A HALL-TYPE CLOSURE PROPERTY FOR CERTAIN FITTING CLASSES

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Abstract

A closure operation connected with Hall subgroups is introduced for classes of finite soluble groups, and it is shown that this operation can be used to give a criterion for membership of certain special Fitting classes, including the so-called ‘central-socle’ classes.


In this note a closure operation connected with Hall subgroups is introduced for classes of finite soluble groups. It is shown that this operation can be used to give a criterion for membership of certain special Fitting classes, namely the so-called ‘central socle’ classes $\mathcal{Z}_\pi$, and the classes $e_\pi(\mathcal{N}^\pi)$: see Section 1 for definitions. Thus, for example, let $G$ be a finite soluble group and let $\sigma$ denote the set of primes which divide $|\text{soc}(G)|$; we show (Theorem 2.6) that $G \in \mathcal{Z}_\pi$ if and only if the Hall $\tau$-subgroups of $G$ belong to $\mathcal{Z}_\pi$ for all sets $\tau$ of the form $\tau = \sigma \cup \{t\}$ where $t$ is a prime.

The paper has three sections. The first consists of preliminaries. In the second, the classes $\mathcal{Z}_\pi$ are investigated, while the classes $e_\pi(\mathcal{N}^\pi)$ form the subject of the third.

1. Preliminaries

All groups considered here will belong to the class $\mathcal{F}$ of finite soluble groups: our classes of groups are isomorphism-closed and contain all groups of order 1. A Fitting class is a class of groups closed under the taking of subnormal subgroups and normal products; a background to Fitting class theory can be found in [6, 10].

If $G$ is a group and $\mathcal{F}$ is a Fitting class then $G_{\mathcal{F}}$ denotes the $\mathcal{F}$-radical of $G$. 

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while $Z(G)$ denotes the centre of $G$. The set of all primes is denoted by $\mathbb{P}$, $p$ will always denote a prime and $\pi$ will always denote some subset of $\mathbb{P}$. Then $\pi$-$\text{soc}(G)$ denotes the product of the minimal normal $\pi$-subgroups of $G$, while $\text{soc}(G)$ denotes $\mathbb{P}$-$\text{soc}(G)$. Let $\mathcal{F}$ be a Fitting class, and define classes of groups as follows:

$$\mathcal{L}_\pi = \{G \in \mathcal{F} : \pi$-$\text{soc}(G) \leq Z(G)\},$$

$$e_\pi(\mathcal{F}) = \{G \in \mathcal{F} : \text{the } G\text{-chief } \pi\text{-factors below } G_{\mathcal{F}} \text{ are central in } G\},$$

$$\mathcal{N} = \{G \in \mathcal{F} : G \text{ is nilpotent}\},$$

$$\mathcal{S}_\pi = \{G \in \mathcal{F} : G \text{ is a } \pi\text{-group}\}.$$  

In addition, we write $\mathcal{L} = \mathcal{L}_p$, while (1) denotes the class of groups of order 1.

It is well-known that both $\mathcal{L}_\pi$ and $e_\pi(\mathcal{F})$ are Fitting classes, and that $\mathcal{L}_\pi$ is subdirect-product-closed while $e_\pi(\mathcal{F})$ is a Fischer class: see [6] for definitions, and [12] for details. Both these families of classes, especially the former, have been extensively studied and have often been used to furnish examples or counterexamples: see, for example, [2, 4, 5, 7, 12].

Write $\text{Hall}_n(G)$ for the set of Hall $\pi$-subgroups of $G$, $\text{Hall}(G)$ for the set of all Hall subgroups of $G$, and $\text{Syl}_p(G)$ for the set of Sylow $p$-subgroups of $G$. Write $C_m$ for the cyclic group of order $m$.

Let $\mathcal{K} \subseteq \mathcal{F}$ be a class of groups and $\mathcal{F}$ be a Fitting class. Define $H_\mathcal{F}\mathcal{K} = \{G \in \mathcal{F} : \exists X \in \mathcal{K} \text{ and } H \in \text{Hall}(X) \text{ with } H \leq X_{\mathcal{F}} \text{ such that } G \simeq H\}$, and write $H\mathcal{K}$ for $H_{\{1\}}\mathcal{K}$. It is not hard to check that $H_\mathcal{F}$ is a closure operation on classes of groups in the sense that (i) $\mathcal{K} \subseteq H_\mathcal{F}\mathcal{K}$, (ii) $H_\mathcal{F}\mathcal{K} \subseteq H_\mathcal{F}\mathcal{V}$ if $\mathcal{K} \subseteq \mathcal{V}$, and (iii) $H_\mathcal{F}\mathcal{K} = H_\mathcal{F}H_\mathcal{F}\mathcal{K}$. If $\mathcal{K} = H_\mathcal{F}\mathcal{K}$, we say that $\mathcal{K}$ is $H_\mathcal{F}$-$\text{closed}$, while an $H$-closed class is called $\text{Hall-closed}$: see [1, 2, 3, 8], and the references contained therein, for results related to Hall-closure.

### 2. The central-socle classes

The section begins with Proposition 2.1, to the effect that $\mathcal{L}_\pi$ is $H_{\mathcal{N}}$-$\text{closed}$, and this is followed by Examples 2.2 to show that $\mathcal{L}_\pi$ is not Hall-closed for $\pi \neq \emptyset$. A converse to Proposition 2.1 is proved as Proposition 2.5, and together these results yield a criterion, Theorem 2.6, for membership of $\mathcal{L}_\pi$. The section ends with a result, not strictly connected with the $H_{\mathcal{F}}$ operation, in a similar spirit to 2.5.

#### 2.1 Proposition. Let $\pi \subseteq \mathbb{P}$ and let $G \in \mathcal{L}_\pi$. Suppose that $H$ is a Hall subgroup of $G$ with $H \geq \text{soc}(G)$. Then $H \in \mathcal{L}_\pi$. Thus $\mathcal{L}_\pi$ is $H_{\mathcal{N}}$-$\text{closed}$. 


PROOF. It is easy to check that \( \mathcal{X}_\pi = \bigcap_{\pi \in \mathcal{P}} \mathcal{X}_\pi \), and so we may without loss of
generality assume that \( \pi = \{p\} \) for some \( p \in \mathcal{P} \).

Suppose for a contradiction that \( G \) is a group of minimal order subject to

(i) \( G \in \mathcal{X}_p \); and
(ii) there exists a Hall subgroup of \( G \) which contains \( \text{soc}(G) \) but does not belong to \( \mathcal{X}_p \).

Let \( H \) be a Hall subgroup of \( G \) with \( H \supseteq \text{soc}(G) \) but \( H \notin \mathcal{X}_p \). Write \( \tau = \{t \in \mathcal{P} : t \mid |H|\} \); then \( H \in \text{Hall}_\tau(G) \), \( F(G) \in \mathcal{S}_\tau \), \( F(G) \leq H \), and \( O_\tau(G) = 1 \). Since \( \mathcal{S}_\tau \subseteq \mathcal{X}_p \), then \( p \in \tau \). Let \( M < G \) with \( F(G) \leq M \) : this is possible because

\( H < G \). Then \( F(M) = F(G) \leq M \cap H \in \text{Hall}_\tau(M) \), and so \( M \cap H \in \mathcal{X}_p \) by minimality. In particular, \( H \not\leq M \). Thus \( G = MH \) and \( |G : M| = q \in \tau \). Because \( H \notin \mathcal{X}_p \), there exists \( L \leq H \) with \( L \in \mathcal{S}_p \) and \( L \nleq Z(H) \). Because \( F(G) \leq H \), then \( [F(G), L] \leq F(G) \cap L \leq H \). Now \( C_G(F(G)) \leq F(G) \), because \( G \) is soluble, and so \( F(G) \cap L > 1 \) because \( L > 1 \). Since \( L \leq H \), it follows that \( L \leq F(G) \). In particular, \( L \leq M \cap H \). Now \( L \) is an irreducible \( H \)-module. Since \( (M \cap H) \leq H \), then by Clifford's Theorem, [9, 3.4.1] or [11, V.17.3], we have

\[
L|_{(M \cap H)} = U_1 \oplus \cdots \oplus U_n,
\]

for some \( n \in \mathbb{N} \), where each \( U_i \) is an irreducible \((M \cap H)\)-module. But this means that, as a normal subgroup of \( M \cap H \), \( L \) is a direct product of minimal normal subgroups. Thus \( L \leq p\text{-soc}(M \cap H) \). But \( M \cap H \in \mathcal{X}_p \) and so

\[
L \leq Z(M \cap H).
\]

But \( L \cdot \subseteq H \) and \( L \nleq Z(H) \); thus

\[
(1) \quad H/(M \cap H) \cong C_q \text{ acts faithfully and irreducibly on } L \in \mathcal{S}_p.
\]

In particular, \( p \neq q \).

Let \( J = \langle L^g : g \in G \rangle \), the normal closure of \( L \) in \( G \). We have \( J \leq F(G) \leq M \cap H \) because \( L \leq F(G) \). Then (1) implies that \( L \leq Z(J) \). But \( Z(J) \leq G \) and so \( J = Z(J) \) is abelian and must now be a \( p \)-group, as it is generated by commuting conjugates of \( L \).

Let \( S_1 \in \text{Hall}_\tau(G) \). By orders we have \( G = HS_1 \) and \( M \geq S_1 \), whence, remembering that \( L \subseteq H \), we have

\[
J = \langle L^h : h \in H, s \in S_1 \rangle = \langle L^s : s \in S_1 \rangle.
\]

By the Frattini argument, using the conjugacy of Hall subgroups, we have \( G = MN_G(S_1) \). But \( |G : M| = q \), and so there exists a \( q \)-element \( n_1 \in N_G(S_1) \) such that
A Hall-type closure property

$G = M \langle n_1 \rangle$. Again by Hall's Theorem, there exists $a \in G$ with $n_1^a \in H$. Write $n = n_1^a \in H \setminus M$ and $S = S_a^g$. Then $n \in N_H(S)$, $G = HS$ and $J = \langle L^s : s \in S \rangle$. It follows that

(3) $L$ is contained in no proper $S$-invariant subgroup of $J$.

We have $S \langle n \rangle \leq G$ because $n \in N_H(S)$; also, $S \langle n \rangle \in \mathcal{I}_p$ because $p \in \tau$, $S \in \mathcal{I}_p$, and $|n| = q^n$ with $q \neq p$. Now $J$ is a normal, abelian $p$-subgroup of $G$ and so by [9,5.2.3] we have

$$J = [J, S \langle n \rangle] \times C_J(S \langle n \rangle).$$

Since $J \leq G$, there exists $J^0 \leq G$ with $J^0 \leq J$. Then $J^0 \leq p \cdot \text{soc}(G) \leq Z(G)$ and so $C_J(S \langle n \rangle) \geq J^0 \geq 1$. Thus $[J, S \langle n \rangle] < J$ by (4). But $[J, S \langle n \rangle]$ is $S \langle n \rangle$-invariant and so $S \langle n \rangle$ centralises the non-trivial group $J/[J, S \langle n \rangle]$. But then any subgroup lying between $[J, S \langle n \rangle]$ and $J$ must be $S$-invariant. By statement (3) above, it follows that $[J, S \langle n \rangle]L = J$. But then

$$1 \neq J/[J, S \langle n \rangle] = [J, S \langle n \rangle]L/[J, S \langle n \rangle] \simeq L/(L \cap [J, S \langle n \rangle]),$$

and since all relevant subgroups here are $\langle n \rangle$-invariant then the isomorphism is an $\langle n \rangle$-isomorphism. But $\langle n \rangle$ centralises $J/[J, S \langle n \rangle]$, and so $\langle n \rangle$ must centralise a non-trivial factor group of $L$. However, $n \in H \setminus M$ whence $H = (M \cap H) \langle n \rangle$ and so by statement (2), $L$ must be a faithful, irreducible module for $\langle n \rangle/\langle n^g \rangle \simeq C_q$, contrary to what we have just seen. This completes the proof.

2.2 EXAMPLES. The main aim of these examples is to show that $\mathcal{L}_\pi$ is not Hall-closed, so that some such condition as `$H \geq \text{soc}(G)$' is needed in 2.1. Examples of classes (i) not $H_{\mathcal{P}}$-closed, and (ii) not $H_{\mathcal{P}}$-closed, will be given in 3.2.

(i) Suppose that $p$, $q$ and $r$ are distinct primes. It is well-known that there exists a group $G$ with a unique chief series whose factors have orders (reading 'from the top') of the form $p, q^n$ and $r^\beta$, respectively. Then $|\text{soc}(G)| = r^\beta$.

(a) Now suppose that $\pi$ with $\emptyset \subset \pi \subset \mathcal{P}$ (proper inclusions) is given. We show that $\mathcal{L}_\pi$ is not Hall-closed. Choose $q \in \pi$ and $r \in \mathcal{P} \setminus \pi$. Then $G \in \mathcal{L}_\pi$. Let $H \in \text{Hall}_{\{p,q\}}(G)$; then $H$ has a non-central $\pi$-socle of order $q^n$, so $H \notin \mathcal{L}_\pi$.

(b) In Proposition 2.1, it is natural to ask whether the conclusion still holds if the condition `$H \geq \text{soc}(G)$' is replaced by `$H \geq \pi$-socle of order $q^n$' so $H \notin \mathcal{L}_\pi$.

(ii) We now show that $\mathcal{L}_\pi = \mathcal{L}_\pi$ is not Hall-closed: the above example is of no avail for this purpose.
Let $S$ denote the group $SL(2, 3)$ and let $Z$ denote $Z(S)$, the centre of $S$. Then $Z = \text{soc}(S)$ has order 2. Let $T$ denote a cyclic group of order 5, and form the regular wreath product $W = S \wr T$ (see [11, §1.15]). We may write $W$ as a semidirect product $W = [S^*]T$, where $S^*$, the ‘base group’, is a direct product of 5 copies of $S$. Then $Z^* = Z(S^*)$ is the corresponding direct product of the respective centres of the 5 copies of $S$, and has order $2^5$. Now $[Z^*, T]$ has order $2^4$ and is normal in $W$. Write $W = W/\langle Z^*, T \rangle$. Then $W$ has a unique minimal normal subgroup, namely $Z^* = Z^*/[Z^*, T]$, and $Z^* = Z(W)$. In particular, $W \in \mathcal{F} = \mathcal{F}_5$. But $W$ has a Hall $\{3, 5\}$-subgroup $H$ of order $3^55$ and $H \simeq C_3 \wr C_5$. Now, $C_3 \wr C_5$ has two minimal normal subgroups: a central subgroup of order 3 and a non-central subgroup of order $3^4$. Thus $H \notin \mathcal{F}$ and so $\mathcal{F}$ is not Hall-closed.

We next prove some results converse in sense to 2.1.

2.3 LEMMA. Suppose that $G \in \mathcal{I}$ and that $M < G$ with $|G : M| = q$ and $M \in \mathcal{L}_p$ where $p, q \in \mathcal{P}$. Suppose that $p \text{-soc}(G) \leq M$. Let $H \in \text{Hall}_r(G)$ where \{p, q\} \subseteq \tau \subseteq \mathcal{P}$. Then if $H \in \mathcal{L}_p$ it follows that $G \in \mathcal{L}_p$.

PROOF. We may suppose that $p \text{-soc}(G) \neq 1$. Let $N \trianglelefteq G$ with $N \in \mathcal{L}_p$. Then $N$ is an irreducible $G$-module; thus by Clifford’s Theorem, $N$ is a completely reducible $M$-module. But then $N \leq p \text{-soc}(M) \leq Z(M)$. Thus $M \leq C_G(N)$ and so $N$ is an irreducible $G/M$-module. Since $|G : M| = q \in \tau$, then $G = MH$, and so $N$ is an irreducible $H/(M \cap H)$-module. But then $N \leq p \text{-soc}(H) \leq Z(H)$. Thus $C_G(N) \geq MH = G$ and the assertion follows.

2.4 NOTATION. If $G \in \mathcal{I}$, write $\sigma_G = \{s \in \mathcal{P} : s \mid |\text{soc}(G)|\}$.

2.5 PROPOSITION. Let $G \in \mathcal{I}$ and $\pi \in \mathcal{P}$. Suppose that $\text{Hall}_r(G) \subseteq \mathcal{L}_\pi$ for all $\tau \subseteq \mathcal{P}$ of the form $\tau = \sigma_G \cup \{t\}$ where $t \in \mathcal{P}$. Then $G \in \mathcal{L}_\pi$.

PROOF. It will suffice to prove that $G \in \mathcal{L}_p$ for all $p \in \pi \cap \sigma_G$. If $\text{soc}(G) = G$ there is nothing to prove and so we assume that $\text{soc}(G) < G$. Let $M < G$ with $M \geq \text{soc}(G)$ and write $|G : M| = q \in \mathcal{P}$.

We claim that $\sigma_G = \sigma_M$. For suppose that $s \in \sigma_M$; then there exists $K \trianglelefteq M$ with $K \in \mathcal{I}$. The normal closure $K^G$ satisfies $\mathcal{I}_s \ni K^G \leq M$, and so there exists $L \trianglelefteq G$ with $L \leq K^G$. Thus $s \in \sigma_G$. Next suppose that $s \in \sigma_G$. Then there exists $K \trianglelefteq G$ with $K \in \mathcal{I}$, and $K \leq M$ because $M \geq \text{soc}(G)$. Thus there exists $L \trianglelefteq M$ with $L \leq K$, whence $s \in \sigma_M$, and $\sigma_G = \sigma_M$.

Let $\tau$ be of the form $\tau = \sigma_M \cup \{t\} = \sigma_G \cup \{t\}$, where $t \in \mathcal{P}$. Let $H_1 \in \text{Hall}_r(M)$ and let $H \in \text{Hall}_r(G)$ with $H_1 = H \cap M$. By hypothesis, $H \in \mathcal{L}_\pi$ and so $H_1 \in \mathcal{L}_\pi$. By the minimality of $G$, it follows that $M \in \mathcal{L}_\pi$. 

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Now write \( \tau_0 = \sigma_G \cup \{q\} \) and fix \( H \in \text{Hall}_{\pi}(G) \). Let \( p \in \pi \cap \sigma_G \) be arbitrary. Then \( H \in \mathcal{Z}_p \), \( M \in \mathcal{Z}_p \), and \( \{p, q\} \subseteq \tau_0 \); it follows from Lemma 2.3 that \( G \in \mathcal{Z}_p \), and the proof is complete.

Putting together Propositions 2.1 and 2.5, we obtain the promised criterion for membership of the central-socle classes as follows.

2.6 THEOREM. Let \( G \in \mathcal{I} \) and \( \pi \subseteq \mathcal{P} \). Then \( G \in \mathcal{Z}_\pi \) if and only if \( \text{Hall}_\tau(G) \subseteq \mathcal{Z}_\pi \) for all \( \tau \subseteq \mathcal{P} \) of the form \( \tau = \sigma_G \cup \{t\} \) with \( t \in \mathcal{P} \).

We now give another result in the spirit of 2.5.

2.7 PROPOSITION. Let \( G \in \mathcal{I} \) and \( \pi \subseteq \mathcal{P} \). Suppose that \( \text{Hall}_\tau(G) \subseteq \mathcal{Z}_\pi \) for all sets of primes \( \tau \) with \( |\tau| \leq 2 \). Then \( G \in \mathcal{Z}_\pi \).

PROOF. Because \( \mathcal{Z}_\pi = \bigcap_{p \in \pi} \mathcal{Z}_p \), we may without loss of generality assume that \( \pi = \{p\} \). Suppose for a contradiction that \( G \) is a counterexample of minimal order. Then \( p\)-soc\((G) \leq G \) and there exists \( M \leq G \) with \( M \supseteq p\)-soc\((G) \). If \( \tau \subseteq \mathcal{P} \) with \( |\tau| = 2 \) and if \( H \in \text{Hall}_\tau(M) \), then \( H = M \cap H_1 \) where \( H_1 \in \text{Hall}_\tau(G) \), and so \( H \in \mathcal{Z}_p \). Thus \( M \in \mathcal{Z}_p \) by minimality. Write \( |G : M| = q \in \mathcal{P} \). Now the Hall \( \{p, q\} \)-subgroups of \( G \) belong to \( \mathcal{Z}_p \) by hypothesis, and the result follows from Lemma 2.3.

3. The classes \( e_\pi(\mathcal{N}^k) \)

This section has a similar structure to Section 2. It is proved in Proposition 3.1 that \( e_\pi(\mathcal{N}^k) \) is \( \text{H}_{\mathcal{K}} \)-closed, and this is followed by some relevant examples (3.2). Proposition 3.3 is a converse to Proposition 3.1, and together these results yield a criterion, Theorem 3.4, for membership of the classes \( e_\pi(\mathcal{N}^k) \). Again the section finishes with a result, Proposition 3.5, not strictly connected with the \( \text{H}_{\mathcal{K}} \) operation, being an analogue for certain classes \( e_\pi(\mathcal{F}) \) of Proposition 2.7.

3.1 PROPOSITION. Let \( \pi \subseteq \mathcal{P} \) and \( k \in \mathbb{N}, k \geq 0 \). Let \( G \in e_\pi(\mathcal{N}^k) \). Suppose that \( H \) is a Hall subgroup of \( G \) with \( H \supseteq G_{\mathcal{N}^k} \). Then \( H \in e_\pi(\mathcal{N}^k) \). It follows that \( e_\pi(\mathcal{N}^k) \) is \( \text{H}_{\mathcal{N}^k} \)-closed.

PROOF. Because \( e_\pi(\mathcal{N}^k) = \bigcap_{p \in \pi} e_p(\mathcal{N}^k) \), we may without loss of generality assume that \( \pi = \{p\} \) where \( p \in \mathcal{P} \).

The proof is by induction on \( k \). If \( k = 0 \) then \( \mathcal{N}^k = 1 \) and \( e_p(1) = \mathcal{I} \); the conclusion clearly holds in this case. We thus suppose that the result holds for all \( G_0 \in e_p(\mathcal{N}^{k_0}) \) for all \( k_0 < k \), and for all \( G_1 \in e_p(\mathcal{N}^k) \) with \( |G_1| < |G| \).
Write $\tau = \{q \in P : q \mid |H|\}$; then $H \in \text{Hall}_\tau(G)$. If $A$ is a group, write $A_\jmath = A_{\jmath r}$, the $\jmath r$-radical of $A$; then $G_\jmath \in \mathcal{J}_\tau$ and $G_\jmath \leq O_\jmath(G) \leq H$, where $O_\jmath(G)$ denotes the $\jmath r$-radical of $G$. Since $\mathcal{J}_p = e_p(\mathcal{N}^k)$, then $H = e_p(\mathcal{N}^k)$ if $p \notin \tau$, and so we may without loss assume that $p \in \tau$.

Choose $M \triangleleft G$ with $G \supseteq G_k$ and write $|G : K| = q \in P$. Then $M \in e_p(\mathcal{N}^k)$, $M \cap H \in \text{Hall}_\tau(M)$ and $M_k = G_k \leq M \cap H$. By the induction hypothesis we have $M \cap H \in e_p(\mathcal{N}^k)$; in particular, $M \cap H \neq H$ and so $G = MH$. Further, all $M \cap H$-chief $p$-factors below $(M \cap H)_k$ are $M \cap H$-central. Since $M \cap H \triangleleft H$ then by Clifford’s Theorem, any $H$-chief $p$-factor, $X/Y$ say, below $(M \cap H)_k$ is completely reducible as an $M \cap H$-module and, being then a sum of $M \cap H$-trivial modules, must itself be $M \cap H$-trivial. Thus,

$$\text{(5)} \quad \text{The } H\text{-chief } p\text{-factors below } (M \cap H)_k \text{ are } M \cap H\text{-central.}$$

There are now two cases to consider.

Case (I). Suppose that $H_k \not\leq M$; then $H = (M \cap H)H_k$. Let $X/Y$ be an $H$-chief $p$-factor in $H_k$ in an $H$-chief series which refines $H \geq H_k \geq H_{k-1} \geq 1$. By the Jordan-Hölder theorem, we may restrict attention to a fixed chief series.

We firstly claim that $X/Y$ is trivial as an $M \cap H$-module. If $X \leq (M \cap H)_k = M \cap H_k$, then $X/Y$ is $M \cap H$-central by (1). If $Y \not\leq M$ then $X/Y \simeq_H (X \cap M)/(Y \cap M)$; the latter is still $H$-chief and so again is $M \cap H$-trivial by (1). In the remaining case we have $Y \leq M$, $X \not\leq M$ and $Y = X \cap M$; then we have $[X, M \cap H] \leq X \cap M = Y$ and again $X/Y$ is $M \cap H$-trivial; this justifies our claim.

Suppose that $X/Y$ lies below $H_{k-1}$; then $X/Y$ is $H$-central because $H \in e_p(\mathcal{N}^{k-1})$ by the induction hypothesis and the fact that $e_p(\mathcal{N}^k) \subseteq e_p(\mathcal{N}^{k-1})$. Suppose, on the other hand, that $X/Y$ lies between $H_k$ and $H_{k-1}$. By Clifford’s Theorem, $X/Y$ is completely reducible as an $H_k$-module and so must be a sum of $H_k$-trivial submodules because $H_k/H_{k-1}$ is nilpotent; thus $X/Y$ is a trivial $H_k$-module. But $H = (M \cap H)H_k$, and since $X/Y$ is trivial for $M \cap H$, it must be trivial for $H$. It follows that $H \in e_p(\mathcal{N}^k)$, as required.

Case (II). Suppose now that $H_k \leq M$; then $H = (M \cap H)H_k$. Now $G_k \leq O_\jmath(G) \cap H_k \leq (O_\jmath(G))_k \leq G_k$, whence $G_k = O_\jmath(G) \cap H_k$.

Let $P \in \text{Syl}_p(H_k)$, and write $J = \langle P^g : g \in G \rangle$, the normal closure of $P$ in $G$; note that $J \leq M$. Let $R$ be a Hall $p$-complement in $G_k$; then $\bar{R} = RG_{k-1}/G_{k-1}$ is the unique $p$-complement in $G_k/G_{k-1} \in \mathcal{N}$, and so $\bar{R} \trianglelefteq G/G_{k-1}$. Now $R \in H_k$ and so, since $H_k/H_{k-1} \in \mathcal{N}$, we have $[R, P] \leq H_{k-1}$. But $[R, P] \leq G_k$ because $R \leq G_k \leq G$, and so

$$[R, P] \leq G_k \cap H_{k-1} = O_\jmath(G) \cap H_k \cap H_{k-1} = G_{k-1}.$$

But then $P \leq C_G(\bar{R}) \trianglelefteq G$ and so $J \leq C_G(\bar{R}) \cap M$. Now let $x \in J$ be a $p'$-element. The $G$-chief $p$-factors between $G_k$ and $G_{k-1}$ are $G$-central because $G \in e_p(\mathcal{N}^k)$, and
so are centralised by \( x \). But then \( x \), being a \( p' \)-element, must centralise the Sylow \( p \)-subgroup of \( G_k/G_{k-1} \), by [9,5.3.2]. But \( x \in J \) already centralises the \( p \)-complement \( RG_{k-1}/G_{k-1} \) of \( G_k/G_{k-1} \), and so \( x \) centralises \( G_k/G_{k-1} \). But \( G_k/G_{k-1} \) is the Fitting subgroup of \( G/G_{k-1} \), and so \( x \in G_k \) by [9,6.1.3]. But this implies that \( JG_k/G_k \) must be a \( p \)-group. Since \( G_k \in \mathcal{F} \), and \( p \in \tau \), it follows that \( J \in \mathcal{F}_\pi \). But now \( J \leq O_\tau(G) \) and \( P \leq O_\tau(G) \cap H_k = G_k \). But then \( P \in \text{Syl}_P(G_k) \) and so \( P \nmid |H_k : G_k| \).

Let \( \mathcal{C}_0 \) be a \( G \)-chief series between \( G_k \) and \( 1 \), and let \( \mathcal{C} \) be an \( H \)-chief series which refines \( H_k \geq G_k \geq 1 \) and which refines \( \mathcal{C}_0 \) below \( G_k \). Now all the \( G \)-chief \( p \)-factors in \( \mathcal{C}_0 \) are \( G \)-central because \( G \in e_p(\mathcal{N}^k) \); thus they all have order \( p \) and so must be \( H \)-chief; moreover, they give us all the \( p \)-factors in \( \mathcal{C} \) because \( p \nmid |H_k : G_k| \). But now \( H \in e_p(\mathcal{N}^k) \), and the proof is complete.

3.2 EXAMPLES. (i) This example is to show that \( e_p(\mathcal{N}^2) \) is not \( H_\mathcal{N} \)-closed. Let \( p, q, r \) and \( s \) be distinct primes. There exists a group \( G \) with a unique chief series whose factors have orders (reading ‘from the top’) of the form \( q, p^a, r^b \) and \( s^c \) respectively. Then \( G \in e_p(\mathcal{N}^2) \) because \( |G_{\mathcal{N}^2}| = s^aq^br^b \). Let \( H \in \text{Hall}(G) \) with \( |H| = s^aq^br^b \). Then \( |H_{\mathcal{N}^2}| = s^aq^br^b \) and \( H \notin e_p(\mathcal{N}^2) \). However, \( H \geq G_\mathcal{N} \), and so \( e_p(\mathcal{N}^2) \) is not \( H_\mathcal{N} \)-closed.

(ii) This example shows that \( e_p(\mathcal{F}_\pi) \) is not \( H_{\mathcal{F}_\pi} \)-closed when \( \pi \subset P \) with \( |\pi| \geq 2 \). Let \( G \) be the group of Example 2.2(i) with \( \{p, q\} \subset \pi, r \notin \pi \), and \( H \in \text{Hall}(G) \). Then \( H \geq O_\tau(G) = 1 \). Now \( G \in e_\pi(\mathcal{F}_\pi) \) while \( H \notin e_\pi(\mathcal{F}_\pi) \). Thus 3.1 is not valid if we replace \( \mathcal{N}^k \) by an arbitrary Fitting class \( \mathcal{F} \).

The next result is an analogue of Proposition 2.5, being converse in sense to 3.1; it is valid for arbitrary \( e_\pi(\mathcal{F}) \) and not just for the classes \( e_\pi(\mathcal{N}^k) \) : as we have just seen, 3.1 is not valid for arbitrary \( e_\pi(\mathcal{F}) \).

3.3 PROPOSITION. Let \( G \in \mathcal{F} \) and \( \pi \subset P \). Let \( \mathcal{F} \) be a Fitting class. Suppose that \( \text{Hall}_\tau(G) \subseteq e_\pi(\mathcal{F}) \) for all \( \tau \subset P \) of the form \( \tau = \rho_G \cup \{t\} \) where \( t \in P \) and \( \rho_G = \{s \in P : s \mid |G_\mathcal{F}| \} \). Then \( G \in e_\pi(\mathcal{F}) \).

PROOF. Suppose for a contradiction that \( G \) is a counterexample of minimal order. Then \( G_\mathcal{F} < G \) as otherwise \( \sigma_G \) contains all primes dividing \( |G| \) and so \( G \in e_\pi(\mathcal{F}) \) by hypothesis. Let \( M \triangleleft G \) with \( M \geq G_\mathcal{F} \), and write \( |G : M| = q \). Then \( M_\mathcal{F} = G_\mathcal{F} \), and so \( \rho_M = \rho_G \). If \( H \in \text{Hall}_\tau(M) \) then \( H = H_1 \cap M \) for some \( H_1 \in \text{Hall}_\tau(G) \) and so \( M \in e_\pi(\mathcal{F}) \) by minimality. Because \( G \notin e_\pi(\mathcal{F}) \), there exists a \( G \)-chief \( \pi \)-factor \( X/Y \) below \( G_\mathcal{F} \) which is not \( G \)-central. By Clifford’s Theorem, \( X/Y \) is completely reducible as an \( M \)-module, and so \( X/Y \) is \( M \)-central because \( M \in e_\pi(\mathcal{F}) \). Thus \( X/Y \) is faithful and irreducible for \( G/M \cong C_q \). Let \( H \in \text{Hall}_\tau(G) \) where \( \tau = \rho_G \cup \{q\} \). Then \( G = MH \). Thus \( X/Y \) is faithful and irreducible for \( H/(H \cap M) \cong G/M \), and
so is non-trivial for $H$. Now $H \geq H_\mathcal{F} \geq G_\mathcal{F} \geq X \geq Y$, and so $X/Y$ is $H$-central because $H \in e_\pi(\mathcal{F})$, in contradiction to the preceding statement. The result follows.

Putting together Propositions 3.1 and 3.3, we obtain our criterion for membership of the classes $e_\pi(\mathcal{N}^k)$ as follows.

3.4 THEOREM. Let $G \in \mathcal{P}$, $\pi \subseteq \mathcal{P}$ and $k \in \mathbb{N}$, $k \geq 0$. Then $G \in e_\pi(\mathcal{N}^k)$ if and only if $\text{Hall}_\pi(G) \subseteq e_\pi(\mathcal{N}^k)$ for all $\pi \subseteq \mathcal{P}$ of the form $\pi = \rho_G \cup \{t\}$ where $t \in \mathcal{P}$ and $\rho_G = \{s \in \mathcal{P} : s \mid |G_{\mathcal{N}^k}|\}$.

The next result is an analogue of Proposition 2.7 for the classes $e_\pi(\mathcal{F})$.

3.5 PROPOSITION. Let $G \in \mathcal{P}$ and $\pi \subseteq \mathcal{P}$. Let $\mathcal{F}$ be a Hall-closed Fitting class. Suppose that $\text{Hall}_\pi(G) \subseteq e_\pi(\mathcal{F})$ for all $\pi \subseteq \mathcal{P}$ with $|\pi| \leq 2$. Then $G \in e_\pi(\mathcal{F})$.

PROOF. The proof is by induction on $|G|$, the result being trivial if $|G| = 1$. If $M \triangleleft G$ and $\tau \subseteq \mathcal{P}$ with $|\tau| \leq 2$ then $\text{Hall}_\tau(M) \subseteq e_\pi(\mathcal{F})$ and so $M \in e_\pi(\mathcal{F})$ by induction. It follows that $G$ contains a unique maximal normal subgroup, which we call $M'$; then $M \geq G'$ and $|G : M'| = q \in \mathcal{P}$. Let now $X/Y$ be a $G$-chief $\pi$-factor below $G_\mathcal{F}$. If $X \not\leq M$ then $X = G$ and $Y = M$ by the unicity of $M \triangleleft G$, and then $X/Y$ is certainly $G$-central. Suppose that $X \leq M$. Then $X/Y$ is below $M_\mathcal{F}$, and by Clifford’s Theorem must be $M$-central. Now $X/Y \in \mathcal{P}_{\pi}$ for some $\pi \in \mathcal{P}$. Let $H \in \text{Hall}_\pi(G)$ where $\pi = \{p, q\}$. Then $G = MH$ and $X/Y$ is a module for $H/H \cap M \simeq G/M$. But $X \leq YH$ and so $X = X \cap TH = Y(X \cap H)$, whence

$$X/Y \simeq_H (X \cap H)/(Y \cap H).$$

Now $M_\mathcal{F} \cap H \in \text{Hall}_\pi(M_\mathcal{F}) \subseteq \mathcal{F}$, the final inclusion because $\mathcal{F}$ is Hall-closed, and so $X \cap H \leq M_\mathcal{F} \cap H \leq H_\mathcal{F}$. But $H \in e_\pi(\mathcal{F})$, and it follows that $X/Y$ is $H$-central and thus $G$-central, as required.

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