

FINITE p -NILPOTENT GROUPS WITH SOME SUBGROUPS c -SUPPLEMENTED

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(Received 10 June 2001; revised 18 February 2004)

Communicated by R. B. Howlett

Abstract

A subgroup H of a finite group G is said to be c -supplemented in G if there exists a subgroup K of G such that $G = HK$ and $H \cap K$ is contained in $\text{core}_G(H)$. In this paper some results for finite p -nilpotent groups are given based on some subgroups of P c -supplemented in G , where p is a prime factor of the order of G and P is a Sylow p -subgroup of G . We also give some applications of these results.

2000 *Mathematics subject classification*: primary 20D10, 20D20.

1. Introduction

Let G be a finite group. The relationship between the properties of the Sylow subgroups of G and the structure of G has been investigated by a number of authors (see, for example, [5, 11, 10, 13, 14]). In particular, Buckley [5] in 1970 proved that a group of odd order is supersolvable if all its minimal subgroups are normal. Srinivassan [13] proved that a finite group is supersolvable if every maximal subgroup of every Sylow subgroup is normal. These two important results on supersolvable groups have been generalized by many authors. One direction of generalization is to replace the normality condition of maximal subgroups or minimal subgroups of Sylow subgroups by a weaker condition; and the other direction of generalization is to minimize the number of maximal subgroups or minimal subgroups of Sylow subgroups.

The research of the first author was partially supported by a grant of the Shanghai Natural Science Foundation (Grant No. 03ZR) and a grant of the National Natural Science Foundation of China (Grant No. 10171086). The research of the second author is partially supported by a UGC(HK) grant #2160126 (2003/2005).

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It has been observed that the property of ‘normality’ for some maximal subgroups or some minimal subgroups of Sylow subgroups gave a lot of useful information on the structure of groups. In this paper, we shall continue to study the structure of finite groups on the assumption that some subgroups are c -supplemented and obtain some interesting results for finite p -nilpotent groups. As an application of our results, we give conditions for a finite group to be in a saturated formation containing the class of finite supersolvable groups.

Throughout this paper, all groups are finite. Our terminology and notation are standard, see, for example, Robinson [12].

2. Preliminaries

A subgroup H of a group G is said to be c -supplemented in G if there exists a subgroup K of G such that $G = HK$ and $H \cap K \leq \text{core}_G(H) = H_G$. We first cite several lemmas for later use in this paper.

LEMMA 2.1 ([4, Lemma 2.1]). *Let H be a subgroup of a group G . Then the following statements hold:*

- (1) *Let K be a subgroup of G such that H is contained in K . If H is c -supplemented in G then H is c -supplemented in K .*
- (2) *Let N be a normal subgroup of G such that N is contained in H . Then H is c -supplemented in G if and only if H/N is c -supplemented in G/N .*
- (3) *Let π be a set of primes. Let N be a normal π' -subgroup of G and H a π -subgroup of G . If H is c -supplemented in G then HN/N is c -supplemented in G/N . Furthermore, if N normalizes H , then the converse statement also holds.*
- (4) *Let L be a subgroup of G and $H \leq \Phi(L)$. If H is c -supplemented in G then H is normal in G and $H \leq \Phi(G)$.*

LEMMA 2.2 ([11, Lemma 2.6]). *Let N be a solvable normal subgroup of a group G ($N \neq 1$). If $N \cap \Phi(G) = 1$, then the Fitting subgroup $F(N)$ of N is the direct product of all minimal normal subgroups of G which are contained in N .*

Recall that a formation of groups is a class of groups \mathcal{F} which is closed under homomorphic images and is such that $G/M \cap N \in \mathcal{F}$ whenever M, N are normal subgroups of a group G with $G/M \in \mathcal{F}$ and $G/N \in \mathcal{F}$. We call a formation \mathcal{F} saturated if $G \in \mathcal{F}$ when $G/\Phi(G)$ is in \mathcal{F} .

Now we let Π be the set of all prime numbers. Then, a function f defined on Π is called a formation function if $f(p)$, possibly empty, is a formation for all $p \in \Pi$. A chief factor H/K of a group G is called f -central in G if $G/C_G(H/K) \in f(p)$ for all prime numbers p dividing $|H/K|$. A formation \mathcal{F} is then called a local formation

if there exists a formation function f such that \mathcal{F} is the class of all groups G for which every chief factor of G is f -central in G . If \mathcal{F} is a local formation defined by a formation function f , then we write $\mathcal{F} = LF(f)$ and we call f a local definition of \mathcal{F} .

Among all the possible local definitions for a local formation \mathcal{F} , it is known that there exists exactly one of them, denoted it by F , such that the formation function F is both integrated (that is, $F(p) \subseteq \mathcal{F}$ for all $p \in \Pi$) and full (that is, $\mathcal{N}_p F(p) = F(p)$ for all $p \in \Pi$), where \mathcal{N}_p is the class of p -groups.

Also it is well known that a formation \mathcal{F} is saturated if and only if \mathcal{F} is a local formation (see [6]).

LEMMA 2.3 ([6, Proposition IV. 3.11]). *Let $\mathcal{F}_1 = LF(F_1)$ and $\mathcal{F}_2 = LF(F_2)$, where each F_i is both an integrated and full formation function of \mathcal{F}_i ($i = 1, 2$). Then the following statements are equivalent:*

- (1) $\mathcal{F}_1 \subseteq \mathcal{F}_2$,
- (2) $F_1(p) \subseteq F_2(p)$ for all $p \in \Pi$.

LEMMA 2.4 ([2, Lemma 2]). *Let \mathcal{F} be a saturated formation. Assume that G is a group such that G does not belong to \mathcal{F} and there exists a maximal subgroup M of G such that $M \in \mathcal{F}$ and $G = MF(G)$, where $F(G)$ is the Fitting subgroup. Then $G^{\mathcal{F}}/(G^{\mathcal{F}})'$ is a chief factor of G , $G^{\mathcal{F}}$ is a p -group for some prime p , $G^{\mathcal{F}}$ has exponent p if $p > 2$ and exponent at most 4 if $p = 2$. Moreover, $G^{\mathcal{F}}$ is either an elementary abelian group or $(G^{\mathcal{F}})' = Z(G^{\mathcal{F}}) = \Phi(G^{\mathcal{F}})$ is an elementary abelian group.*

3. Main results

We now establish our main theorems for p -nilpotent groups.

THEOREM 3.1. *Let p be the smallest prime dividing the order of a group G and P a Sylow p -subgroup of G . If every minimal subgroup of P is c -supplemented in G and when $p = 2$, either every cyclic subgroup of order 4 of P is also c -supplemented in G or P is quaternion-free, then G is p -nilpotent.*

PROOF. Suppose that the theorem is false and let G be a counterexample of the smallest order. Then G is not p -nilpotent. Since all Sylow p -subgroups of G are conjugate in G , we see that the hypotheses of our theorem is subgroup-closure by Lemma 2.1. Therefore G is a minimal non- p -nilpotent group (that is, every proper subgroup of a group is p -nilpotent but itself is not p -nilpotent). By a result of Itô [12, Theorem 10.3.3], we know that G must be a minimal non-nilpotent group. Also by a

result of Schmidt [12, Theorem 9.1.9 and Exercises 9.1.11], we see that G is of order $p^\alpha q^\beta$, where q is a prime distinct from p , P is normal in G and any Sylow q -subgroup Q of G is cyclic. Furthermore, P is of exponent p if p is odd and of exponent at most 4 if $p = 2$. Let A be a minimal subgroup of P . Then by our hypotheses, there exists a subgroup K of G such that $G = AK$ and $A \cap K \leq \text{core}_G(A)$.

If A is not normal in G then $A \cap K = 1$ and therefore K is a maximal subgroup of G with index p . Since p is the smallest prime dividing the order of G , we see that K is normal in G . Also since K is a proper subgroup of G , K is nilpotent. It follows that the Sylow q -subgroup of K is normal in G and therefore G is nilpotent, a contradiction. Hence, we may assume that every minimal subgroup of P must be normal in G and therefore every minimal subgroup of P must be in the center of G . If p is odd, then G is p -nilpotent by Itô's lemma, a contradiction. So there remains the case when $p = 2$.

Now let $p = 2$. By the above proof, we can see that every minimal subgroup of P lies in the center of G . If P is quaternion-free, then by applying [7, Theorem 2.8], we have $\Omega_1(P) \leq P \cap G^{\mathcal{N}} \cap Z(G) = 1$, where $G^{\mathcal{N}}$ is the nilpotent residual of G , a contradiction. Now let every cyclic subgroup of order 4 of P be also c -supplemented in G and let $B = \langle b \rangle$ be a cyclic subgroup of P with order 4. Then, by our hypotheses, there exists a subgroup K of G such that $G = BK$ and $B \cap K \leq \text{core}_G(B)$. Since $\langle b^2 \rangle$ lies in the center of G , we may replace K by $K\langle b^2 \rangle$ if necessary and we may assume that $[G : K] \leq 2$. If $[G : K] = 2$, then K is normal in G and K is nilpotent. Since the normal p -complement of K is the normal p -complement of G , G is nilpotent, a contradiction. Hence, $K = G$ and B must be normal in G . If $B \neq P$, then, since G is a minimal non-nilpotent group and the exponent of P is at most 4, we have $P \leq C_G(Q)$ and therefore $G = P \times Q$, a contradiction. If $P = B$, then it is clear that G is p -nilpotent, another contradiction. Thus, by all the above contradictions, we conclude that the theorem is true. □

COROLLARY 3.2. *Let N be a normal subgroup of a group G and p the smallest prime dividing the order of G . Also let \mathcal{F} be a saturated formation containing the class \mathcal{N}_p of the all p -nilpotent groups and $G/N \in \mathcal{F}$. If every minimal subgroup of P is c -supplemented in G , and when $p = 2$, either every cyclic subgroup of order 4 of P is also c -supplemented in G or P is quaternion-free, then $G \in \mathcal{F}$, where P is a Sylow p -subgroup of N .*

PROOF. It is easy to see from Lemma 2.1 that every minimal subgroup of P is c -supplemented in N , and when $p = 2$ either every cyclic subgroup of order 4 of P is also c -supplemented in N or P is quaternion-free. By Theorem 3.1, N is p -nilpotent. Let H be the normal p -complement of N . Then it is clear that H is normal in G and $(G/H)/(N/H) \simeq G/N \in \mathcal{F}$. By Lemma 2.1 again, G/H

satisfies the hypotheses of the corollary for normal subgroup N/H . Now if $H \neq 1$, by induction, we see that $G/H \in \mathcal{F}$. Let F_i ($i = 1, 2$) be the full and integrated formation function such that $\mathcal{N}_p = LF(F_1)$ and $\mathcal{F} = LF(F_2)$, respectively. Then, it is clear that $G/C_G(K_1/K_2) \in F_1(q)$ for every chief factor K_1/K_2 of G with $K_1 \leq H$ and every prime q dividing the order of $|K_1/K_2|$. By Lemma 2.3, we see that $G/C_G(K_1/K_2) \in F_2(q)$ for every chief factor K_1/K_2 of G with $K_1 \leq H$ and every prime q dividing the order of $|K_1/K_2|$. It follows that $G \in \mathcal{F}$. Hence, we may assume that $H = 1$ and $N = P$ is a p -group. In this case, for any prime q dividing the order of G with $q \neq p$ and $Q \in \text{Syl}_q(G)$, it is clear that PQ is a subgroup of G and hence PQ is p -nilpotent by Theorem 3.1, and therefore we have $PQ = P \times Q$. It follows that $G/C_G(K_1/K_2)$ is a p -group for every chief factor K_1/K_2 of G with $K_1 \leq P$. Now by using Lemma 2.3 again, we see that $G \in \mathcal{F}$. □

REMARK 3.3. The hypotheses that p is the smallest prime dividing the order of a group G in Theorem 3.1 and Corollary 3.2 cannot be removed. For example, $G = S_3$, the symmetric group of order three, is an example for $p = 3$.

THEOREM 3.4. *Let p be the smallest prime dividing the order of a group G and P a Sylow p -subgroup of G . If every maximal subgroup of P is c -supplemented in G , then G is p -nilpotent.*

PROOF. It is easy to see that every maximal subgroup of every Sylow p -subgroup of G is c -supplemented in G . Thus, in the following proof, we may make a choice among Sylow p -subgroups of G . Now, assume that the theorem is false and let G be a counterexample of minimal order. Then we prove the theorem by making the following claims:

(1) $O_{p'}(G) = 1$.

If $O_{p'}(G) \neq 1$, then we may choose a minimal normal subgroup N of G such that $N \leq O_{p'}(G)$. It is clear that PN/N is a Sylow p -subgroup of G/N . For every maximal subgroup P_1N/N of PN/N , we may assume that P_1 is a maximal subgroup of P . Thus, by Lemma 2.1 (3), every maximal subgroup of PN/N is c -supplemented in G/N . Hence, by the minimality of G , we know that G/N is p -nilpotent and so G is p -nilpotent, a contradiction.

(2) $O_p(G) \neq 1$.

If G is odd, then G is solvable by the well-known odd order theorem of Feit and Thompson [8] and therefore $O_p(G) \neq 1$. Now let G be a group of even order and $O_p(G) = O_2(G) = 1$. Let P_1 be a maximal subgroup of P . By hypotheses there exists a subgroup K of G such that $G = P_1K$ and $P_1 \cap K = 1$. Since $[P : P_1] = 2$, it follows that the Sylow 2-subgroups of K are cyclic of order 2 and therefore K is 2-nilpotent. Let K_2 be the Hall 2'-subgroup of K . Then $G = PK_2$ and K_2 is a

Hall 2'-subgroup of G . Assume that G is a non-abelian simple group. Then, by [1, Corollary 5.6], G is isomorphic to $PSL(2, r)$ with r a Mersenne prime. In this case, by [1, Corollary 5.8], every subgroup of G of 2-power index is the normalizer of a Sylow r -subgroup of G . In particular, K and K_2 have the same order, a contradiction. Hence G is not a non-abelian simple group.

Let N be a minimal normal subgroup of G with $N \neq G$. Then N is neither a 2-group nor a 2'-group. Since G satisfies E_2 (existence of Hall 2'-subgroups), we assume that N_2 is a Hall 2'-subgroup of N and N_2 a Sylow 2-subgroup of N . If $P = N_2$, then N clearly satisfies the hypotheses of our theorem by Lemma 2.1 (1). Thus, by the minimality of G , we know that N is 2-nilpotent and hence $O_2(G) \neq 1$, which contradicts to (1). On the other hand, if N_2 is not a Hall 2'-subgroup of G , then PN is a proper subgroup of G and PN also satisfies the hypotheses of our theorem. Now, by the minimality of G again, PN is 2-nilpotent and therefore N itself is 2-nilpotent. It follows that $O_2(G) \neq 1$, a contradiction again. Hence we conclude that $N_2 < P$ and N_2 is a Hall 2'-subgroup of G . Since G satisfies E_2 , we can see that both G and N satisfy C_2 (all Hall 2'-subgroups are conjugate) by Gross' theorem [9, Main Theorem]. Now by using the Frattini argument, we have

$$G = N_2 N_G(N_2).$$

Now let $P^* \in \text{Syl}_2(N_G(N_2))$ with $P^* \leq P$. Then, by our choice of G , we know that $N_G(N_2) < G$. Thus $P^* < P$ and therefore there exists a maximal subgroup P_1 of P such that $P^* \leq P_1$. By our hypotheses again, there exists a subgroup K of G such that $G = P_1 K$ and $P_1 \cap K = 1$. It is now clear that the order of Sylow 2-subgroups of K is 2 and therefore K is 2-nilpotent. Let H be a normal 2-complement of K . Then, H is a Hall 2'-subgroup of G . Thus there exists an element g of G such that $H^g = N_2$. Since $G = P_1 K$ and H is a normal subgroup of K , we may choose $g \in P_1$. We also see that K^g normalizes $H^g = N_2$ and therefore $K^g \leq N_G(N_2)$. Thus, it follows that $G = G^g = (P_1 K)^g = P_1 N_G(N_2)$. This leads to $P = P \cap G = P_1(P \cap N_G(N_2)) = P_1 P^* = P_1 < P$, a contradiction. Thus, our claim (2) is established.

$$(3) \Phi(O_p(G)) = 1$$

If $\Phi(O_p(G)) \neq 1$, then we may consider the quotient group $G/\Phi(O_p(G))$. Obviously, by Lemma 2.1 (2), every maximal subgroup of $P/\Phi(O_p(G))$ is c -supplemented in $G/\Phi(O_p(G))$. Thus, by the minimality of G , we see that $G/\Phi(O_p(G))$ has a normal p -complement $T/\Phi(O_p(G))$. By the Schur-Zassenhaus Theorem, there exists a Hall p' -subgroup H of T such that $T = H\Phi(O_p(G))$. By using the Frattini argument again, we see that $G = \Phi(O_p(G))N_G(H) = N_G(H)$ since $\Phi(O_p(G)) \leq \Phi(G)$, a contradiction. Thus $\Phi(O_p(G)) = 1$ and $O_p(G)$ is an elementary abelian group.

$$(4) O_p(G) \text{ is a minimal normal subgroup of } G.$$

Let N be a minimal normal subgroup of G such that $N \leq O_p(G)$. Then it

is easy to see that G/N satisfies the hypotheses of our theorem. The minimality of G implies that G/N is p -nilpotent. Similarly, if N_1 is another minimal normal subgroup of G with $N_1 \leq O_p(G)$, then we see that G/N_1 is also p -nilpotent. Now it follows that $G \simeq G/N \cap N_1$ is p -nilpotent, a contradiction. Hence N must be the unique minimal normal subgroup of G which is contained in $O_p(G)$. By using the arguments similar to the proof in (3), we have $G = NN_G(H)$, where H is a Hall p' -subgroup of G . Since $N_G(H) < G$, it follows that $N \not\leq N_G(H)$ and then, since $O_p(G) \cap N_G(H)$ is normal in G , $O_p(G) \cap N_G(H) = 1$. Finally, by Dedekind's law, we have $O_p(G) = N(O_p(G) \cap N_G(H)) = N$. This proves (4).

(5) The final contradiction.

From the above proof, we see that $G/O_p(G)$ is p -nilpotent. By using the arguments similar to the proof in (3), we have $G = NN_G(H)$, where H is a Hall p' -subgroup of G . Let P^* be a Sylow p -subgroup of $N_G(H)$. Then by our choice of G , we have $N_G(H) < G$ and therefore $P^* < P$. Let P_1 be a maximal subgroup of P with $P^* \leq P_1$. Since $O_p(G)$ is a minimal normal subgroup of G and $O_p(G) \not\leq P_1$, we have $\text{core}_G(P_1) = 1$. By our hypotheses, there exists a subgroup K_1 of G such that $G = P_1K_1$ and $P_1 \cap K_1 = 1$. It is clear that the order of Sylow p -subgroups of K_1 is p and therefore K_1 has a normal p -complement H_1 . Then there exists an element $g \in O_p(G)P^*$ such that $(H_1)^g = H$. Noticing that P_1 is normal in $O_p(G)P^*$, we have that $G = P_1K_1 = (P_1K_1)^g = P_1(K_1)^g$ and $P_1 \cap (K_1)^g = 1$. Since $(K_1)^g \simeq K_1$ has normal p -complement and $H = (H_1)^g \leq (K_1)^g$, it follows that $(K_1)^g \leq N_G(H)$ and consequently $G = P_1(K_1)^g = P_1N_G(H)$. Hence $O_p(G)P^* = (O_p(G)P^*) \cap G = (O_p(G)P^*) \cap (P_1N_G(H)) = P_1((O_p(G)P^*) \cap N_G(H)) = P_1P^* = P_1$, which contradicts to the fact that $O_p(G)P^*$ is a Sylow p -subgroup of G . Thus our theorem is proved. □

COROLLARY 3.5. *If every maximal subgroup of every Sylow subgroup of a group G is c -supplemented in G , then G is a Sylow tower group of supersolvable type.*

PROOF. Let p be the smallest prime dividing the order of G and P a Sylow p -subgroup of G . By Corollary 3.2 G is p -nilpotent. Let N be a normal p -complement of G . Clearly N satisfies the hypotheses of G and therefore by induction N is a Sylow tower group of supersolvable type. This proves that G is a Sylow tower group of supersolvable type. □

By using the arguments similar to the proof in Corollary 3.2, we can prove the following corollary.

COROLLARY 3.6. *Let N be a normal subgroup of a group G and p the smallest prime dividing the order of G . Also let \mathcal{F} be a saturated formation containing the*

class \mathcal{N}_p of the all p -nilpotent groups and $G/N \in \mathcal{F}$. If every maximal subgroup of P is c -supplemented in G , then $G \in \mathcal{F}$, where P is a Sylow p -subgroup of N .

4. Applications

As an application of Theorem 3.1 and Theorem 3.4, we establish the following theorems for a group to be in the saturated formation containing the class of supersolvable groups.

THEOREM 4.1. *Let \mathcal{F} be a saturated formation containing the class of supersolvable groups \mathcal{U} . Let N be a normal subgroup of a group G such that G/N is in \mathcal{F} . If for every prime p dividing the order of N and for every Sylow p -subgroup P of N , every minimal subgroup of P is c -supplemented in G and when $p = 2$, either every cyclic subgroup of order 4 of P is also c -supplemented in G or P is quaternion-free, then G is in \mathcal{F} .*

PROOF. Assume that the theorem is false and let G be a counterexample of minimal order. By Lemma 2.1 and Theorem 3.1, we know that N is a Sylow tower group of supersolvable type. Let q be the largest prime dividing the order of N and Q a Sylow q -subgroup of N . Then Q is normal in G and every minimal subgroup of Q is c -supplemented in G . It is clear that $(G/Q)/(N/Q) \simeq G/N \in \mathcal{F}$ and that G/Q satisfies the hypotheses of our theorem by Lemma 2.1. The minimality of G implies that G/Q is in \mathcal{F} . It follows that $G^{\mathcal{F}} \leq Q$ and $G^{\mathcal{F}}$ is a q -group, where $G^{\mathcal{F}}$ is the \mathcal{F} -residual of G . By [3, Theorem 3.5], there exists a maximal subgroup M of G such that $G = MF'(G)$, where $F'(G) = \text{Soc}(G \text{ mod } \Phi(G))$ and $G/\text{core}_G(M)$ is not in \mathcal{F} . Then $G = MG^{\mathcal{F}}$ and therefore $G = MF(G)$ since $G^{\mathcal{F}}$ is a q -group, where $F(G)$ is the Fitting subgroup of G . It is now clear that M satisfies the hypotheses of our theorem for its normal subgroup $M \cap Q$. Hence, by the minimality of G , it leads to M must be in \mathcal{F} .

Now, by Lemma 2.4, $G^{\mathcal{F}}$ has exponent q when $q \neq 2$ and exponent at most 4 when $q = 2$. If $G^{\mathcal{F}}$ is an elementary abelian group, then $G^{\mathcal{F}}$ is a minimal normal subgroup of G . For any minimal subgroup A of $G^{\mathcal{F}}$, we know that A is c -supplemented in G by our hypotheses. Hence there exists a subgroup K of G such that $G = AK$ and $A \cap K \leq \text{core}_G(A)$. If A is not normal in G , then $A \cap K = 1$. It is clear that $K \cap G^{\mathcal{F}}$ is normal in G . The minimality of $G^{\mathcal{F}}$ implies that $K \cap G^{\mathcal{F}} = 1$ and A is normal in G , a contradiction. Hence A is normal in G and $G^{\mathcal{F}} = A$ is cyclic of order q . If $G^{\mathcal{F}}$ is not an elementary abelian group, then $(G^{\mathcal{F}})' = Z(G^{\mathcal{F}}) = \Phi(G^{\mathcal{F}})$ is an elementary abelian group by Lemma 2.4. Noticing that $\Phi(G^{\mathcal{F}}) \leq \Phi(G)$, we know that every minimal subgroup of $(G^{\mathcal{F}})'$ is not complemented in G . It now follows from our hypotheses that every minimal subgroup of $(G^{\mathcal{F}})'$ must be normal in G .

For any minimal subgroup \bar{A} of $G^{\mathcal{F}}/(G^{\mathcal{F}})'$, there exists a subgroup A of $G^{\mathcal{F}}$ such that $\bar{A} = A(G^{\mathcal{F}})'/(G^{\mathcal{F}})'$. Assume that A is of order q . If A is not normal in G , then, by our hypotheses, there exists a subgroup K of G such that $G = AK$ and $A \cap K = 1$. Noticing that $(G^{\mathcal{F}})' = \Phi(G^{\mathcal{F}}) \leq \Phi(G)$, we see that $K/(G^{\mathcal{F}})'$ is a complement of \bar{A} . The minimality of $G^{\mathcal{F}}/(G^{\mathcal{F}})'$ implies that $\bar{A} = G^{\mathcal{F}}/(G^{\mathcal{F}})'$ is normal in $G/(G^{\mathcal{F}})'$, and therefore $G^{\mathcal{F}}/(G^{\mathcal{F}})'$ is a cyclic group of order q . Hence we may assume that $q = 2$ and every generated element of $G^{\mathcal{F}}$ is of order 4. It follows immediately that $\Omega_1(G^{\mathcal{F}}) = (G^{\mathcal{F}})' = \Phi(G^{\mathcal{F}})$ and therefore every minimal subgroup of $\Omega_1(G^{\mathcal{F}})$ is normal in G . Hence $\Omega_1(G^{\mathcal{F}}) \leq Z(G)$. If Q is quaternion-free, then, by [7, Lemma 2.15], every 2'-element of G acts trivially on $G^{\mathcal{F}}$. Since $G^{\mathcal{F}}/(G^{\mathcal{F}})'$ is a chief factor of G , we see that $G^{\mathcal{F}}/(G^{\mathcal{F}})'$ is a cyclic group of order 2. Assume that every cyclic group of order 4 of Q is c -supplemented in G . Let $B = \langle b \rangle$ be a cyclic group of order 4 of $G^{\mathcal{F}}$. Then $\langle b^2 \rangle$ is normal in G . If B is not normal in G , then there exists a subgroup K of G such that $G = BK$ and $B \cap K = \langle b^2 \rangle$. It is clear that $(G^{\mathcal{F}})' = \Phi(G^{\mathcal{F}}) \leq K$ and $G^{\mathcal{F}}/(G^{\mathcal{F}})' \cap K/(G^{\mathcal{F}})'$ is normal in $G/(G^{\mathcal{F}})'$. The minimality of $G^{\mathcal{F}}/(G^{\mathcal{F}})'$ implies that $G^{\mathcal{F}}/(G^{\mathcal{F}})' \cap K/(G^{\mathcal{F}})'' = 1$ and therefore $G^{\mathcal{F}}/(G^{\mathcal{F}})'$ is a cyclic group of order 2. We have now shown that for all cases, $G^{\mathcal{F}}/(G^{\mathcal{F}})'$ is always a cyclic group of prime order. Noticing that $G^{\mathcal{F}}/(G^{\mathcal{F}})'$ is G -isomorphic to $\text{Soc}(G/\text{core}_G(M))$, it follows that $G/\text{core}_G(M)$ is supersolvable, a contradiction. Thus, our proof is completed. \square

THEOREM 4.2. *Let \mathcal{F} be a saturated formation containing the class of supersolvable groups \mathcal{U} . Let N be a normal subgroup of a group G such that G/N is in \mathcal{F} . If for every prime p dividing the order of N and for every Sylow p -subgroup P of N , every maximal subgroup of P is c -supplemented in G , then G is in \mathcal{F} .*

PROOF. Let $F_i (i = 1, 2)$ be the full and integrated formation functions such that $\mathcal{U} = LF(F_1)$ and $\mathcal{F} = LF(F_2)$. Assume that the theorem is false and we may let G be a minimal counterexample. Then, by applying Lemma 2.1 and Corollary 3.5, we know that N has a Sylow tower of supersolvable type. Let p be the largest prime dividing the order of N and $P \in \text{Syl}_p(N)$. Then P must be a normal subgroup of G . Clearly, $(G/P)/(N/P) \simeq G/N \in \mathcal{F}$. It is easy to see that G/P satisfies our hypotheses of the theorem for the normal subgroup N/P . By the minimality of G , we see that $G/P \in \mathcal{F}$, and of course, every maximal subgroup of P is c -supplemented in G .

Let L be a minimal normal subgroup of G with $L \leq P$. Then, it is easy to see that the quotient group G/L satisfies the hypotheses of our theorem for the normal subgroup of P/L . By our choice of G , we have $G/L \in \mathcal{F}$. Since \mathcal{F} is a saturated formation, L is the unique minimal normal subgroup of G which is contained in P and also L is complemented in G . In particular, we have $P \cap \Phi(G) = 1$ and therefore

$L = F(P) = P$ is an abelian minimal normal subgroup of G by Lemma 2.2.

Let P_1 be a maximal subgroup of P . By our hypotheses, there exists a subgroup K of G such that $G = P_1K$ and $P_1 \cap K = 1$ since L is the unique minimal normal subgroup of G contained in P with $L \not\leq P_1$. Thus $P = P_1(P \cap K)$. It is clear that $P \cap K$ is normal in K and is normalized by P_1 since P is abelian. Therefore $P \cap K$ is a normal subgroup of G . Since $P \cap K \neq 1$ and P is a minimal normal subgroup of G , it follows that $P \cap K = P$ and P is a cyclic group of order p . Since $\text{Aut}(P)$ is a cyclic group of order $p - 1$ and $G/C_G(P) \leq \text{Aut}(P)$, we have $G/C_G(P) \in F_1(p) \subseteq F_2(p)$, by Lemma 2.3. Therefore, $G \in \mathcal{F}$, a contradiction. Thus, our proof is completed. \square

REMARK 4.3. Let \mathcal{F} be the class of groups G whose derived group G' is nilpotent. Then it is easy to see that \mathcal{F} is a saturated formation containing the class \mathcal{U} . Now, by applying our Theorems 4.1 and 4.2, we also obtain some sufficient conditions for a group to be a \mathcal{F} -group.

Acknowledgement

The authors would like to thank the referee for his comments contributed to the final draft of this paper.

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