inequality by y^2, we see that $(y^2f(y))' > 0$. Hence $y^2f(y)$ is strictly increasing for $y \geq y_0$. Hence $f(y) > f(y_0)(y_0/y)^2 \geq 0$ when $y > y_0$.

We now are in a position to prove our second main lemma.

LEMMA 7. *If* $\alpha > 0$, $0 \leqslant x \leq 1/2$, $x^2+y^2 \geq 1$, *then*

$$\frac{\partial}{\partial y}\theta(\alpha; x, y) \geq 0$$

with equality if and only if (x, y) *is one of the points* $(0, 1)$, $(1/2, \sqrt{3}/2)$.

If Theorem 1 were our only goal, then it would suffice to establish this lemma in the special case $x = 1/2$. The main interest of Lemmas 4 and 7 is that they provide information concerning the directional derivative of $\theta(\alpha; x, y)$.

Proof. In view of the previous two lemmas, it suffices to show that

$$\frac{\partial}{\partial y}\theta(\alpha; x, y) \geq 0$$

when $x^2+y^2=1$, $0 \leq x \leq 1/2$, with equality if and only if x is at one of the endpoints. To this end, let

$$g(r) = \theta(\alpha; r\cos\phi, r\sin\phi),$$

where ϕ is fixed, $0 < \phi < \pi$. Then $g(r) = g(1/r)$. Hence $g'(1) = 0$. But

$$g'(1) = \frac{\partial\theta}{\partial x}\cos\phi + \frac{\partial\theta}{\partial y}\sin\phi,$$

so that

$$\frac{\partial\theta}{\partial x}\cos\phi = -\frac{\partial\theta}{\partial y}\sin\phi$$

when $(x, y) = (\cos\phi, \sin\phi)$. From Lemma 4 we see that $\dfrac{\partial\theta}{\partial x} \leq 0$ when $\pi/3 \leq \phi \leq \pi/2$, with equality only at the endpoints. Hence $\dfrac{\partial\theta}{\partial y} \geq 0$, and the proof is complete.

6. Proof of theorem 2. By logarithmic differentiation we see that

$$\frac{U'}{U}(t) = \alpha \sum_i \frac{\theta'}{\theta}(c_i^t\alpha)c_i^t \log c_i.$$

Hence

$$\left(\frac{U'}{U}\right)'(t) = \sum_i \left(\frac{\theta'}{\theta}(c_i^t\alpha)\alpha c_i^t + \left(\frac{\theta'}{\theta}\right)(c_i^t\alpha)(\alpha c_i^t)^2\right)(\log c_i)^2$$
$$= \sum_i T(c_i^t\alpha)(\log c_i)^2$$

in the notation of Lemma 3. Hence by this lemma, $(U'/U)(t)$ is strictly increasing. But

$$\frac{U'}{U}(0) = \sum_i \log c_i = 0,$$

so that $(U'/U)(t) > 0$ for $t > 0$ and $(U'/U)(t) < 0$ for $t < 0$. Since $U(t) > 0$ for all t, this gives the result.

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