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THE ASYMPTOTIC SHAPE OF THE BRANCHING RANDOM WALK

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Abstract

In a supercritical branching random walk on $\mathbb{R}^p$, a Galton–Watson process with the additional feature that people have positions, let $\mathcal{I}^{(n)}$ be the set of positions of the nth-generation people, scaled by the factor n^{-1}. It is shown that when the process survives $\mathcal{I}^{(n)}$ looks like a convex set $\mathcal{I}$ for large n. An analogous result is established for an age-dependent branching process in which people also have positions. In certain cases an explicit formula for the asymptotic shape is given.

SHAPE; BRANCHING PROCESSES; BRANCHING RANDOM WALK

1. Introduction

Let $\mathcal{X}$ be the vector space $\mathbb{R}^p$ for some finite p. In the branching random walk on $\mathcal{X}$ an initial ancestor is born at the origin. His children, who form the first generation, have positions which form a point process on $\mathcal{X}$. Let $\{Z_r^{(1)}\}$ be the set of positions of the first-generation people. The people in the nth generation give birth independently of one another and of $\mathfrak{F}^n$, the σ-field generated by all of the births in the first n generations. Given $\mathfrak{F}^n$ the point process formed by the children of an nth-generation person at X has the same distributions as the process with points $\{Z_r^{(1)}+X\}$; thus if the origin were moved to X it would have the same distributions as $Z_r^{(1)}$. Let the set of positions of the nth-generation people be $\{Z_r^{(n)}\}$. Ney (1965) considered a process similar to this one; however, the asymptotic properties that he was concerned with are quite different from those considered here.

The generation sizes, $\#\{Z_r^{(n)}: r\}$, in a branching random walk form a Galton–Watson process. Let S be the event that there are people in every generation. To ensure that S has positive probability we will assume that the expected number of people in the first generation is strictly greater than one.

Throughout this article lower-case letters will be used for real numbers, capital letters for elements of $\mathcal{X}$ and script letters for subsets of $\mathcal{X}$.

Let $I_r^{(n)} = Z_r^{(n)}/n$ and for each n let $\mathcal{I}^{(n)}$ be the set of points $\{I_r^{(n)}: r\}$; thus $\mathcal{I}^{(n)}$ is the set of positions of the nth-generation people scaled by the factor n^{-1}. The convex hull of $\mathcal{I}^{(n)}$ will be denoted by $\mathcal{H}^{(n)}$. The main aim of this paper is to show that on S both $\mathcal{H}^{(n)}$ and $\mathcal{I}^{(n)}$ 'look like' a convex set $\mathcal{I}$ when n is large. The set $\mathcal{I}$ is the asymptotic shape of the branching random walk. When

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$\mathscr{X} = \mathbb{R}\ \mathscr{H}^{(n)}$ is just an interval and the asymptotic behaviour of its end points has been studied in a series of papers by Hammersley (1974), Kingman (1975) and Biggins (1976), (1977b). The description of the behaviour of $\mathscr{I}^{(n)}$ when $\mathscr{X} = \mathbb{R}^p$ will be approached through the one-dimensional result, the appropriate version of which is given in Section 3.

Mollison (1977) has written a review article on the spread of spatial processes and the problems considered here fall within this broad category; a discussion of related problems can be found in that article. There are, as is explained there, other approaches to the problem of asymptotic shape. For example, in the one-dimensional case, if we let ${}^s\mathscr{H}^{(n)}$ be the smallest interval with at most s points of $\mathscr{I}^{(n)}$ to the left of it and at most s points to the right of it, so that ${}^0\mathscr{H}^{(n)} = \mathscr{H}^{(n)}$, then we could consider the asymptotic shape of ${}^s\mathscr{H}^{(n)}$. It is easy to establish, using the results contained in Biggins (1977b), that $\mathscr{H}^{(n)}$ and ${}^s\mathscr{H}^{(n)}$ have the same asymptotic shape. Other definitions, in terms of expected numbers, of quantities describing, in some sense, the shape of the nth generation can be formulated. Thus, still thinking of the one-dimensional case, we could examine, for fixed s, the behaviour of $\inf\{x : E[\#\{r : I_r^{(n)} \geqq x\}] \geqq s\}$ as n gets large, and its relationship with the right endpoint of ${}^s\mathscr{H}^{(n)}$. These questions will not be pursued here. Daniels (1977a,b) has some results on questions of this kind.

If X and Y are in $\mathscr{X}$ their inner product will be written as $\langle X, Y\rangle$ and the Euclidean norm of X as $\|X\|$. The unit sphere, $\{X : \|X\| = 1\}$, will be denoted by $\mathscr{S}$ and the closed ball of radius r, $\{X : \|X\| \leqq r\}$, by $\mathscr{B}_r$. The function $k(\Theta)$ is defined on $\mathscr{X}$ by

$$k(\Theta) = \log E\left[\sum_r \exp\langle -\Theta, Z_r^{(1)}\rangle\right].$$

It is possible that $k(\Theta)$ is always infinite. Notice that $k(0)$, where 0 is the origin of $\mathscr{X}$, is the logarithm of the expected number of people in the first generation; thus we are assuming that

$$k(0) > 0.$$

Let the measure g on $\mathscr{X}$ be defined by $g(\mathscr{D}) = E[\#\{r : Z_r^{(1)} \in \mathscr{D}\}]$ where $\mathscr{D} \subset \mathscr{X}$ (the adjective measurable will always be omitted), then for any h: $\mathscr{X} \to \mathbb{R}^+$

$$E\left[\sum_r h(Z_r^{(1)})\right] = \int_{\mathscr{X}} h(X)\,dg(X) \tag{1.1}$$

and in particular $\exp k(\Theta)$ is the multivariate Laplace–Stieltjes transform of g.

The next section is a collection of the various results on multivariate

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Laplace–Stieltjes transforms that we will need. In the third section the sets $\mathscr{H}^{(n)}$ are shown to have the asymptotic shape $\mathscr{I}$. The fourth section contains the main result, that the sets $\mathscr{I}^{(n)}$ also have the asymptotic shape $\mathscr{I}$. A continuous-time analogue of the main result is obtained in the fifth section as a consequence of it. In the sixth section a special case of this continuous-time result, when the number of people alive form a continuous-time Galton–Watson process, is discussed and its relationship with some results of Mollison (1978) on 'the velocity of the contact birth process' is explained.

2. Multivariate Laplace–Stieltjes transforms

As was mentioned in the introduction we will need various results about multivariate Laplace–Stieltjes transforms. These results, which are all connected with the convexity properties of such transforms, are discussed in this section.

For this section let g be a measure on $\mathscr{X}(=\mathbb{R}^p)$ and let

$$\exp k(\Theta) = \int \exp\langle -\Theta, X\rangle dg(X). \tag{2.1}$$

Then, by Hölder's inequality,

$$\exp k(\alpha\Theta_1 + \beta\Theta_2) \leqq \exp \alpha k(\Theta_1) \exp \beta k(\Theta_2)$$

for α, $\beta \geqq 0$ and $\alpha + \beta = 1$, so that $k(\Theta)$ is a convex function and is finite on a possibly empty convex set $\mathscr{T}$.

For any fixed $W \in \mathscr{S}$ (the unit sphere in $\mathscr{X}$) and Y in $\mathscr{X}$ the function

$$\exp k(\theta W + Y) \tag{2.2}$$

of the real number θ is a Laplace–Stieltjes transform. In fact

$$\begin{aligned} \exp k(\theta W + Y) &= \int_{\mathscr{X}} \exp\langle -\theta W - Y, X\rangle dg(X) \\ &= \int_{\mathscr{X}} \exp(-\theta\langle W, X\rangle) \exp\langle -Y, X\rangle dg(X) \\ &= \int_{\mathbb{R}} \exp(-x\theta) d\tilde{g}(x) \end{aligned} \tag{2.3}$$

where the measure $\tilde{g}$ on $\mathbb{R}$ is defined by

$$\tilde{g}(\mathscr{D}) = \int_{\{X : \langle X, W\rangle \in \mathscr{D}\}} \exp\langle -Y, X\rangle dg(X) \quad \text{for} \quad \mathscr{D} \subset \mathbb{R}.$$

This provides us with much information about the one-dimensional sections of $k(\Theta)$.

The function ξ on $\mathscr{X}$ given by

$$\xi(A) = \inf\{k(\Theta) + \langle \Theta, A\rangle : \Theta\} \tag{2.4}$$

will arise naturally later in the paper. As ξ is the infimum of a set of linear functions it must be concave. Also, if $\xi(A)$ is finite and $A_i \to A$ then for small ε, suitable Θ_ε, and large i,

$$\xi(A) + \varepsilon \geqq k(\Theta_\varepsilon) + \langle \Theta_\varepsilon, A\rangle \geqq k(\Theta_\varepsilon) + \langle \Theta_\varepsilon, A_i\rangle - \varepsilon \geqq \xi(A_i) - \varepsilon;$$

hence ξ is upper semicontinuous. In fact, as can be seen from Chapter 12 of Rockafellar (1970), $-\xi(-A)$ is the convex conjugate function of k so that the concavity and upper semicontinuity of ξ follow from the general theory of conjugate convex functions. Also, since (2.2) is lower semicontinuous in θ, it is not hard to see that Theorem 7.5 of Rockafellar (1970) implies that k is a closed (i.e. lower semicontinuous on $\mathscr{X}$) convex function. Then, by Theorem 12.2 we have, as a dual relationship to (2.4), that

$$k(\Theta) = \sup\{\xi(A) - \langle \Theta, A\rangle : A\}. \tag{2.5}$$

Let $\mathscr{A}$ be the smallest closed convex set such that $g(\mathscr{X}\backslash\mathscr{A}) = 0$; thus $\mathscr{A}$ is the closure of the convex hull of the support of g. The sets

$$\mathscr{I}(a) = \{A : \xi(A) \geqq a\} \cap \mathscr{A} \tag{2.6}$$

will be of particular interest to us. The interior and closure of a set $\mathscr{D}$ will be denoted by int $\mathscr{D}$ and cl $\mathscr{D}$, and if $\mathscr{D}$ is convex its relative interior will be denoted by rint $\mathscr{D}$.

Lemma 1.

(i) For any $c < k(0)$ and $A \in \operatorname{rint} \mathscr{A}$ the set $\{\Theta : k(\Theta) + \langle \Theta, A\rangle \leqq c\}$ bounded.

(ii) $\xi(A) > -\infty$ on rint $\mathscr{A}$.

(iii) $\xi(A) = -\infty$ on $\mathscr{X}\backslash\mathscr{A}$ when $\mathscr{T}$ is non-empty.

(iv) $\operatorname{rint}\{A : \xi(A) > -\infty\} = \operatorname{rint} \mathscr{A}$ when $\mathscr{T}$ is non-empty.

Proof.

(i) If $A \in \operatorname{rint} \mathscr{A}$ then for any $W \in \mathscr{S}$ either there is a $d > 0$ such that $\{X : \langle W, A - X\rangle \geqq d, X \in \operatorname{rint} \mathscr{A}\}$ is non-empty or $\mathscr{A} \subset \{X : \langle W, A - X\rangle = 0\}$. In the first case,

$$\exp(k(\theta W) + \theta\langle W, A\rangle) = \int \exp(\theta\langle W, A - X\rangle)dg(X)$$
$$\geqq e^{\theta d} g(\{X : \langle W, A - X\rangle \geqq d\}) > 0$$

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and so $k(\theta W)+\theta\langle W, A\rangle\to\infty$ as $\theta\to\infty$, and in the second case $k(\theta W)+\theta\langle W, A\rangle = k(0)$ for all θ. Therefore the convex set $\{\Theta: k(\Theta)+\langle\Theta, A\rangle\leqq c\}$ must be bounded.

(ii) This is immediate from (2.4) and (i).

(iii) Suppose that $Y\in\mathcal{T}$ and that $A\in\mathcal{X}\backslash\mathcal{A}$. There is a plane through A not touching $\mathcal{A}$, that is there is a $W\in\mathcal{S}$ for which $\mathcal{A}\subset\{X:\langle W, X-A\rangle\geqq d>0\}$. Hence

$$\begin{aligned}\exp(k(\theta W+Y)+\langle\theta W+Y, A\rangle) &= \int_{\mathcal{A}}\exp\langle\theta W+Y, A-X\rangle dg(X)\\ &\leqq e^{-\theta d}\int_{\mathcal{A}}\exp\langle Y, A-X\rangle dg(X)\\ &= e^{-\theta d}\exp(k(Y)+\langle Y, A\rangle)\\ &\to 0 \text{ as } \theta\to\infty\end{aligned}$$

and so, from (2.4), $\xi(A)=-\infty$.

(iv) This follows from (ii) and (iii).

Lemma 2. $\mathcal{I}(a)=\{A:\xi(A)\geqq a\}$ when $\mathcal{T}$ is non-empty and $\mathcal{I}(a)=\mathcal{A}$ otherwise.

Proof. This follows from the definition (2.6) and Lemma 1 (iv).

Lemma 3.

(i) $\mathcal{I}(a)$ is a closed convex set and $\mathcal{I}(a)=\bigcap_{d<a}\mathcal{I}(d)$.

(ii) If $a<k(0)$ then $\mathcal{I}(a)$ is non-empty and $\operatorname{rint}\mathcal{I}(a)\subset\bigcup_{d>a}\mathcal{I}(d)$.

(iii) If $a<k(0)$ then $\operatorname{int}\mathcal{I}(a)$ is non-empty if and only if $\operatorname{int}\mathcal{A}$ is non-empty.

Proof. All the assertions hold when $\mathcal{I}(a)=\mathcal{A}$ for all a; thus we can assume, by Lemma 2, that $\mathcal{I}(a)=\{A:\xi(A)\geqq a\}$.

(i) This follows from the concavity and upper semicontinuity of ξ.

(ii) By (2.5) $\sup\{\xi(A):A\}=k(0)$ so that $\{A:\xi(A)\geqq a\}$ is non-empty when $a<k(0)$, and by Theorem 7.6 of Rockafellar $\operatorname{rint}\{A:\xi(A)\geqq a\}\subset\{A:\xi(A)>a\}$.

(iii) Again by Theorem 7.6 when $a<k(0)$ $\operatorname{int}\{A:\xi(A)\geqq a\}$ is non-empty if and only if $\operatorname{int}\{A:\xi(A)>-\infty\}$ is non-empty, and by Corollary 13.4.2 of Rockafellar $\operatorname{int}\{A:\xi(A)>-\infty\}$ is empty if and only if $k(\theta W+Y)=a\theta+b$ for some $W\in\mathcal{S}$, $Y\in\mathcal{X}$ and $a, b\in\mathbb{R}$, and for all real θ; but then, from (2.3), g must be concentrated on the plane $\{X:\langle X, W\rangle=a\}$ and $\operatorname{int}\mathcal{A}$ is empty.

Lemma 4. If $0\in\operatorname{int}\mathcal{T}$ then $\mathcal{I}(a)$ is compact.

Proof. This follows from Corollary 14.2.2 of Rockafellar.

The next lemma, which is not needed until the fifth section, uses only the facts that $\{\mathcal{I}(a):a\}$ are nested closed convex sets. The convex set

$$\mathcal{K}(a)=\{X : cX\in\mathcal{I}(a) \text{ for some } c>0\}\cup\{0\}$$

is called the cone generated by $\mathcal{I}(a)$. The recession cone of $\mathcal{I}(a)$ is the set

$$\{Y:\{Z+\lambda Y:\lambda\geqq 0\}\subset\mathcal{I}(a)\quad\text{for all}\quad Z\in\mathcal{I}(a)\}.$$

Lemma 5. If $0\notin\mathcal{I}(a)$ and $a<k(0)$ then

$$\bigcap_{d<a}\operatorname{cl}\mathcal{K}(d)=\operatorname{cl}\mathcal{K}(a).$$

Proof. Again by Lemma 2 we need only consider the case when $\mathcal{T}$ is non-empty. Obviously $\mathcal{K}(a)\subset\bigcap_{d>a}\mathcal{K}(d)$, and $\mathcal{K}(a)$ is non-empty because $\mathcal{I}(a)$ is non-empty when $a<k(0)$. By Theorem 6.5 of Rockafellar $\operatorname{cl}\bigcap\mathcal{K}(d)=\bigcap\operatorname{cl}\mathcal{K}(d)$ thus it suffices to show that $\bigcap\mathcal{K}(d)\subset\operatorname{cl}\mathcal{K}(a)$. If $X\in\mathcal{K}(d)$ then $c_dX\in\mathcal{I}(d)$ for some $c_d>0$ and either $\{c_d\}$ has an accumulation point c or $c_d\to\infty$, as $d\to a$. In the former case, because the sets $\mathcal{I}(d)$ are closed and nested, $cX\in\mathcal{I}(a)$ thus $c\neq 0$ and so $X\in\mathcal{K}(a)$. In the latter case, since the sets $\mathcal{I}(d)$ are nested, the ray $\{\lambda X:\lambda\geqq c_d\}\subset\mathcal{I}(d)$ and so, by Theorem 8.3 of Rockafellar, X is in the recession cone of $\mathcal{I}(d)$. Then, because the sets $\mathcal{I}(d)$ are closed, Corollary 8.3.3 implies that X is in the recession cone of $\mathcal{I}(a)$ and so Theorem 9.6 implies that $X\in\operatorname{cl}\mathcal{K}(a)$.

For the remainder of this section we will assume that int $\mathcal{T}$ is non-empty. If $Y\in\operatorname{int}\mathcal{T}$ then, from (2.3), $\exp k(\theta W+Y)$ is differentiable at $\theta=0$ for any $W\in\mathcal{S}$; thus

$$\frac{\partial\exp k(\Theta)}{\partial\Theta_i}=\int -X_i\exp\langle-\Theta,X\rangle dg(X)$$

where the subscript i indicates the ith component with respect to some basis. The integral on the left has the same form as (2.1), as can be seen by incorporating X_i into $g(X)$, and so $\exp k(\Theta)$ and hence $k(\Theta)$ is C^∞ on int $\mathcal{T}$. Let

$$\mathcal{A}^*=\{-\nabla k(\Theta):\Theta\in\operatorname{int}\mathcal{T}\}.$$

A point E in the convex set $\mathcal{D}$ is called an exposed point if there exists a supporting plane, $\{Y:\langle Y,W\rangle=\kappa\}$, to $\mathcal{D}$ for which $\mathcal{D}\cap\{Y:\langle Y,W\rangle=\kappa\}=E$. Obviously E is on the relative boundary of $\mathcal{D}$ which will be denoted by $r\partial\mathcal{D}$.

Lemma 6.

(i) All points in $r\partial\mathcal{I}(a)\cap\mathcal{A}^*$ are exposed points of $\mathcal{I}(a)$.

(ii) If $\mathcal{I}(a)\subset\mathcal{A}^*$ then every point in $r\partial\mathcal{I}(a)$ is exposed.

Proof. This lemma is an immediate consequence of the definitions and Corollary 25.1.2 of Rockafellar.

Lemma 7.

(i) If $0 \in \text{int}\,\mathcal{T}$ and $A \neq -\nabla k(0)$ with $\xi(A) > -\infty$ then $k(0) > \xi(A)$ and $\xi(\lambda A - (1-\lambda)\nabla k(0))$ is a strictly decreasing function of λ for $0 \leqq \lambda \leqq 1$.

(ii) $\mathcal{A}^* \subset \mathcal{A}$ and if $\mathcal{T} = \mathcal{X}$ then $\text{rint}\,\mathcal{A} \subset \mathcal{A}^*$.

Proof.

(i) From (2.5) $k(0) = \sup\{\xi(A) : A\}$ and, by Theorem 27.1(e) of Rockafellar this supremum is achieved at the unique point $-\nabla k(0)$; the stated result follows.

(ii) If $-\nabla k(\Theta_0) = A_0 \in \mathcal{A}^*$ then, by calculus, $\xi(A_0) = k(\Theta_0) - \langle A_0, \Theta_0 \rangle$ so that $\xi(A_0) > -\infty$. Therefore, using Lemma 1 (iv), $\mathcal{A}^* \subset \mathcal{A}$. If $\mathcal{T} = \mathcal{X}$ and $A_0 \in \text{rint}\,\mathcal{A}$ then either $\xi(A_0) = k(0)$, in which case $A_0 = -\nabla k(0) \in \mathcal{A}^*$, or $\xi(A_0) < k(0)$. In the latter case, by Lemma 1 (i), the infimum in (2.4) is attained at some finite Θ_0 and then $A_0 = -\nabla k(\Theta_0) \in \mathcal{A}^*$.

The remaining lemmas will not be needed to prove the main results in this paper but they are connected with certain peripheral matters that will also be discussed. Let

$$\mathcal{E}(a) = \{\Theta : k(\Theta) + \langle \Theta, -\nabla k(\Theta) \rangle > a,\ \Theta \in \text{int}\,\mathcal{T}\}. \tag{2.7}$$

We will assume from now on in this section that $\text{int}\,\mathcal{A}$ is non-empty; by Lemma 3(iii) $\text{int}\,\mathcal{I}(a)$ is then non-empty when $a < k(0)$. The boundary of a set $\mathcal{D}$ will be denoted by $\partial\mathcal{D}$.

Lemma 8.

(i) $-\nabla k$ is a C^∞-diffeomorphism from $\text{int}\,\mathcal{T}$ to $\mathcal{A}^*$.

(ii) $\mathcal{E}(a) = (-\nabla k)^{-1}(\text{int}\,\mathcal{I}(a) \cap \mathcal{A}^*)$ when $a < k(0)$.

Proof.

(i) It is easy to verify that

$$\frac{\partial^2 k(\Theta)}{\partial\Theta_i \partial\Theta_j} = \int \left(X_i - \int X_i dg^*(X)\right)\left(X_j - \int X_j dg^*(X)\right) dg^*(X) \tag{2.8}$$

where $dg^*(X)$ is the probability distribution $\exp(\langle -\Theta, X \rangle - k(\Theta))dg(X)$; hence (2.8) is the covariance matrix of this distribution. Since $\mathcal{A}$ has a non-empty interior this distribution is truly p-dimensional and so the matrix (2.8) must be positive definite. The function $k(\Theta)$ has, therefore, a positive definite Hessian matrix on $\text{int}\,\mathcal{T}$ and so is strictly convex there. (This argument is taken from Lemma 2.1.1 of Brown (1971).) It follows that $-\nabla k$ is one–one from $\text{int}\,\mathcal{T}$ on to $\mathcal{A}^*$. Since the determinant of the matrix (2.8) is non-zero and $-\nabla k$ is C^∞ the inverse function theorem implies that $(-\nabla k)^{-1}$ is C^∞ also.

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(ii) By Lemma 3 $\text{int}\,\mathscr{I}(a)\cap\mathscr{A}^*=\{A:\xi(A)>a\}\cap\mathscr{A}^*$, but when $A\in\mathscr{A}^*$ $\xi(A)=k((-\nabla k)^{-1}(A))+\langle A,(-\nabla k)^{-1}(A)\rangle$, by calculus. Therefore

$$\text{int}\,\mathscr{I}(a)\cap\mathscr{A}^*=\{A:k((-\nabla k)^{-1}(A))+\langle A,(-\nabla k)^{-1}(A)\rangle>a,\ A\in\mathscr{A}^*\}$$

and the result follows from the definition (2.7).

Lemma 9. If $\mathscr{I}(a)\subset\mathscr{A}^*$ and $a<k(0)$ then
(i) $\partial\mathscr{E}(a)=(-\nabla k)^{-1}(\partial\mathscr{I}(a))$;
(ii) the equations

$$k(\Theta)+\langle\Theta,A\rangle=a\quad\text{and}\quad A=-\nabla k(\Theta)$$

are satisfied only when $A\in\partial\mathscr{I}(a)$ and $\Theta\in\partial\mathscr{E}(a)$; furthermore, to each point in $\partial\mathscr{I}(a)$ (and to each point in $\partial\mathscr{E}(a)$) there corresponds exactly one solution;
(iii) $\partial\mathscr{I}(a)$ is a C^∞-manifold of dimension $(p-1)$.

Proof. The interior of the closed set $(-\nabla k)^{-1}\mathscr{I}(a)$ is $\mathscr{E}(a)$, by Lemma 8(ii), and hence $(-\nabla k)^{-1}(\partial\mathscr{I}(a))=\partial\mathscr{E}(a)$. By Lemma 8(i) $\mathscr{A}^*$ is open. The function ξ, being concave, is continuous on $\text{int}\,\{A:\xi(A)>-\infty\}$ and by Lemmas 1(ii) and 7(ii) this set contains $\mathscr{A}^*$. Therefore, using the definition (2.6) and Lemma 3(ii) $\partial\mathscr{I}(a)=\{A:\xi(A)=a\}$ and so by (i)

$$\partial\mathscr{E}(a)=\{\Theta:k(\Theta)+\langle\Theta,-\nabla k(\Theta)\rangle=a\}.$$

Combining this formula with (i) proves (ii). On $\mathscr{A}^*$

$$\xi(A)=k((-\nabla k)^{-1}(A))+\langle A,(-\nabla k)^{-1}(A)\rangle$$

and so is C^∞ there. If $\nabla\xi(A)$ is zero then ξ attains its maximum, $k(0)$, at A. Therefore $\nabla\xi(A)$ is non-zero on $\{A:\xi(A)=a\}=\partial\mathscr{I}(a)$, since $a<k(0)$, and this proves (iii).

3. The shape of $\mathscr{H}^{(n)}$

Let the particular convex set $\mathscr{I}(0)$ be denoted by $\mathscr{I}$. In this section the sense will be shown to be an 'upper bound' for the sets $\mathscr{I}^{(n)}$ in the sense that for any $a<0$

(3.1) $$\mathscr{I}^{(n)}\subset\mathscr{I}(a)\text{ for all but finitely many } n \text{ on } S$$

where, by Lemma 3(i), $\bigcap_{a<0}\mathscr{I}(a)=\mathscr{I}$. This upper bound will be combined with the known behaviour of the branching random walk on $\mathbb{R}$ to show that $\mathscr{I}$ is the asymptotic shape of $\mathscr{H}^{(n)}$; in fact we will show that

(3.2) $$\text{rint}\,\mathscr{I}\subset\liminf\mathscr{H}^{(n)}\subset\limsup\mathscr{H}^{(n)}\subset\mathscr{I}\text{ a.s. on } S$$

where

$$\liminf\mathscr{H}^{(n)}=\bigcup_m\bigcap_{n>m}\mathscr{H}^{(n)}\quad\text{and}\quad\limsup\mathscr{H}^{(n)}=\bigcap_{n>m}\bigcup_m\mathscr{H}^{(n)}.$$

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If $\{\mathcal{F}^{(n)}\}$ are subsets of $\mathcal{X}$ then $\{F^{(n)}\}$, where for each n $F^{(n)}\in\mathcal{F}^{(n)}$, will be called an $\mathcal{F}^{(n)}$-section, and if $\|F^{(n)}-F\|\to 0$ as $n\to\infty$ it will be called an $\mathcal{F}^{(n)}$-section for F.

It follows from (1.1) and the description of the process that for any $h:\mathcal{X}\to\mathbb{R}^+$

$$E\Big[\sum_r h(Z_r^{(n)})\mid \mathfrak{F}^{n-1}\Big]=\sum_r\int h(Z_r^{(n-1)}+X)dg(X).$$

In particular

$$E\Big[\sum_r \exp\langle -\Theta, Z_r^{(n)}\rangle\mid \mathfrak{F}^{n-1}\Big]=\exp k(\Theta)\sum_r \exp\langle -\Theta, Z_r^{(n-1)}\rangle \tag{3.3}$$

and so

$$E\Big[\sum_r \exp\langle -\Theta, Z_r^{(n)}\rangle\Big]=\exp nk(\Theta).$$

Hence when $\Theta\in\mathcal{T}$

$$E\Big[\sum_r \frac{1}{(\exp(k(\Theta)+\langle\Theta, I_r^{(n)}\rangle))^n}\Big]=1 \text{ for each } n. \tag{3.4}$$

Let us assume that $\mathcal{T}$ is non-empty and let Ω_n be the event that $\mathcal{I}^{(n)}\backslash\mathcal{I}(a)$ is non-empty, where $a<0$ is fixed. We may enumerate $\mathcal{I}^{(n)}$ so that $I_1^{(n)}\in\mathcal{I}^{(n)}\backslash\mathcal{I}(a)$ when Ω_n occurs and then, from Lemma 2 and the definition (2.4),

$$\frac{1}{\exp(k(\Theta)+\langle\Theta, I_1^{(n)}\rangle)}\leqq\frac{1}{\exp\xi(I_1^{(n)})}\leqq e^{-a}.$$

Thus, from (3.4),

$$P[\Omega_n]e^{-na}\leqq 1 \quad\text{and so}\quad P[\Omega_n]\leqq e^{na}.$$

The Borel–Cantelli lemma now applies to prove (3.1) in this case. When $\mathcal{T}$ is empty $\mathcal{I}(a)=\mathcal{A}$ for all a by Lemma 2. The set $\mathcal{A}$ is the smallest closed convex set such that $\mathcal{I}^{(1)}\subset\mathcal{A}$ a.s. and then $\mathcal{I}^{(n)}\subset\mathcal{A}$ a.s.; thus (3.1) holds in this case also.

It is clear that any accumulation point of an $\mathcal{I}^{(n)}$-section must lie in $\bigcap_{a<0}\mathcal{I}(a)$ almost surely, and that lim sup $\mathcal{H}^{(n)}$ must be contained in $\bigcap_{a<0}\mathcal{I}(a)$ $(=\mathcal{I})$.

Digressing for a moment notice, from (3.3), that when $\Theta\in\mathcal{T}$

$$W^{(n)}(\Theta)=\sum_r\frac{\exp\langle -\Theta, Z_r^{(n)}\rangle}{\exp nk(\Theta)}$$

is a positive martingale with respect to the σ-fields $\mathfrak{F}^n$ and so has an almost sure limit, $W(\Theta)$. When $\mathscr{X}=\mathbb{R}$ the question 'when is $E[W(\Theta)]=1$' is examined in Biggins (1977a). The same proofs work in this context. If we assume that $\operatorname{int}\mathscr{T}$ is non-empty and let

$$\mathscr{E}(0)=\mathscr{E}$$

(where $\mathscr{E}(a)$ is defined by (2.7)) then when $\Theta\in\operatorname{int}\mathscr{T}$

$$E[W(\Theta)]=1 \quad \text{if} \quad E[W^{(1)}(\Theta)\log^+ W^{(1)}(\Theta)]<\infty \quad \text{and} \quad \Theta\in\mathscr{E},$$

and

$$W(\Theta)=0 \quad \text{if} \quad E[W^{(1)}(\Theta)\log^+ W^{(1)}(\Theta)]=\infty \quad \text{or} \quad \Theta\notin\mathscr{E}.$$

When $\operatorname{int}\mathscr{A}$ is non-empty Lemma 8(ii) shows that

$$\mathscr{E}=(-\nabla k)^{-1}(\operatorname{int}\mathscr{I}\cap\mathscr{A}^*).$$

It also seems worth noting that $E[W^{(1)}(\Theta)\log^+ W^{(1)}(\Theta)]$ is finite if and only if

$$E\left[\sum_r \exp\langle -\Theta, Z_r^{(1)}\rangle \log^+\left(\sum_r \exp\langle -\Theta, Z_r^{(1)}\rangle\right)\right]<\infty,$$

and since $\sum_r \exp\langle -\Theta, Z_r^{(1)}\rangle$ is convex in Θ and $x\log^+ x$ is increasing and convex in x we can see that $E[W^{(1)}(\Theta)\log^+ W^{(1)}(\Theta)]$ is finite for all Θ in some convex set.

To show that $\mathscr{I}$ is the asymptotic shape of $\mathscr{H}^{(n)}$ we will need the following result. Suppose that $\mathscr{X}=\mathbb{R}$ and $k(\Theta)<\infty$ for some $\Theta>0$. Let $\log\mu(a)=\inf\{\Theta a+k(\Theta):\Theta\geqq 0\}$, $\gamma=\inf\{a:\mu(a)>1\}$ and $I_{\min}^{(n)}=\inf\{I_r^{(n)}:r\}$; then $I_{\min}^{(n)}\to\gamma$ a.s. on S. A proof of this based on Kingman (1975) can be found in Biggins ((1976), Section 6) and a self-contained proof is given in Biggins (1977b).

The important observation now is that the projection of the branching random walk on $\mathscr{X}$ onto any subspace of $\mathscr{X}$ gives another branching random walk. In particular we can, for any $W\in S$, project the original process onto the one-dimensional subspace spanned by W; thus the nth generation in the new process have the positions $\{\langle W, Z_r^{(n)}\rangle\}$. (This observation was made by Professor Kingman. The same idea has been used by Mollison (1978).) Let us suppose, for the moment, that $0\in\operatorname{int}\mathscr{T}$ so that by Lemma 4 $\mathscr{I}$ is compact. Also

$$\log E\left[\sum_r \exp(-\theta\langle W, Z_r^{(1)}\rangle)\right]=k(\theta W)<\infty \quad \text{for some} \quad \theta>0$$

so that for any W the associated projected branching random walk satisfies the theorem quoted above. Thus if we let

$$\gamma(W)=\inf\{a:\inf\{k(\theta W)+\theta a:\theta\geqq 0\}>0\}$$

then we know that, almost surely on S, there exists an $\mathscr{I}^{(n)}$-section $\{I^{(n)}\}$ such that

$$\langle I^{(n)}, W\rangle \to \gamma(W). \tag{3.5}$$

Suppose that $\{Y:\langle Y, W\rangle = \kappa\}$ is a supporting plane to $\mathscr{I}$ such that $\mathscr{I} \subset \{Y:\langle Y, W\rangle \geqq \kappa\}$. For any $\varepsilon > 0$ $\mathscr{I}(a) \subset \{Y:\langle Y, W\rangle \geqq \kappa - \varepsilon\}$ for $a < 0$ sufficiently small. Therefore, from (3.1) and (3.5), $\gamma(W) \geqq \kappa - \varepsilon$. Now take $A \in \operatorname{rint} \mathscr{I}$; then by Lemma 3(ii) and the definition of ξ $0 < \xi(A) \leqq k(\theta W) + \langle \theta W, A\rangle$ for all real θ, hence $\gamma(W) \leqq \langle W, A\rangle$. Thus $\kappa = \gamma(W)$ and any supporting plane to $\mathscr{I}$ has the form $\{Y:\langle Y, W\rangle = \gamma(W)\}$ for some $W \in S$.

Let E be an exposed point of $\mathscr{I}$, then there is a $W_0 \in \mathscr{S}$ such that

$$\mathscr{I} \cap \{Y:\langle Y, W_0\rangle = \gamma(W_0)\} = E. \tag{3.6}$$

If we now take $\{I^{(n)}\}$ satisfying (3.5) with $W = W_0$, then from (3.1) and Lemma 4 this sequence is bounded, and any accumulation point of it must lie in $\mathscr{I}$. Let A be an accumulation point of the sequence: then along a suitable subsequence $\langle I^{(n)}, W_0\rangle \to \langle A, W_0\rangle = \gamma(W_0)$ and so, by (3.6), $A = E$. Hence the whole sequence must converge to E. This establishes that, when $0 \in \operatorname{int} \mathscr{T}$, for any exposed point E of $\mathscr{I}$ there is almost surely on S an $\mathscr{I}^{(n)}$-section for E.

Let $E_1, \cdots, E_N$ be exposed points of $\mathscr{I}$ and let $\mathscr{H}(E_1, \cdots, E_N)$ be their convex hull, then by choosing an $\mathscr{I}^{(n)}$-section for each of them we see that

$$\operatorname{rint} \mathscr{H}(E_1, \cdots, E_N) \subset \liminf \mathscr{H}^{(n)} \text{ a.s. on } S. \tag{3.7}$$

By Theorem 18.7 of Rockafellar (1970) any compact convex set is the closure of the convex hull of its exposed points, and so if we let $\{E_i\}$ be a countable set of exposed points dense in the set of all exposed points of $\mathscr{I}$ then we may let N tend to infinity in (3.7) to establish that when $0 \in \operatorname{int} \mathscr{T}$

$$\operatorname{rint} \mathscr{I} \subset \liminf \mathscr{H}^{(n)} \text{ a.s. on } S. \tag{3.8}$$

A straightforward truncation argument removes the restriction that $0 \in \operatorname{int} \mathscr{T}$. We first relax it to '$\mathscr{T}$ is non-empty'. For each integer $s > 0$ a new branching random walk is constructed from the original one in the following way. Only those people born at a distance less than or equal to s from their parent occur in the new process. The same procedure is applied to their children and so on. This will be called the bounded modification. Quantities in the sth modification will be denoted by a subscript, s. Then

$$\exp k_s(\Theta) = \int_{-\mathscr{B}_s} \exp\langle -\Theta, X\rangle dg(X)$$

so that $\mathscr{T}_s = \mathscr{X}$, (3.8) holds for each modified process and

$$k_s(\Theta) \uparrow k(\Theta).$$

As $\mathscr{H}_s^{(n)} \subset \mathscr{H}_{s+1}^{(n)} \subset \mathscr{H}^{(n)}$ for each s,

$$\bigcup_{s=1}^{\infty} \operatorname{rint} \mathscr{I}_s \subset \liminf \mathscr{H}^{(n)} \text{ a.s. on } \bigcup_{s=1}^{\infty} S_s.$$

Suppose that $A \in \operatorname{rint} \mathscr{I}$, then by Lemma 3(ii) $c = \xi(A) > 0$. Let $\tau(x) = x$ if $x < c$ and $\tau(x) = c$ if $x \geqq c$. For large s $A \in \operatorname{rint} \mathscr{A}_s$ and, by Lemma 1(i), $\tau(k_s(\Theta) + \langle \Theta, A \rangle) = c$ when $\Theta \in \mathscr{X} \backslash \mathscr{B}_r$ for large r. As $s \to \infty$

$$\tau(k_s(\Theta) + \langle \Theta, A \rangle) \uparrow c,$$

Dini's theorem implies that this convergence is uniform and so $\xi_s(A) \geqq c - \varepsilon > 0$ for large s. Therefore $\operatorname{rint} \mathscr{I} \subset \bigcup_{s=1}^{\infty} \mathscr{I}_s$, but $\{\mathscr{I}_s\}$ are nested convex sets so $\bigcup_{s=1}^{\infty} \mathscr{I}_s$ is convex and $\operatorname{rint} \bigcup_{s=1}^{\infty} \mathscr{I}_s = \bigcup_{s=1}^{\infty} \operatorname{rint} \mathscr{I}_s$ which implies that $\operatorname{rint} \mathscr{I} \subset \bigcup_{s=1}^{\infty} \operatorname{rint} \mathscr{I}_s$. An argument like those given by Kingman ((1975), Section 6) shows that $P[S \backslash \bigcup_{s=1}^{\infty} S_s] = 0$ and so $\operatorname{rint} \mathscr{I} \subset \liminf \mathscr{H}^{(n)}$ a.s. on S when $\mathscr{T}$ is non-empty. The remaining restriction can be removed by an application of the sterilization modification as described by Kingman ((1975), Section 6). (In this each person's children are assigned an order; in the Nth modification only the first N children of the initial ancestor are allowed to appear, and then only their first N children and so on.) This completes the proof of (3.2).

4. The shape of $\mathscr{I}^{(n)}$

In this section we will complete the proof of the following theorem, the first part of which was proved in the preceding section.

Theorem A. The following hold almost surely on S:

(i) for any $a < 0$ $\mathscr{I}^{(n)} \subset \mathscr{I}(a)$ for all sufficiently large n,

(ii) there is an $\mathscr{I}^{(n)}$-section for every point in $\mathscr{I}(= \bigcap_{a<0} \mathscr{I}(a))$.

We already know from the preceding section that when $0 \in \operatorname{int} \mathscr{T}$ and E is an exposed point of $\mathscr{I}$ there is, almost surely on S, an $\mathscr{I}^{(n)}$-section for E. Part (ii) of the theorem will be established by using this fact and truncation arguments. This theorem provides the justification for the informal statement '$\mathscr{I}^{(n)}$ has the asymptotic shape $\mathscr{I}$'.

Let us first suppose that $\mathscr{T} = \mathscr{X}$, so that by Lemma 7(ii), $\operatorname{rint} \mathscr{A} \subset \mathscr{A}^*$, and let $I \in \mathscr{I} \backslash \{-\nabla k(0)\}$. Then $\xi(I) \geqq 0$ and by Lemma 7(i) we may take a strictly monotonic sequence $\{p_i\}$ in $(0, 1)$ such that

$$\exp - k(0) < p_i < \exp - \xi(I) \quad \text{and} \quad p_i \uparrow \exp - \xi(I). \tag{4.1}$$

For each p_i construct a new branching random walk from the original one in the following way. Let each first generation person appear in the new process with probability p_i independently of the others. Apply the same procedure to the survivors' children and so on. Denote quantities in this new process by a

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subscript, i. Clearly we can arrange that $\mathcal{I}_i^{(n)} \subset \mathcal{I}_j^{(n)}$ whenever $i<j$. Since $k_i(\Theta)=k(\Theta)+\log p_i$ and $\xi_i(A)=\xi(A)+\log p_i$ it follows from (4.1) that

$$\xi_i(-\nabla k_i(0))=k_i(0)>0 \quad \text{and} \quad \xi_i(I)<0$$

for each i, and obviously $-\nabla k_i(0)=-\nabla k(0)$. Let I_i be the point on the line segment $\{\lambda I-(1-\lambda)\nabla k(0): 0<\lambda<1\}$ for which $\xi_i(I_i)=0$, by Lemma 7(i) there is just one and $I_i \to I$ as $i\to\infty$. Then $I_i \in r\partial\mathcal{I}_i \cap \operatorname{rint}\mathcal{A}=r\partial\mathcal{I}_i\cap\mathcal{A}^*=r\partial\mathcal{I}_i\cap\mathcal{A}_i^*$ and so by Lemma 6(i) I_i is an exposed point of $\mathcal{I}_i$; thus we know that on S_i there is an $\mathcal{I}_i^{(n)}$-section for I_i. Since $\mathcal{I}_i^{(n)}\subset\mathcal{I}_{i+1}^{(n)}$ for each i we can construct from these sequences, by a subsequence argument, an $\mathcal{I}^{(n)}$-section $\{I^{(n)}\}$ such that

$$I^{(n)}\to I \text{ a.s.} \quad \text{on} \quad S_1,$$

(actually this construction works on $\cup_{i=1}^{\infty} S_i$ but on S_1 suffices) where, since $k_1(0)>0$, $P[S_1]>0$.

Let f be the generating function for the Galton–Watson process consisting of the generation sizes in the original process, and let f^s be its sth iterate. The initial ancestor has the property that there exists an $\mathcal{I}^{(n)}$-section for I in the branching random walk emanating from him whenever one of his sth-generation offspring has this property. The probability of the latter event is greater than $1-f^s(1-P[S_1])$ and since $P[S_1]>0$ this tends to $P[S]$ as s tends to infinity by Theorem II.7.2 of Harris (1963). Hence the initial ancestor must have the stated property almost surely on S.

This establishes that when $\mathcal{T}=\mathcal{X}$ and $I\in\mathcal{I}\setminus\{-\nabla k(0)\}$ there exists, almost surely on S, an $\mathcal{I}^{(n)}$-section for I. Obviously this must also hold at $-\nabla k(0)$, as can be seen by taking a sequence in $\mathcal{I}\setminus\{-\nabla k(0)\}$ tending to $-\nabla k(0)$ and using their $\mathcal{I}^{(n)}$-sections to construct an $\mathcal{I}^{(n)}$-section for $-\nabla k(0)$. The condition that $\mathcal{T}=\mathcal{X}$ is easily removed using the bounded modification, then the sterilization modification, and a subsequence argument. The details are straightforward and are omitted.

Suppose now that we consider only those points of $\mathcal{I}$ in some countable dense subset of $\mathcal{I}$. Then we may say that, on $S\setminus N$ where N is null, for any I in this subset there is an $\mathcal{I}^{(n)}$-section for I. These $\mathcal{I}^{(n)}$-sections can then be used to approximate any point in $\mathcal{I}$ by an $\mathcal{I}^{(n)}$-section, on $S\setminus N$. This completes the proof of Theorem A.

When $\mathcal{T}$ is empty and $\mathcal{A}=\mathcal{X}$ or $\mathcal{T}=\{0\}$ the theorem tells us that as n tends to infinity $\mathcal{I}^{(n)}$ fills $\mathcal{X}$, on S. This suggests that, in these cases, the scaling factor n^{-1} in the definition of $\mathcal{I}^{(n)}$ does not decrease sufficiently quickly. It would be interesting to know whether an alternative scaling can ever produce a genuine asymptotic shape for these processes. Renshaw (1977) is looking at an example of this. When $\mathcal{T}$ is non-empty an obvious question is what is the asymptotic

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density of the sets $\mathscr{I}^{(n)}$ near a point A in $\mathscr{I}$. The results in Biggins (1977b) suggest that, on S, there are about $\exp(n\xi(A))$ points of $\mathscr{I}^{(n)}$ near A.

When int $\mathscr{T}$ and int $\mathscr{A}$ are non-empty and $\mathscr{I} \subset \mathscr{A}^*$, as will be the case whenever $\mathscr{T} = \mathscr{X} = \mathscr{A}$ for example, then by Lemma 3(iii) int $\mathscr{I}$ is non-empty and by Lemma 9(ii) the equations

$$k(\Theta) + \langle \Theta, A \rangle = 0 \quad \text{and} \quad A = -\nabla k(\Theta)$$

are satisfied only when $A \in \partial\mathscr{I}$ and $\Theta \in \partial\mathscr{E}$. Furthermore to each point of $\partial\mathscr{I}$ (and to each point of $\partial\mathscr{E}$) there corresponds exactly one solution. (These equations appear, in a different form, in both Kingman ((1975), Equation (3.7)) and Biggins ((1976), Equation (7.1)).) What is possibly more interesting is that, by Lemmas 6(ii) and 9(iii), in this case $\mathscr{I}$ is a strictly convex C^∞-manifold; thus the branching random walk has a smooth rounded shape.

When $0 \in$ int $\mathscr{T}$, as we shall now assume, Theorem A has a neater formulation in terms of the Hausdorff metric on the compact subsets of $\mathscr{X}$. For any set $\mathscr{D}$ let $\mathscr{N}_\varepsilon(\mathscr{D}) = \{X : \|X - D\| < \varepsilon, D \in \mathscr{D}\}$. For any two compact sets $\mathscr{D}_1$ and $\mathscr{D}_2$ let $\varepsilon_1 = \inf\{\varepsilon : \mathscr{D}_2 \subset \mathscr{N}_\varepsilon(\mathscr{D}_1)\}$ and $\varepsilon_2 = \inf\{\varepsilon : \mathscr{D}_1 \subset \mathscr{N}_\varepsilon(\mathscr{D}_2)\}$, and let $\Delta(\mathscr{D}_1, \mathscr{D}_2) = \max\{\varepsilon_1, \varepsilon_2\}$; then Δ is the Hausdorff metric on the compact subsets of $\mathscr{X}$ (a discussion of it can be found in Rogers ((1970), p. 90)).

By Lemma 4 $\mathscr{I}(a)$ is compact for every a; since these sets are nested and $\mathscr{I} = \bigcap_{a<0} \mathscr{I}(a)$ it is easy to see that for any $\varepsilon > 0$ $\mathscr{I}(a) \subset \mathscr{N}_\varepsilon(\mathscr{I})$ when a is sufficiently small. Thus when (i) of Theorem A holds we have that for any $\varepsilon > 0$

$$\mathscr{I}^{(n)} \subset \mathscr{N}_\varepsilon(\mathscr{I}) \quad \text{for sufficiently large } n.$$

Now let $\{I_i\}$ be a finite $\frac{1}{2}\varepsilon$-net for $\mathscr{I}$. Then when (ii) of the Theorem holds there is an $\mathscr{I}^{(n)}$-section $\{I_i^{(n)}\}$ for each I_i; thus for large n $\|I_i^{(n)} - I_i\| < \frac{1}{2}\varepsilon$ for each I_i and so

$$\mathscr{I} \subset \mathscr{N}_\varepsilon(\mathscr{I}^{(n)}) \quad \text{for sufficiently large } n.$$

Hence we have arrived at the following corollary.

Corollary to Theorem A. When $0 \in$ int I

$$\Delta(\mathscr{I}^{(n)}, \mathscr{I}) \to 0 \quad \text{a.s. on } S.$$

It is also not hard to show that (3.2) is equivalent to

$$\Delta(\mathscr{H}^{(n)}, \mathscr{I}) \to 0 \quad \text{a.s. on } S$$

when $0 \in$ int $\mathscr{T}$.

Suppose that $\|\cdot\|^*$ is any norm on $\mathscr{X}$ and that

$$I_{\min}^{(n)} = \inf\{\|I_r^{(n)}\|^* : r\} = \inf\{n^{-1}\|Z_r^{(n)}\|^* : r\}$$

so that $nI_{\min}^{(n)}$ is the minimum distance, with respect to $\|\cdot\|^*$, from the origin to an nth-generation person. This work arose from consideration of the following

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question. Does $I^{(n)}_{\min} \to \gamma$ a.s. on S for some constant γ? This question can now be answered quite simply and there is no loss in giving a slightly more general formulation. Let $h:\mathscr{X}\to\mathbb{R}$ satisfy:

(i) for some $a<0$ h is continuous on $\mathscr{I}(a)$,

(ii) for some δ $h(cX)=c^{\delta}h(X)$ for all $c>0$,

(iii) for some $a<0$ and some $b>\inf\{h(I):I\in\mathscr{I}\}$ the set $\mathscr{I}(a)\cap\{X:h(X)\leqq b\}$ is compact.

Now let

$$B^{(n)}=\inf\{h(Z_r^{(n)}):r\} \quad \text{and} \quad \gamma=\inf\{h(I):I\in\mathscr{I}\}$$

then we will show that

$$\frac{B^{(n)}}{n^{\delta}}\to\gamma \quad \text{a.s. on } S. \tag{4.2}$$

Using the conditions (i) and (iii) imposed on h we see that for some $I^{+}\in\mathscr{I}$ $h(I^{+})=\gamma$. Let $\{I^{(n)}\}$ be an $\mathscr{I}^{(n)}$-section for I^{+}; then, using Condition (ii),

$$\limsup\frac{B^{(n)}}{n^{\delta}}\leqq\lim h(I^{(n)})=\gamma.$$

For $a<0$ and large n, on S,

$$\mathscr{I}^{(n)}\subset\mathscr{I}(a)=(\mathscr{I}(a)\cap\{X:h(X)\geqq b\})\cup(\mathscr{I}(a)\cap\{X:h(X)\leqq b\})$$

and then. using (i), (ii), (iii),

$$\liminf(B^{(n)}/n^{\delta})\geqq\inf\{h(X):X\in\mathscr{I}(a)\cap\{X:h(X)\leqq b\}\}\to\gamma \quad \text{as} \quad a\to 0.$$

If we take $h(X)=\|X\|^*$ then (4.2) holds with $\delta=1$, answering the question posed above. If $0\in\operatorname{int}\mathscr{T}$ so that $\mathscr{I}(a)$ is compact then the condition (iii) on h holds automatically; in this case if we take $h(X)=-\|X\|^*$ then (4.2) holds, again with $\delta=1$, and $-B^{(n)}$ is the maximum distance from the origin to an nth-generation person.

5. Continuous time

In this section we will obtain a continuous-time analogue of Theorem A. Let us consider a branching random walk on $\mathscr{X}=\mathscr{Y}\times\mathbb{R}$ with the nth-generation peoples' positions denoted by $(Y_r^{(n)}, t_r^{(n)})$ where $Y_r^{(n)}\in\mathscr{Y}$ and $t_r^{(n)}\in\mathbb{R}$. If we assume that $t_r^{(1)}\in\mathbb{R}^+$ (i.e. $t_r^{(1)}>0$) for all r, then $t_r^{(n)}$ may be considered to be the birth time of a person at $Y_r^{(n)}$; thus people are now thought of as having positions in $\mathscr{Y}$. For the moment, people remain for ever in their positions from the time of their birth. Let

$$\tilde{\mathscr{I}}^{(t)}=\{Y_r^{(n)}/t:t_r^{(n)}\leqq t\}$$

so that $\tilde{\mathscr{I}}^{(t)}$ is the set of (scaled) positions of the people born before time t. The aim of this section is to show that, subject to a mild side condition, $\tilde{\mathscr{I}}^{(t)}$ has an asymptotic shape. This problem, in a slightly less general form, was mentioned in Section 6 of Biggins (1976). The notation of the previous sections will still be used for the branching random walk on $\mathscr{X}$, and the idea is to use the known results about the shape of $\mathscr{I}^{(n)}$ to deduce asymptotic results about $\tilde{\mathscr{I}}^{(t)}$. We will assume in this section that the initial ancestor has only finitely many children, that is that

(5.1) $$\#\{(Y_r^{(1)}, t_r^{(1)}):r\}<\infty.$$

This assumption prevents the offspring in early generations from having a great effect on $\tilde{\mathscr{I}}^{(t)}$ when t is large.

Notice that if we forget the peoples' position and consider only their birth times then the process obtained is an age-dependent branching process of the kind considered by Jagers (1975), except that people do not die. This process could be said to have survived if for any t there are births after the time t. It is not hard to show that the event that this occurs is equal, up to a null set, to the event that the Galton–Watson process of generation sizes survives. Hence we may, without real ambiguity, continue describing S as the survival set.

We will define

$$\mathscr{I}^*(a)=\{Y:(Y,1)\in \mathrm{cl}\,\mathscr{K}(a)\}.$$

Then $\mathscr{I}^*(a)$ is the projection onto $\mathscr{Y}$ of the intersection of the closed convex sets $\mathrm{cl}\,\mathscr{K}(a)$ and $\{(Y,1):Y\in\mathscr{Y}\}$. It is a closed convex set. If $(\Phi_0,\phi_0)\in\mathscr{T}$ then, since $t_r^{(1)}>0$ for all r, $(\Phi_0,\phi)\in\mathscr{T}$ for all $\phi\geqq\phi_0$ and then $k(\Phi_0,\phi)\to-\infty$ as $\phi\to\infty$. Thus, for $A\in\mathscr{Y}$,

$$\xi((A,0))\leqq k(\Phi_0,\phi)+\langle(A,0),(\Phi_0,\phi)\rangle\to-\infty \quad \text{as} \quad \phi\to\infty$$

and so $\mathscr{I}(a)\subset\mathscr{Y}\times\mathbb{R}^+$ for all a. Therefore, by Lemma 5,

(5.2) $$\bigcap_{d<a}\mathscr{I}^*(d)=\mathscr{I}^*(a)$$

for $a<k(0)$. Notice that, since $\mathscr{A}\cap(\mathscr{Y}\times\mathbb{R}^+)$ is non-empty, $\mathscr{I}^*(a)$ is non-empty when $\mathscr{T}$ is empty and clearly (5.2) holds in this case also.

Now let $\tilde{\mathscr{I}}(a)$ be the closure of the convex hull of $\mathscr{I}^*(a)$ and the origin of $\mathscr{Y}$, that is

(5.3) $$\tilde{\mathscr{I}}(a)=\mathrm{cl}\,\{Y:cY\in\mathscr{I}^*(a) \quad \text{for some} \quad c\geqq 1\}\cup\{0\}=\mathrm{cl}\,\mathscr{H}(\mathscr{I}^*(a),0).$$

An argument just like that given in Lemma 5 shows that for $a<k(0)$

(5.4) $$\bigcap_{d<a}\tilde{\mathscr{I}}(d)=\tilde{\mathscr{I}}(a).$$

(The final part of the proof is completed by noting that if X is in the recession cone of $\mathscr{I}^*(a)$ then, by Theorem 8.3 of Rockafellar (1970), X is in the recession cone of $\mathrm{cl}\,\mathscr{H}(\mathscr{I}^*(a),0)$ and so $X\in\mathrm{cl}\,\mathscr{H}(\mathscr{I}^*(a),0)$.)

Let

$$\tilde{\mathscr{I}} = \tilde{\mathscr{I}}(0);$$

we will show that $\tilde{\mathscr{I}}$ is the asymptotic shape of $\tilde{\mathscr{I}}^{(t)}$.

Fix $a<0$; then we know that, on S, for some n_0, $\mathscr{I}^{(n)} \subset \mathscr{I}(a)$ when $n \geqq n_0$. Now if $(Y_r^{(n)}/n, t_r^{(n)}/n) \in \mathscr{I}(a)$ then $Y_r^{(n)}/t_r^{(n)} \in \mathscr{I}^*(a)$, thus

$$\{Y_r^{(n)}/t_r^{(n)} : r\} \subset \mathscr{I}^*(a) \quad \text{for } n \geqq n_0, \text{ on } S.$$

From (5.3), if $Y_r^{(n)}/t_r^{(n)} \in \mathscr{I}^*(a)$ then $Y_r^{(n)}/t \in \tilde{\mathscr{I}}(a)$ for all $t \geqq t_r^{(n)}$. Also, by the assumption (5.1), $\{Y_r^{(n)} : n \leqq n_0\}$ is a finite set and so

$$\{Y_r^{(n)}/t : n \leqq n_0\} \subset \{Y : \|Y\| \leqq -a\}$$

for large t. Combining these remarks we can see that, on S, there is a T such that

$$\tilde{\mathscr{I}}^{(t)} \subset \tilde{\mathscr{I}}(a) \cup \{Y : \|Y\| \leqq -a\} \quad \text{for all} \quad t \geqq T,$$

and by (5.4),

$$\bigcap_{a<0} (\tilde{\mathscr{I}}(a) \cup \{Y : \|Y\| \leqq -a\}) = \tilde{\mathscr{I}}.$$

To complete the proof of an analogue of Theorem A we will now show that, on S, there is an $\tilde{\mathscr{I}}^{(t)}$-section for every point in $\tilde{\mathscr{I}}$. If $Y \in \mathscr{H}(\mathscr{I}^*(0), 0)$ then $cY \in \mathscr{I}^*(0)$ for some $c \geqq 1$. If $\{\tilde{I}^{(t)}\}$ is an $\tilde{\mathscr{I}}^{(t)}$-section for cY, then $\{c^{-1}\tilde{I}^{(t)}\}$ is an $\tilde{\mathscr{I}}^{(t)}$-section for Y because $c^{-1}\tilde{I}^{(t)} \to Y$ and $c^{-1}\tilde{I}^{(t)} \in \tilde{\mathscr{I}}^{(ct)}$. It is therefore sufficient to show that there is an $\tilde{\mathscr{I}}^{(t)}$-section for any point in $\mathscr{I}^*(0)$.

Take $Y \in \mathscr{I}^*(0)$, so $(c'Y, c') \in \mathscr{I}$ for some $c'>0$, then there is an $\mathscr{I}^{(n)}$-section, $\{(Y^{(n)}/n, t^{(n)}/n)\}$, for $(c'Yc')$. Obviously $Y^{(n)}/t^{(n)} \to Y$. Let $\{(R^{(s)}, \tau^{(s)})\}$ be a subsequence of the sequence $\{(Y^{(n)}, t^{(n)})\}$ with the property that $\tau^{(s)} < {}^{(s+1)}$ for all s; such subsequences certainly exist because $c'>0$. We can now define an $\tilde{\mathscr{I}}^{(t)}$-section corresponding to this subsequence by

$$\tilde{I}^{(t)} = \{R^{(s)}/t\} \quad \text{for} \quad \tau^{(s)} \leqq t < \tau^{(s+1)}$$

when $t \geqq \tau^{(1)}$. Then, for $\tau^{(s)} \leqq t < \tau^{(s+1)}$

$$\|\tilde{I}^{(t)} - Y\| \leqq \left\| \frac{R^{(s)}}{t} - \frac{R^{(s)}}{\tau^{(s)}} \right\| + \left\| \frac{R^{(s)}}{\tau^{(s)}} - Y \right\|$$

$$\leqq \left\| \frac{R^{(s)}}{\tau^{(s)}} \right\| \left| \frac{\tau^{(s)}}{\tau^{(s+1)}} - 1 \right| + \left\| \frac{R^{(s)}}{\tau^{(s)}} - Y \right\|$$

therefore

$$\limsup_{t\to\infty} \|\tilde{I}^{(t)} - Y\| \leqq \|Y\| \limsup_{s} \left| \frac{\tau^{(s)}}{\tau^{(s+1)}} - 1 \right|. \tag{5.5}$$

Take $a>1$ and $\varepsilon=\frac{1}{2}((a-1)/(a+1))c'$, then for s large enough $[a^s](c'-\varepsilon)\geqq [a^{s-1}](c'+\varepsilon)$ ($[a^s]$ is the integer part of a^s) whilst for n large enough $n(c'-\varepsilon)< t^{(n)}<n(c'+\varepsilon)$. Hence, for large s_0, the subsequence $\{t^{(n)}: n=[a^s]\, s\geqq s_0\}$ is strictly monotonic. Let $\{(R^{(s)}, \tau^{(s)})\}$ be the sequence $\{(Y^{(n)}, t^{(n)}): n=[a^s]\, s\geqq s_0\}$, then, because $t^{(n)}/n\to c'$, $\tau^{(s)}/\tau^{(s+1)}\to a^{-1}$. Therefore, by (5.5), the $\tilde{\mathscr{I}}^{(t)}$-section corresponding to this subsequence ultimately lies inside a ball of radius $\|Y\|\,|a^{-1}-1|$ centred on Y. We can perform this construction for a sequence of values of a tending down to one and then use a subsequence argument on the $\tilde{\mathscr{I}}^{(t)}$-sections obtained in this way to obtain an $\tilde{\mathscr{I}}^{(t)}$-section for Y. This completes the proof of the following theorem.

Theorem B.

The following hold almost surely on S:

(i) for any $a<0$ $\tilde{\mathscr{I}}^{(t)}\subset\tilde{\mathscr{I}}(a)\cup\{Y::\|Y\|\leqq -a\}$ for all sufficiently large t,

(ii) there is an $\tilde{\mathscr{I}}^{(t)}$-section for every point in $\tilde{\mathscr{I}}(=\bigcap_{a<0}\tilde{\mathscr{I}}(a))$.

Again, in certain circumstances, a reformulation of this result in terms of the Hausdorff metric is possible; the proof of the Corollary to Theorem A will work here whenever $\tilde{\mathscr{I}}(a)$ is compact for some $a<0$. Let

$$m(\Phi)=E\Big[\sum_r \exp\langle -\Phi, Y_r^{(1)}\rangle\Big]\quad\text{for}\quad \Phi\in\mathscr{Y}$$

and let $\mathscr{T}'=\{\Phi: m(\Phi)<\infty\}$. If $0\in$ int $\mathscr{T}'$ and $E[\sum_r \exp(-\theta t_r^{(1)})]$ is finite for some θ_0 (and hence for all $\theta\geqq\theta_0$) it is not too hard to show that $\mathscr{I}^*(a)$, and so $\tilde{\mathscr{I}}(a)$, is compact for $a<0$. The first condition implies that the projection of $\mathscr{I}(a)$ onto $\mathscr{Y}$ is bounded and the second implies that the projection of $\mathscr{I}(a)$ onto $\mathbb{R}$ ($\mathscr{X}=\mathscr{Y}\times\mathbb{R}$) is contained in $[c,\infty)$ for some $c>0$, and together these imply that $\mathscr{I}^*(a)$ must be bounded and hence compact. Then

$$\Delta(\tilde{\mathscr{I}}^{(t)}, \tilde{\mathscr{I}})\to 0\quad\text{a.s. on } S$$

as $t\to\infty$.

Let us now include the additional feature of death in our model. The initial ancestor has a lifetime, $d>0$, and the process is built up, as usual, by associating an independent copy of $\{Y_r^{(1)}, t_r^{(1)}, d: r\}$ with each person to give the relative positions of his children, their birth times relative to his, and his lifetime; thus the person at $Y_r^{(n)}$ has an associated lifetime $d_r^{(n)}$ and is alive at those times t satisfying $t_r^{(n)}\leqq t<t_r^{(n)}+d_r^{(n)}$. The set of points

$$\bar{\mathscr{I}}^{(t)}=\{Y_r^{(n)}/t: t_r^{(n)}\leqq t<t_r^{(n)}+d_r^{(n)}\}$$

is the set of (scaled) positions of those people alive at the time t. We have shown that $\bar{\mathscr{I}}^{(t)}$ has an asymptotic shape when $d=\infty$ and $\#\{Y_r^{(1)}: r\}$ is finite; it is reasonable to expect that weaker conditions suffice. The general question is left open; here a simple result needed in the next section is given.

Suppose that the natural requirements

(5.6) $$d \geqq t_r^{(1)} \quad \text{for all } r \text{ and } d < \infty$$

hold almost surely; then (5.1) must hold. Suppose too that

(5.7) $$0 \in \{Y_r^{(1)} : r\} \quad \text{a.s.};$$

then $P[S] = 1$ and every person has a child at his own position before he dies. The sets $\bar{\mathscr{I}}^{(t)}$ and $\tilde{\mathscr{I}}^{(t)}$ now contain the same points of $\mathscr{Y}$, though with different multiplicities, and so we have the following result.

Corollary to Theorem B. When (5.6) and (5.7) hold Theorem B holds with $\tilde{\mathscr{I}}^{(t)}$ replaced by $\bar{\mathscr{I}}^{(t)}$.

6. Continuous time — the Markov case

A special case of the corollary to Theorem B, when the age-dependent branching process formed by those people alive at time t is a continuous-time Galton–Watson process, is of particular interest. The result is then closely related to certain results of Mollison (1978) on the 'contact birth process' and a discrete skeleton argument allows the results of the earlier sections to be used to give a more explicit formula for $\tilde{\mathscr{I}}$.

Let us suppose that d, the initial ancestor's lifetime, has an exponential distribution with the parameter α, that $d = t_r^{(1)}$ for all r and that d is independent of the point process $\{Y_r^{(1)} : r\}$. We will also assume that (5.7) holds, so that the corollary to Theorem B holds. In this situation $\#\bar{\mathscr{I}}^{(t)}$ is a continuous-time Galton–Watson process and if we sample the continuous-time process formed by $\{Y_r^{(n)} : t_r^{(n)} \leqq t < t_r^{(n)} + d_r^{(n)}\}$ only at those time points t with integer values then we obtain a discrete-time branching random walk. All quantities associated with this branching random walk will be denoted by a subscript, I. Obviously $\mathscr{I}_I^{(n)} = \bar{\mathscr{I}}^{(n)}$ and so $\tilde{\mathscr{I}}$ and $\mathscr{I}_I$ must be the same. However $\mathscr{I}_I$ is given by the formula (2.6) with $a = 0$ which will allow us to give a similar formula for $\tilde{\mathscr{I}}$ in terms of $m(\Phi)$, α and $\mathscr{A}'$, where $\mathscr{A}'$ is the smallest closed convex set containing $\{Y_r^{(1)} : r\}$ almost surely.

Let

$$\exp \bar{k}_t(\Phi) = E\left[\sum \exp \langle -\Phi, Y_r^{(n)} \rangle\right] \quad \text{for} \quad \Phi \in \mathscr{Y},$$

where the sum is taken over those $Y_r^{(n)}$ for which $t_r^{(n)} \leqq t < t_r^{(n)} + d_r^{(n)}$. By considering the possible time of death of the initial ancestor we can see that

$$\exp \bar{k}_t(\Phi) = e^{-\alpha t} + m(\Phi) \int_0^t \exp \bar{k}_{t-\tau}(\Phi) \alpha e^{-\alpha\tau} \, d\tau,$$

and hence, by standard renewal theory, that

$$\bar{k}_t(\Phi) = \alpha(m(\Phi) - 1)t.$$

Therefore

$$k_I(\Phi) = \alpha(m(\Phi) - 1)$$

and so by Lemma 2 when $\mathcal{T}'$ is non-empty

$$\tilde{\mathcal{J}} = \{A : \inf\{\alpha(m(\Phi) - 1) + \langle\Phi, A\rangle : \Phi\} \geqq 0\}.$$

To cover the remaining case, when $\mathcal{T}'$ is empty, we have only to identify $\mathcal{A}_I$; in fact

$$\mathcal{A}_I = \text{cl}\{A : cA \in \mathcal{A}' \text{ for some } c > 0\},$$

the closure of the cone generated by $\mathcal{A}'$. This formula will be established under the condition that $0 \in \text{int}\,\mathcal{T}'$, a truncation argument would then show that it always holds. If $A \in \text{rint}\,\mathcal{A}'$ then by Lemma 1 $\inf\{\log m(\Phi) + \langle\Phi, A\rangle : \Phi\} = b > -\infty$ and $wA \in \mathcal{A}_I$ if $\xi_I(wA) > -\infty$. Now

$$\begin{aligned} k_I(\Phi) + \langle\Phi, wA\rangle &= \exp\langle-\Phi, A\rangle(\alpha m(\Phi)\exp\langle\Phi, A\rangle)) + w\langle\Phi, A\rangle - \alpha \\ &\geqq \alpha e^b \exp\langle-\Phi, A\rangle + w\langle\Phi, A\rangle - \alpha \end{aligned}$$

which is bounded below whenever $w > 0$, then $\xi_I(wA) > -\infty$ and $wA \in \mathcal{A}_I$. Hence $\text{cl}\{A : cA \in \mathcal{A}' \text{ for some } c > 0\} \subset \mathcal{A}_I$. The opposite inclusion is obvious because $\text{cl}\{A : cA \in \mathcal{A}' \text{ for some } c > 0\}$ is closed under addition and so contains $Y_r^{(n)}$ for every n and r. Notice that whenever $0 \in \text{int}\,\mathcal{A}'\ \mathcal{A}_I = \mathcal{Y}$.

In Mollison's (1978) contact birth process a person at the origin in $\mathcal{Y}$ has children according to a Poisson process of rate α. Each child is given a position in $\mathcal{Y}$ using independent copies of a random vector, U. Each child, from birth, produces children according to a Poisson process of rate α and their positions, relative to their parents', are again given by independent copies of U, and so on. This process falls into the framework above when considered in a slightly different way. Suppose that d is exponentially distributed with parameter α and that

$$\{Y_r^{(1)} : r\} = \{0, U\}.$$

The resulting process is the contact birth process if a parent is identified with his offspring at his own position. Let

$$v(\Phi) = E[\exp\langle-\Phi, U\rangle];$$

then $m(\Phi) = \alpha(v(\Phi) + 1 - 1) = \alpha v(\Phi)$ and $\mathcal{A}'$ is the closure of the convex hull of possible values of U and the origin of $\mathcal{Y}$. If $\{\Phi : v(\Phi) < \infty\}$ is non-empty then the asymptotic shape of $\bar{\mathcal{J}}^{(t)}$ is

$$\tilde{\mathcal{J}} = \{A : \inf\{\alpha v(\Phi) + \langle\Phi, A\rangle : \Phi\} \geqq 0\}. \tag{6.1}$$

If we further assume that $0 \in \operatorname{int}\{\Phi : v(\Phi) < \infty\}$ then, using the remarks following Theorem B

$$\Delta(\bar{\mathscr{I}}^{(t)}, \tilde{\mathscr{I}}) \to 0 \text{ a.s.} \quad \text{as} \quad t \to \infty.$$

Mollison ((1978), 5.7) proves that for certain processes including the contact birth process, the convex hull of $\bar{\mathscr{I}}^{(t)}$ has an asymptotic shape which he calls Γ. When $\mathscr{T}'$ is non-empty Mollison ((1978), 3.10) also gives an upper bound, which he calls C, on the asymptotic shape of $\mathscr{I}^{(t)}$ which is

$$\bigcap_{W \in \mathscr{G}} \left\{ A : \frac{1}{\alpha} \langle A, W \rangle \leqq \inf \left\{ \frac{v(-\phi W)}{\phi} : \phi > 0 \right\} \right\} \tag{6.2}$$

where $\mathscr{G}$ is a countable dense subset of the unit sphere in $\mathscr{Y}$. He conjectures that $\Gamma = C$ for the contact birth process. We are now in a position to prove this conjecture since we know the asymptotic shape of $\bar{\mathscr{I}}^{(t)}$ to be $\tilde{\mathscr{I}}$ and have a formula for $\tilde{\mathscr{I}}$; thus we have only to check that (6.1) is equal to (6.2). We may assume that $-W \in \mathscr{G}$ whenever $W \in \mathscr{G}$ and rearrange (6.2) to get

$$\bigcap_{W \in \mathscr{G}} \{A : \inf\{\alpha v(\phi W) + \langle A, \phi W \rangle : \phi \in \mathbb{R}\} \geqq 0\}$$
$$= \{A : \inf\{\inf\{\alpha v(\phi W) + \langle A, \phi W \rangle : \phi \in \mathbb{R}\} W \in \mathscr{G}\}.$$

Now, because $\alpha v(\Phi) + \langle A, \Phi \rangle$ is convex in Φ this equals (6.1) as it should.

Concluding comments

One of the most unrealistic features of branching-process models for populations is that there is no interaction between neighbouring individuals. More realistic models incorporate the effects of competition and a bounded population density. The population in any branching-process model that does not die out almost surely grows geometrically with time. Typically the population will grow less quickly in more realistic models. Hence one would expect branching-process models to provide an upper bound on the rate of spread of the more realistic models.

Mollison (1977) defines a Markovian contact process to be a population process with a contact birth process underlying it. (The formal definition is given in that paper.) Usually this means that given a realization of the contact birth process the application of some set of rules for removing people results in a corresponding realization of the Markovian contact process. Quite a broad class of models, incorporating the interaction of neighbours, can be obtained in this way. For example, in a contact birth process one could delete any person, and all of his descendants, born at a distance less than s from any person already present. As the births in the contact birth process are ordered almost surely, this proscription works. The result is a Markovian contact process. A model of this kind might well be appropriate for describing the spread of some plant colonies. (It seems likely that the process just described has an asymptotic shape; however this is, as far as I know, unproven.) Obviously for models of this kind Mollison's upper bound, C, provides a computable upper bound on their rate of spread.

It is clear that this idea can be extended; we can consider processes with an underlying branching process (this includes a variety of 'non-Markovian' models when they are thought of in the way described in Section 5). Now, as in the simpler situation, the asymptotic shape of the underlying process provides an upper bound on the rate of spread of the process of interest. Notice that for any given, or postulated, $k(\Theta)$ the corresponding sets $\mathscr{I}$, $\mathscr{I}^*$ and $\tilde{\mathscr{I}}$ can be computed. Furthermore we know that this computable upper bound is actually attained by the branching process.

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