WHEN IS A REGULAR SEQUENCE SUPER REGULAR?

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Let \((B, \mathcal{F})\) be a filtered, noetherian ring. A sequence \(x = x_1, \ldots, x_n\) in \(B\) is called super regular if the sequence of initial forms
\[
\xi_1 = L(x_1), \ldots, \xi_n = L(x_n)
\]
is a regular sequence in \(\text{gr}_s(B)\).

If \(B\) is local and the filtration \(\mathcal{F}\) is \(\mathfrak{m}\)-adic then any super regular sequence is also regular, see [6], 2.4.

In [3], Prop. 6 Hironaka shows that in a local ring \((B, \mathcal{M})\) an element \(x \in \mathcal{M}\setminus\mathcal{M}^2\) is super regular (with respect to the \(\mathcal{M}\)-adic filtration) if and only if \(x\) is regular in \(B\) and \((x) \cap \mathcal{M}^{n+1} = (x)\mathcal{M}^n\) for every integer \(n\).

This result is extended to a more general situation in [6], 1.1. In the present paper we will characterize super regular sequences in a relative case:

Let \(A\) be a regular complete local ring, \(B = A/I\) an epimorphic image of \(A\) and \(x = x_1, \ldots, x_n\) a regular sequence in \(B\) which is part of a minimal system of generators of the maximal ideal of \(B\). Let \(y = y_1, \ldots, y_n\) be a sequence in \(A\) which is mapped onto \(x\). Then \(y\) is part of a regular system of parameters of \(A\). Therefore \(y\) is a super regular sequence in \(A\).

We put \(\overline{A} = A/(y)A\), \(\overline{I} = I/(y)I\) and \(\overline{B} = B(x)B\). Then \(\overline{B} = \overline{A}/\overline{I}\), since \(x\) is a \(B\)-sequence.

As a consequence of our main result, the following conditions are equivalent:

(a) \(x\) is a super regular sequence in \(B\)

(b) For all elements \(g \in \overline{I}\) there exists \(f \in I\), such that
\[
\overline{f} = g \quad \text{and} \quad \nu(f) = \nu(g).
\]

(Here \(\overline{f}\) denotes the image of \(f\) in \(\overline{I}\) and \(\nu(f)\) the degree of the initial form.

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of \( f \).)

The equivalence of (a) and (b) can also be expressed in terms of Hironaka’s numerical character \( \nu^*(J, R) \): \( x \) is a super regular sequence in \( B \) if and only if \( \nu^*(I, A) = \nu^*(\overline{I}, \overline{A}) \).

In the applications we will use this characterization to show that the tangent cone of certain algebras is CM (Cohen-Macaulay). Our examples contain some results of J. Sally [4], [5] in a more special case.

§1. Notations and remarks

In the following we fix our notations and recall some basic facts about filtrations. For a more detailed information about filtrations we refer to N. Bourbaki [1].

Let \( (A, \mathcal{F}) \) be a noetherian filtered ring such that \( \mathcal{F}_0A = A \) and \( \mathcal{F}_{i+1}A \subseteq \mathcal{F}_iA \) for \( i \geq 0 \) and let \( (M, \mathcal{G}) \) be a filtered \( (A, \mathcal{F}) \)-module. Then \( \text{gr}_i(M) = \bigoplus_{i \geq 0} \mathcal{F}_iM/\mathcal{F}_{i+1}M \) is a graded \( \text{gr}_*A = \bigoplus_{i \geq 0} \mathcal{F}_iA/\mathcal{F}_{i+1}A \)-module.

If \( x \in M \) we define \( \nu(x) = \sup \{ n/x \in \mathcal{G}_nM \} \) to be the degree of \( x \) and call

\[
L(x) = x + \mathcal{G}_{\nu(x)}M
\]

the initial form of \( x \).

Let \( \varphi: M \rightarrow N \) be a homomorphism of filtered modules then \( \varphi \) induces a homogeneous homomorphism

\[
\text{gr}(\varphi): \text{gr}(M) \rightarrow \text{gr}(N).
\]

If \( \varphi \) is an epimorphism, we always will assume that \( N \) admits the canonical filtration induced from the filtration of \( M \). Then

\[
\text{Ker}(\text{gr}(\varphi)) = \{ L(x)/x \in \text{Ker} \varphi \}.
\]

We call a sequence \( (x_1, \cdots, x_n), x_i \in \text{Ker} \varphi \) a standard base of \( \text{Ker} \varphi \) if

\[
\text{Ker}(\text{gr}(\varphi)) = (L(x_1), \cdots, L(x_n)).
\]

In the particular case that \( \varphi: A \rightarrow B \) is an epimorphism of filtered rings, we now give a slightly different but useful description of a standard base: Corresponding to a sequence \( (x_1, \cdots, x_n), x_i \in \text{Ker} \varphi \), we define a filtration on \( A^n \):

\[
\mathcal{F}_iA^n = \{(a_1, \cdots, a_n)|a_j \in \mathcal{F}_{i-\nu(x_j)}A\}.
\]

Now
(1) \[ A^n \xrightarrow{(x_1, \ldots, x_n)} A \xrightarrow{\varphi} B \rightarrow 0 \]
is a complex of filtered \(A\)-modules inducing a complex of \(gr(A)\)-modules

(2) \[ gr(A^n) \xrightarrow{(L(x_1), \ldots, L(x_n))} gr(A) \xrightarrow{gr(\varphi)} gr(B) \rightarrow 0 \]
and \((x_1, \ldots, x_n)\) is a standard base of \(\text{Ker}\varphi\) if and only if the complex (2) is exact.

If \(A\) is complete and separated then any standard base of \(\text{Ker}\varphi\) is also a base of \(\text{Ker}\varphi\). However the converse is false in general.
Consider the following case:
Let \(B = A/xA\), where \(x\) is not a zero-divisor on \(A\) and let \(\varphi: A \rightarrow B\) be the canonical epimorphism and \(\xi = L(x)\).

**Lemma.** (a) If \(x\) is super regular then

\[(*) \quad gr(A) \xrightarrow{\xi} gr(A) \xrightarrow{gr(\varphi)} gr(B) \rightarrow 0\]
is exact, i.e. \((x)\) is a standard base of \(\text{Ker}\varphi = (x)\).

(b) If \(A\) is complete and separated and the sequence \((*)\) is exact then \(x\) is super regular.

The lemma shows that a non-zero-divisor \(x\) in a complete separated ring forms a standard base of \((x)\) if and only if it is super regular.

**Proof of the lemma.** (a) Let \(\alpha \neq 0\) be a homogeneous element of \(\text{Ker}\,(gr(\varphi))\). Then \(\alpha = L(\alpha a)\) for some \(a \in A\). Since \(\xi L(a) \neq 0\), we have \(\xi L(a) = L(\alpha a) = \alpha\).

(b) Let \(\alpha \in gr(A)\) be a homogeneous element such that \(\xi \alpha = 0\).
We construct a convergent series \((a_n)\) such that for all \(n \geq 1\) we have \(L(a_n) = \alpha\) and \(\nu(\alpha a_n) \geq \nu(\alpha) + \nu(a_n) + n\).

Let \(a = \lim a_n\), then \(\alpha = L(a)\) and \(\alpha a \in \cap \mathcal{F}_1 A = \{0\}\). Therefore \(\alpha = 0\) and consequently \(\alpha = 0\). Construction of the sequence \((a_n)\) by induction on \(n\):

Let \(a_n \in A\) such that \(\alpha = L(a_n)\). Since \(\xi \alpha = 0\) we have \(\nu(\alpha a_n) \geq \nu(\alpha) + \nu(a_n) + 1\).

Suppose we have already constructed \(a_1, \ldots, a_n\). By induction hypothesis we have \(\nu(\alpha a_n) \geq \nu(\alpha) + \nu(a_n) + n\). Since \(L(\alpha a_n) \in \text{Ker}\,(gr(\varphi))\) and since we suppose that \((*)\) is exact we find a homogeneous element \(\gamma_n\) such that \(\xi \gamma_n = L(\alpha a_n)\).
Choose \( g_n \in A \) such that \( \gamma_n = L(g_n) \), then \( \nu(g_n) = \nu(xa_n) - \nu(x) \geq \nu(a_n) + n \) and \( \nu(xa_n - g_n) \geq \nu(x) + \nu(a_n)(n + 1) \). The element \( a_{n+1} = a_n - g_n \) is the next member of the sequence.

\[\text{§2. The main result}\]

Let \( \varepsilon: A \to B \) be an epimorphism of complete and separated filtered rings. As before we assume that \( B \) admits the induced filtration. Then \( \text{Ker} \varepsilon \) is a closed ideal of \( A \).

Suppose we are given a super regular sequence \( y = y_1, \ldots, y_n \) on \( A \) and let \( x_i = \varepsilon(y_i) \). Suppose that \( x = x_1, \ldots, x_n \) is a regular sequence on \( B \) and that

\[\nu(x_i) = \nu(y_i) > 0\]

for \( i = 1, \ldots, n \).

Let \( \bar{\varepsilon} = A/(y)A, \bar{B} = B/(x)B, I = \text{Ker} \varepsilon \) and \( \bar{I} = I/(y)I \). We have \( \bar{I} \subseteq \bar{\varepsilon} \) and \( \bar{B} = \bar{\varepsilon}/I \), since \( x \) is a regular sequence on \( B \). If \( f \) is an element of \( A \) or of \( B \) we denote its image in \( \bar{\varepsilon} \) or \( \bar{B} \) by \( \bar{f} \).

**Theorem 1.** 1) The following properties are equivalent:

a) For each \( g \in I \) there exists \( f \in I \) such that \( \bar{f} = g \) and \( \nu(f) = \nu(g) \).

b) There exists a standard base \( g_1, \ldots, g_m \in I \) and elements \( f_i \in I \) such that \( \bar{f}_i = g_i \) and \( \nu(f_i) = \nu(g_i) \) for \( i = 1, \ldots, m \).

c) \( x \) is a super regular sequence.

2) If the equivalent conditions of 1) hold and the \( f_i \) are chosen as in b), then \( (f_1, \ldots, f_m) \) is a standard base of \( I \).

**Proof.** It is sufficient to consider the case that the sequence \( x \) consists only of one element. The general case follows by induction on the length of the sequence.

1) a) \( \Rightarrow \) b): is obvious

b) \( \Rightarrow \) c): Let \( (g_1, \ldots, g_m) \) be a standard base of \( I \) and \( f_i \in I \) be such that \( \bar{f}_i = g_i \) and \( \nu(f_i) = \nu(g_i) \).

We define on \( A^n \) and \( \bar{A}^n \) filtrations

\[\mathcal{F}_i A^n = \{(a_1, \ldots, a_n)/a_j \in \mathcal{F}_i - \nu(f_j) A\}\]

\[\mathcal{F}_i \bar{A}^n = \{(a_1, \ldots, a_n)/a_j \in \mathcal{F}_i - \nu(g_j) \bar{A}\}\]

and obtain a commutative diagram of filtered modules.
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\[ \begin{array}{ccc}
0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow \\
A^n & A & B \\
\downarrow & \downarrow & \downarrow \\
(f_1, \ldots, f_n) & y & x \\
\downarrow & \downarrow & \downarrow \\
A^n & (f_1, \ldots, f_n) & z \\
\downarrow & \downarrow & \downarrow \\
(\varepsilon_1, \ldots, \varepsilon_n) & A & \varepsilon \\
\downarrow & \downarrow & \downarrow \\
0 & 0 & 0
\end{array} \]

(1)

inducing a commutative diagram of graded modules

\[ \begin{array}{ccc}
0 & & \\
\downarrow & & \\
\text{gr}(A^n) & \text{gr}(A) & \text{gr}(B) \\
\downarrow & \downarrow & \downarrow \\
\eta & \eta & \xi \\
\downarrow & \downarrow & \downarrow \\
\text{gr}(A^n) & \text{gr}(A) & \text{gr}(B) \\
\downarrow & \downarrow & \downarrow \\
\phi & \text{gr}(\varepsilon) & \text{gr}(\varepsilon) \\
\downarrow & \downarrow & \downarrow \\
\text{gr}(A^n) & \text{gr}(A) & \text{gr}(B) \\
\downarrow & \downarrow & \downarrow \\
0 & 0 & 0
\end{array} \]

(2)

\( \xi = L(x), \ \eta = L(y). \)

The lowest row is exact since \((g_1, \ldots, g_n)\) is a standard base. Also the middle column is exact since \(y\) is super regular.

By diagram chasing we find, that also the sequence

\[ \text{gr}(B) \xrightarrow{\xi} \text{gr}(B) \xrightarrow{\eta} \text{gr}(B) \xrightarrow{\xi} 0 \]

is exact.

By the lemma it follows that \(x\) is super regular. c) \(\Rightarrow\) a): Let \(g \in \bar{I}\), then we can find an element \(f \in A\) such that \(\bar{f} = g\) and \(\nu(f) = \nu(g)\).

However we would like to find such an element \(f\) in \(I\). To do this we consider

\[ \sigma \text{gr}(\varepsilon)(L(f)) = \text{gr}(\varepsilon)(L(g)) = 0. \]
Since we assume that \( x \) is super regular it follows from the lemma that \( \text{gr}(\varepsilon)(L(f)) = \beta \xi \). Therefore \( L(f) = \alpha \gamma + \gamma \), where \( \alpha, \gamma \) are homogeneous and \( \gamma \in \ker(\text{gr}(\varepsilon)) \).

Hence we can choose \( a, f, h \in I \) such that
\[
  f = a_x y + h_x + f_x,
\]
\[
  \nu(f) = \nu(a_x y) = \nu(h_x) < \nu(f_x).
\]

From this we obtain \( g = \bar{f} = h_x + f_x \in I \), hence \( f_x \in I \). Repeating the same reasoning for \( f_x \), we can find \( a, f, h \in I \) such that
\[
  f_x = a_x y + h_x + f_x,
\]
\[
  \nu(f_x) = \nu(a_x y) = \nu(h_x) < \nu(f_x).
\]

This time it may happen that \( \nu(f_x) < \nu(\bar{f_x}) \), but that doesn't matter and we can take \( h_x = 0 \) in that case. Proceeding that way we construct sequences \((a_i), (h_i)\) and \((f_i)\) such that \( h_i \in I \) and
\[
  f_i = a_i y + h_i + f_{i+1},
\]
\[
  \nu(f_i) = \nu(a_i y) = \nu(h_i) < \nu(f_{i+1}).
\]

Put \( a = \sum_{i=1}^\infty a_i, h = \sum_{i=1}^\infty h_i \). Then \( f = a y + h, h \in I \) and \( \nu(h) = \nu(h_i) \).
\[
  \nu(f) = \nu(g).
\]

Thus \( h \) is the desired element.

2) Consider again the diagram (2). We have to show that if \( \alpha \in \ker(\text{gr}(\varepsilon)) \) is a homogeneous element then there exists \( \gamma \in \text{gr}(A^x) \) such that \( \varphi(\gamma) = \alpha \). We prove this by induction on the degree of \( \alpha \). If \( \deg \alpha < 0 \), then \( \alpha = 0 \). Thus suppose that \( \deg \alpha > 0 \). By assumption all columns and the lowest row are exact. By diagram chasing we can find homogeneous elements \( \beta, \delta \) such that
\[
  \alpha = \beta \eta + \delta,
\]
where \( \delta \in \text{Im} \varphi \) and \( \beta \in \ker(\text{gr}(\varepsilon)) \). Since by assumption \( \deg \eta > 0 \), we have \( \deg \beta < \deg \alpha \). From the induction hypothesis the assertion follows.

§ 3. Some applications

(a) Let \( B = \mathbb{k}[x_1, \ldots, x_n]/I \) be a 1-dimensional complete algebra over an algebraically closed field \( \mathbb{k} \). In the following we consider only the \( m_x \)-adic filtration of \( B \).

Suppose that the residue class \( x_1 \) of \( X_1 \) is not a zero-divisor and a superficial element of \( B \), then \( \text{gr}(B) \) is a CM-ring (Cohen-Macaulay) if
and only if $x_1$ is super regular on $B$.

Applying Theorem 1 we find:

$gr(B)$ is a CM-ring if and only if for all $F \in I$ there exists $G \in k[[X_1, \cdots, X_n]]$ such that

$$F(0, X_2, \cdots, X_n) + GX_1 \in I \quad \text{and} \quad \nu(G) \geq \nu(F(0, X_2, \cdots, X_n)) - 1.$$ 

Next we restrict our attention to the more special case that $B$ is a monomial ring:

Let $H \subset N$ be a numerical semigroup generated minimally by $n_1 < n_2 < \cdots < n_i$, see [2].

To $H$ belongs the monomial ring $B = k[t^{n_1}, \cdots, t^{n_i}]$, whose maximal ideal is $m_B = (t^{n_1}, \cdots, t^{n_i})$. We want to describe in terms of the semigroup when $gr_{m_B}(B)$ is a CM-ring.

$t^{n_1}$ is a superficial element of $B$. Let $\bar{B} = B/t^{n_1}B \cong k[[X_2, \cdots, X_n]]/I$. It is easy to see that a standard base of $\bar{I}$ can be chosen such that the elements of the base are either monomials $X_i^{n_1} \cdots X_i^{n_l}$ or differences of monomials

$$X_i^{n_1} \cdots X_i^{n_l} - X_i^{n_1} \cdots X_i^{n_l},$$

with

$$\sum_{i=1}^{l} \mu_i n_i = \sum_{i=1}^{l} \mu_i^* n_i.$$ 

Let $n_i + H = \{n_i + h/h \in H\}$. A monomial $X_i^{n_1} \cdots X_i^{n_l}$ is an element of $\bar{I}$ if and only if

$$\sum_{i=1}^{l} \nu_i n_i \in n_i + H.$$ 

Thus we find:

$gr(B)$ is a CM-ring if and only if for all integers $\nu_1 \geq 0, \nu_2 \geq 0, \cdots, \nu_l \geq 0$ such that

$$\sum_{i=1}^{l} \nu_i n_i \in n_i + H,$$

there exist $\nu_i^* > 0, \nu_2^* \geq 0, \cdots, \nu_l^* \geq 0$ such that

$$\sum_{i=1}^{l} \nu_i n_i = \sum_{i=1}^{l} \nu_i^* n_i \quad \text{and} \quad \sum_{i=1}^{l} \nu_i \leq \sum_{i=1}^{l} \nu_i^*.$$ 

It is not difficult to see that it suffices to consider only such $\nu_i$ with
the extra condition that \( n_i > \nu_i \). Therefore only a finite number of conditions are to be checked.

If in addition \( \bar{I} \) is generated only by monomials, then there is a unique minimal system of generators of \( \bar{I} \) consisting of monomials \( M_1, \cdots, M_k \). These monomials form a standard base of \( \bar{I} \).

Thus \( \text{gr}(B) \) is a CM-ring if and only if to each such monomial

\[
M_i = X_{i_1}^{r_i} \cdots X_{i_l}^{r_l}
\]

we can find

\[
F_i = X_{i_1}^{r_i} \cdots X_{i_l}^{r_l} - X_{i_1'}^{r_i'} \cdots X_{i_l'}^{r_l'} \in I,
\]

with

\[
\nu_i^* > 0 \quad \text{and} \quad \sum_{i=2}^l \nu_i \leq \sum_{i=1}^l \nu_i^*.
\]

In particular if \( \text{gr}(B) \) is a CM-ring then \( F_1, \cdots, F_k \) forms a standard base of \( I \) and also a minimal base of \( I \).

We now discuss in more detail monomial rings of embedding dimension 3. These examples were first studied by G. Valla and R. Robbiano in [7] and communicated to me, when I was visiting Genova. Using different methods they are able to construct in all cases a standard base. Here we restrict ourselves to the question whether \( \text{gr}(B) \) is a CM-ring.

Let \( B = k[[t^{r_1}, t^{r_2}], t^{r_3}] \) and assume first that \( B \) is not a complete intersection. In [2] it is shown that \( I = (F_1, F_2, F_3) \) with

\[
\begin{align*}
F_1 &= X_{i_1}^{r_1} - X_{i_1}^{r_{11}} X_{i_2}^{r_2} \\
F_2 &= X_{i_1}^{r_1} - X_{i_1}^{r_{21}} X_{i_3}^{r_3} \\
F_3 &= X_{i_1}^{r_1} - X_{i_1}^{r_{31}} X_{i_3}^{r_3}
\end{align*}
\]

where \( r_{ij} > 0 \) and \( c_1 = r_{21} + r_{31}, \quad c_2 = r_{12} + r_{32} \) and \( c_3 = r_{13} + r_{23} \). It follows that \( \bar{I} \) is generated by monomials and therefore \( \text{gr}(B) \) is a CM-ring if and only if

\[
\begin{align*}
c_1 &\geq r_{12} + r_{13} \\
c_2 &\leq r_{21} + r_{23} \\
c_3 &\leq r_{31} + r_{32}.
\end{align*}
\]

The first inequality is always satisfied since

\[
c_1 n_1 = r_{12} n_2 + r_{13} n_3
\]
and
\[ n_1 < n_2 < n_3. \]

Similarly the third inequality is always true. Our final result is therefore:
\[ \text{gr}(B) \text{ is a CM-ring if and only if } c_2 \leq r_{21} + r_{23}. \]

<table>
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<th>n_2</th>
<th>n_3</th>
<th>c_2</th>
<th>r_{21}</th>
<th>r_{23}</th>
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We now assume that \( B = k[[t^n, t^m, t^a]] \) is a complete intersection. Then \( I \) can be generated by two elements \( F_1, F_2 \). We have to distinguish several case:

**Case a).** \( \bar{I} = (X_i^{c_1}, X_i^{c_2}) \) is generated by monomials. Since \( c_i > c_2 \) and \( c_i > c_3 \), it follows that \( B \) is a strict complete intersection.

**Case b).** \( \bar{I} = (X_i^{c_2}, X_i^{c_3}, X_i^{c_4}, X_i^{c_5}) \).

We want to find a standard base of \( \bar{I} \):
\[ X_i^{c_2+c_3}, X_i^{c_3}, X_i^{c_4}, X_i^{c_5} \]

are relations of \( \text{gr} (\bar{B}) \). We easily compute the length \( l \) of
\[ k[[X_i, X_j]]/(X_i^{c_2+c_3}, X_i^{c_3}, X_i^{c_4}, X_i^{c_5}) \]
to be
\[ l = r_{12}c_3 + r_{13}c_2. \]

On the other hand we have
\[ n_2 = c_2c_1, \ n_3 = c_3c_1 \]

and
\[ c_1n_1 = r_{12}n_2 + r_{13}n_3. \]
therefore
\[ n_1 = r_1 c_1 + r_1 c_2 = l . \]

Since
\[ n_1 = 1(B/t^n B) = l(gr(B/t^n B)) , \]
it follows that
\[ X_2^{r_2+r_1}, X_3^{r_3} - X_3^{r_3}, X_4^{r_3}X_5^{r_5} \]
is a standard base of \( \tilde{I} \).

There is only one way to lift these equations:
\[ X_2^{r_2+r_1} - X_3^{r_3}, X_4^{r_3} - X_4^{r_3}X_5^{r_5} . \]

Since \( c_1 \geq r_1 + r_2 \), we find that \( gr(B) \) is a CM-ring if and only if
\[ c_2 + r_2 \leq c_3 - r_3 + c_1 . \]

However \( B \) is never a strict complete intersection.

\( \gamma \) \( \tilde{I} = (X_3^{r_3}, X_5^{r_5}) \) is generated by monomials. Thus \( B \) is a strict complete intersection if and only if \( c_1 \leq r_1 + r_3 \).

\( \delta \) \( \tilde{I} = (X_3^{r_3}, X_5^{r_5}) \) is generated by monomials and \( c_3 < r_1 + r_2 \), therefore \( B \) is always a strict complete intersection.

**Theorem 2.** Let \( B = k[[X_1, \ldots, X_n]]/I \) be a complete \( k \)-algebra and suppose that \( I \) admits a standard base \( F_1, \ldots, F_m \) such that:

1. \( \nu(F_i) = 2 \) for \( i = 1, \ldots, m \).
2. For each homomorphism \( \varphi: I/I^e \to B \) the elements \( \varphi(F_i + I^e) \), \( i = 1, \ldots, m \) are not units in \( B \) (equivalently, \( B \) is not a direct summand of \( I/I^e \)).

Then for any complete algebra \( \tilde{B} = k[[Y_1, \ldots, Y_k]]/J \) and any regular \( \tilde{B} \)-sequence \( t_1, \ldots, t_k \) such that \( \tilde{B}/(t_1, \ldots, t_k)\tilde{B} = B \) it follows that \( (t_1, \ldots, t_k) \) is a super regular sequence on \( \tilde{B} \).

**Proof.** We may write
\[ \tilde{B} \cong k[[X_1, \ldots, X_n, T_1, \ldots, T_k]]/J \]
such that \( t_i = T_i + J \), \( i = 1, \ldots, k \). Then \( J = (G_1, \ldots, G_m) \) with
\[ G_i = F_i + \sum_{j=1}^k F_i^{(j)}T_j + H_i , \]
$H_t \in (T_1, \cdots, T_n)$ and $F_t^{(j)} \in k[[X_1, \cdots, X_n]]$. Since $t_1, \cdots, t_n$ is a regular $B$-sequence, we obtain $B$-module homomorphisms

$$
\varphi_j : I/I^2 \to B,
$$

$$
F_t + I^2 \mapsto F_t^{(j)} + I
$$

By assumption 2) it follows that $\nu(F_t^{(j)}) \geq 1$ and by assumption 1) it follows that $\nu(F_t) = \nu(F_t^{(j)})$ for $i = 1, \cdots, m$.

From our criterion of section 2 the assertion follows.

We use this theorem to derive two results of J. Sally in a slightly more special case.

We introduce the following notations: $e(B)$ = embedding dimension of $B$, $d(B)$ = Krull dimension of $B$ and $m(B)$ = multiplicity of $B$.

**THEOREM 3 ([4], [5]).** Let $B = k[[X_1, \cdots, X_n]]/I$ be a complete CM-algebra and suppose that either

a) $m(B) \leq e(B) - d(B) + 1$

or

b) $m(B) \leq e(B) - d(B) + 2$ and $B$ is a Gorenstein ring

then $\text{gr}(B)$ is a CM-ring.

**Proof.** We may assume that $k$ is algebraically closed.

a) There exists a regular sequence $(t_1, \cdots, t_n)$ such that

$$
l(B/(t_1, \cdots, t_n)B) = m(B).
$$

This sequence is part of a minimal system of generators of $m_B$. Let $\bar{B} = B/(t_1, \cdots, t_n)B$, then $e(\bar{B}) = e(B) - d(B) = m(B) - 1 = l(\bar{B}) - 1$. It follows that $m_{\bar{B}} = 0$, and $\bar{B} = k[[X_1, \cdots, X_n]]/I$ with $I = (X_1, \cdots, X_n)^2$. We may assume that $m \geq 2$ and show that $\bar{B}$ satisfies the conditions of Theorem 2.

Condition 1) is obviously satisfied since $\bar{I}$ is generated by the monomials $X_iX_j$ of degree 2, which form a standard base of $\bar{I}$.

Suppose there exists a $\bar{B}$-module homomorphism $\varphi : \bar{I}/\bar{I}^2 \to \bar{B}$ and integers $i, j$ such that $\varphi(X_iX_j + \bar{I}^2)$ is a unit.

1st Case. If $i = j$, then for any $k \neq i$ we have

$$
x_i\varphi(X_i^2 + \bar{I}^2) + x_k\varphi(X_kX_k + \bar{I}^2),
$$

a contradiction since $(x_1, \cdots, x_n)$ is a minimal base of $m_B$. 
2nd Case. If $i \neq j$, then $x_i\varphi(X_iX_j + \bar{I}^3) = x_j\varphi(X_i^3 + \bar{I}^3)$, again a contradiction.

β) As in the case α) we can reduce $B$ to an algebra $\bar{B}$ such that $l(\bar{B}) = e(\bar{B}) + 2$. It follows that $m_\bar{B} = 0$ and that $\bar{B}$ is a graded ring with Hilbert function $1 + e(\bar{B})t + \bar{t}$. Let $\sigma$ be generator of $\bar{B}_2$. The multiplication on $\bar{B}$ induces a non singular quadratic form $q: \bar{B}_1 \times \bar{B}_1 \rightarrow k$ defined by

$$q(u, w)\sigma = u \cdot w$$

Since we assume that $k$ is algebraically closed we can choose a $k$-vectorspace base $x_1, \ldots, x_m$ of $\bar{B}$, such that $x_i = \sigma$ for $i = 1, \ldots, m$ and $x_i x_j = 0$ for $i \neq j$.

We treat the case $m = 2$ separately, since in that case $\bar{B}$ is a complete intersection and Theorem 2 is not applicable. However then we have $B = k[[X_1 \cdots X_n]]/(F_1, F_2)$ with $\bar{F}_1 = X_1^2 - X_2^2, \bar{F}_2 = X_1X_2$. If $\nu(F_1) = \nu(F_2) = 2$, then the assertion follows from Theorem 1. Otherwise, say $\nu(F_1) = 1$, then $B$ is a hypersurface ring and the assertion follows again.

Now if $m > 2$ we apply Theorem 2: Again the first condition is satisfied. We check condition 2):

1st Case. Suppose there exists a $\bar{B}$-module homomorphism $\varphi: \bar{I}/\bar{I}^3 \rightarrow \bar{B}$ such that $\varphi(X_i^3 - X_i^2 + \bar{I}^3)$ is a unit, then

$$\sigma\varphi(X_i^3 - X_i^2 + \bar{I}^3) = x_i^2(\varphi(X_i^3 - X_i^2 + \bar{I}^3) = \varphi(X_i^3 - X_i^2 + \bar{I}^3) = 0,$$

since $X_i^2 - X_iX_i^2 \in \bar{I}^3$. This is a contradiction.

2nd Case. Suppose there exists a $\bar{B}$-module homomorphism $\varphi: \bar{I}/\bar{I}^3 \rightarrow \bar{B}$ such that $\varphi(X_iX_j + \bar{I}^3)$ is a unit, then $\sigma\varphi(X_iX_j + \bar{I}^3) = x_i^2(\varphi(X_iX_j + \bar{I}^3) = \varphi((X_iX_j)(X_iX_j) + \bar{I}^3) = 0$ since $(X_iX_j)(X_iX_j) \in \bar{I}^3$. This is again a contradiction.

**Literature**


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