ENDPOINT ESTIMATES FOR COMMUTATORS OF
RIESZ TRANSFORMS ASSOCIATED WITH
SCHRÖDINGER OPERATORS

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Abstract
In this paper, we discuss the $H^1_1$-boundedness of commutators of Riesz transforms associated with the Schrödinger operator $L = -\Delta + V$, where $H^1_1(R^n)$ is the Hardy space associated with $L$. We assume that $V(x)$ is a nonzero, nonnegative potential which belongs to $B^q$ for some $q > n/2$. Let $T_1 = V(x)(-\Delta + V)^{-1}$, $T_2 = V^{1/2}(-\Delta + V)^{-1/2}$ and $T_3 = \nabla(-\Delta + V)^{-1/2}$. We prove that, for $b \in \text{BMO}(R^n)$, the commutator $[b, T_3]$ is not bounded from $H^1_1(R^n)$ to $L^1(R^n)$ as $T_3$ itself. As an alternative, we obtain that $[b, T_i]$, ($i = 1, 2, 3$) are of $(H^1_1, L^1_{\text{weak}})$-boundedness.

Keywords and phrases: commutator, $H^1_1$, BMO, Schrödinger operator, Riesz transform.

1. Introduction

Let $L = -\Delta + V$ be the Schrödinger operator on $R^n$, $n \geq 3$. Throughout this paper, we assume that $V$ is a nonzero, nonnegative potential which belongs to $B_q$ for some $q > n/2$. Let $T_i$ ($i = 1, 2, 3$) be the Riesz transform associated with Schrödinger operators, specifically, $T_1 = V(-\Delta + V)^{-1}$, $T_2 = V^{1/2}(-\Delta + V)^{-1/2}$ and $T_3 = \nabla(-\Delta + V)^{-1/2}$. The $L^p$-boundedness of $T_i$ ($i = 1, 2, 3$) was widely studied in [7, 9]. In [3], using a pointwise estimate of the kernel of $T_i$ ($i = 1, 2, 3$), the authors proved the $L^p$-boundedness of commutators $[b, T_i]$ ($i = 1, 2, 3$) for some $p > 1$. In this paper, we discuss the boundedness of $[b, T_i]$ ($i = 1, 2, 3$) at the endpoint $p = 1$.

A nonnegative locally $L^q$ integrable function $V(x)$ on $R^n$ is said to belong to $B_q$ ($1 < q < \infty$), if there exists $C > 0$, such that the reverse Hölder inequality

$$\left(\frac{1}{|B|} \int_B V^q(x) \, dx\right)^{1/q} \leq C\left(\frac{1}{|B|} \int_B V(x) \, dx\right)$$

(1.1)

holds for every ball $B$ in $R^n$. 

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By Hölder’s inequality, we have $B_{q_1} \subseteq B_{q_2}$ for $q_1 \geq q_2 > 1$. One remarkable feature of the $B_q$ class is that if $V \in B_q$ for some $q > 1$, then there exists an $\epsilon > 0$ which depends only on $n$ and the constant $C$ in (1.1) such that $V \in B_{q+\epsilon}$. It is also well known that if $V \in B_q$, $q > 1$, then $V(x) \, dx$ is a doubling measure, namely for any $r > 0, x \in \mathbb{R}^n$ and some constant $C_0$,

$$\int_{B(x,2r)} V(y) \, dy \leq C_0 \int_{B(x,r)} V(y) \, dy. \tag{1.2}$$

For such a Schrödinger operator $L$, Shen [7] studied the $L^p$-boundedness of Riesz transforms associated with $L$. He obtained the following result.

**Theorem 1.1** [7, Theorem 0.5, Theorem 3.1, Theorem 5.10].

(i) Suppose that $V \in B_q$ and $q \geq n/2$. Then for $q' \leq p < \infty$,

$$\|(-\Delta + V)^{-1} Vf\|_p \leq C_p \|f\|_p.$$

(ii) Suppose that $V \in B_q$ and $q \geq n/2$. Then for $(2q)' \leq p < \infty$,

$$\|(-\Delta + V)^{-1/2} V^{1/2} f\|_p \leq C_p \|f\|_p.$$

(iii) Suppose that $V \in B_q$ and $n/2 \leq q < n$. Then for $p_1' \leq p < \infty$,

$$\|(-\Delta + V)^{-1/2} \nabla f\|_p \leq C_p \|f\|_p$$

where $1/p_1 = 1/q' - 1/n$.

By duality, we can easily obtain the $L^p$-boundedness of $T_i$ ($i = 1, 2, 3$). Take $T_3 = \nabla (-\Delta + V)^{-1/2}$ for example; using (iii) of Theorem 1.1, we find that $T_3$ is bounded on $L^p(\mathbb{R}^n)$, $1 < p \leq p_1$. So an interesting problem is the boundedness of $T_i$ ($i = 1, 2, 3$) at the endpoint $p = 1$. In Section 2, we prove that the $T_i$ ($i = 1, 2, 3$) are bounded from $L^1(\mathbb{R}^n)$ to $L^1_{\text{weak}}(\mathbb{R}^n)$. It was pointed out in [7] that if $V \in B_n$, then $T_3$ is a Calderón–Zygmund operator. So when considering $[b, T_3]$, we restrict ourselves to the case where $V \in B_q$ ($n/2 < q < n$).

In [3] the authors proved that for $b \in \text{BMO}(\mathbb{R}^n)$, the commutators $[b, T_i]$ ($i = 1, 2, 3$) are bounded on $L^p(\mathbb{R}^n)$ for some $p > 1$. Another problem we are interested in is the boundedness of commutators $[b, T_i]$ ($i = 1, 2, 3$) at endpoint $p = 1$ for $b \in \text{BMO}(\mathbb{R}^n)$. In [6] Pérez proved that if $b \in \text{BMO}(\mathbb{R}^n)$, the commutator $[b, T]$ may not be of weak-type $(1, 1)$ where $T$ is a Calderón–Zygmund operator. In [4] Harboure et al. proved that, even if we restrict $f \in H^1(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$, $[b, T]f$ still may not be in $L^1(\mathbb{R}^n)$.

In [2] Dziubanski and Zienkiewicz studied the Hardy space $H^1_L$ associated with the Schrödinger operator $L = -\Delta + V$, for $V \in B_q$, $q > n/2$. Actually they showed that if $f \in H^1_L(\mathbb{R}^n)$, then $T_3 f \in L^1(\mathbb{R}^n)$. So a natural question is whether the commutator $[b, T_3]$ is bounded from $H^1_L(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$ when $b \in \text{BMO}(\mathbb{R}^n)$? Unfortunately, in Section 3, we get a negative result. We give a counterexample to imply that the commutators $[b, T_i]$ ($i = 1, 2, 3$) may not be bounded from $H^1_L(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$.
These facts imply that, in order to get the $H^1_L$-boundedness of the commutators $[b, T_i]$ ($i = 1, 2, 3$), we need to replace of the space $L^1(R^n)$ by a larger class. In Section 4, we prove that, if $b \in \text{BMO}(R^n)$, the commutators $[b, T_i]$ ($i = 1, 2, 3$) are bounded from $H^1_L(R^n)$ into $L^1_{\text{weak}}(R^n)$.

In the rest of this section, we list some notation and properties for later use.

**Definition 1.2.** For $x \in R^n$, the function $m(x, V)$ is defined by

$$
\frac{1}{m(x, V)} = \sup_{r > 0} \left\{ r : \frac{1}{r^{n-2}} \int_{B(x, r)} V(y) \, dy \leq 1 \right\}.
$$

(1.3)

Clearly, $0 < m(x, V) < \infty$ for every $x \in R^n$ and if $r = 1/m(x, V)$, then $1/r^{n-2} \int_{B(x, r)} V(y) \, dy = 1$. For simplicity, we sometimes denote $1/m(x, V)$ by $\rho(x)$ in proofs.

The function $m(x, V)$ has many useful properties. We list them in the following lemmas.

**Lemma 1.3** [7, Lemma 1.4]. There exist $C > 0$, $c > 0$ and $k_0 > 0$ such that for $x, y \in R^n$:

1. $m(x, V) \sim m(y, V)$, if $|x - y| \leq C/m(x, V)$;
2. $m(y, V) \leq C(1 + |x - y|m(x, V))^{k_0}m(x, V)$;
3. $m(y, V) \geq cm(x, V)/(1 + |x - y|m(x, V))^{k_0/(k_0+1)}$.

**Lemma 1.4** [7, Lemma 1.8]. There exist $C > 0$ and $k_0 > 0$ such that if $Rm(x, V) \geq 1$, then

$$
\frac{1}{R^{n-2}} \int_{B(x, R)} V(y) \, dy \leq C(Rm(x, V))^{k_0}.
$$

When we estimate the integral of the kernels of $T_i$ ($i = 1, 2, 3$), we need the following lemma.

**Lemma 1.5** [3, Lemma 1]. Suppose that $V \in B_q$ for some $q > n/2$. Let $N > \log_2 C_0 + 1$, where $C_0$ is the constant in (1.2). Then for any $x_0 \in R^n$ and $R > 0$,

$$
\frac{1}{\{1 + m(x_0, V)R\}^N} \int_{B(x_0, R)} V(\xi) \, d\xi \leq R^{n-2}.
$$

### 2. The $(L^1, L^1_{\text{weak}})$-boundedness of $T_i$ ($i = 1, 2, 3$)

In this section, we discuss the $(L^1, L^1_{\text{weak}})$-boundedness of $T_i$ ($i = 1, 2, 3$). For the operator $T_3 = \nabla(-\Delta + V)^{-1/2}$, Li [5] proved the $(L^1, L^1_{\text{weak}})$-boundedness of the Riesz transform $X_{\lambda}L^{-1/2}$ associated with a Schrödinger operator on a nilpotent group. So we need only give the proof of $T_i$ for $i = 1, 2$. For the proof, we need the well-known Calderón–Zygmund decomposition as follows.

**Lemma 2.1** [8]. Let $f \in L^1$ and $\alpha > 0$; there exist a decomposition of $f$ as $f = g + b$, where $b = \sum_k b_k$, and a sequence of balls $\{B_k^*\}$ such that:
Now we estimate

Using (i) and (iv) of Lemma 2.1,

Then by (i) of Theorem 1.1 and 1 < p < q,

Now we estimate \(|x : |T_1b(x)| > \alpha/2| |

\[ \leq \sum_k |16B^*_k| + |\{x \in (\bigcup 16B^*_k)^c : |T_1b(x)| > \alpha/2\}| \]

\[ \leq \frac{c}{\alpha} \int |f(x)| \, dx + |\{x \in (\bigcup 16B^*_k)^c : |T_1b(x)| > \alpha/2\}|. \]
By the cancelling property of $b_k$, we let $B^*_k = B(x_k, r_k)$. Then

$$\{x \in (\cup 16B^*_k)^c : |T_1 b(x)| > \alpha/2\}$$

$$\leq \frac{c}{\alpha} \int_{(\cup 16B^*_k)^c} |T_1 b(x)| \, dx$$

$$\leq \frac{c}{\alpha} \sum_k \int_{B^*_k} \int_{(\cup 16B^*_k)^c} |K_1(x, y) - K_1(x, x_k)| b_k(y) \, dy \, dx$$

$$\leq \frac{c}{\alpha} \sum_k \int_{B^*_k} |b_k(y)| \, dy \int_{(\cup 16B^*_k)^c} |K_1(x, y) - K_1(x, x_k)| \, dx.$$

Because $y \in B^*_k$, then $|y - x_k| < r_k < |x - x_k|/16$. In Lemma 2.3, set $h = |y - x_k|$. Then

$$|K_1(x, y) - K_1(x, x_k)| \leq \frac{C_K}{(1 + m(x_k, V)|x - x_k|)^K} \frac{|y - x_k|^\delta}{|x - x_k|^{n-2+\delta}} V(x).$$

By Lemma 1.5,

$$\int_{(\cup 16B^*_k)^c} |K_1(x, y) - K_1(x, x_k)| \, dx$$

$$\leq \int_{(\cup B^*_k)^c} \frac{C_K}{(1 + m(x_k, V)|x - x_k|)^K} \frac{|y - x_k|^\delta}{|x - x_k|^{n-2+\delta}} V(x) \, dx$$

$$\leq \sum_{j=4}^{\infty} \int_{2jr_k \leq |x - x_k| < 2jr_k} \frac{C_K}{(1 + m(x_k, V)|x - x_k|)^K} \frac{|y - x_k|^\delta}{|x - x_k|^{n-2+\delta}} V(x) \, dx$$

$$\leq \sum_{j=4}^{\infty} \frac{C_K}{(1 + m(x_k, V)2jr_k)^K} \frac{r_k^\delta}{(2jr_k)^{n-2+\delta}} \int_{|x - x_k| < 2^j r_k} V(x) \, dx$$

$$\leq \sum_{j=4}^{\infty} \frac{r_k^\delta}{(2jr_k)^{n-2+\delta}} (2^j r_k)^{n-2} \leq C.$$

Finally, we obtain

$$\{|x : |T_1 b(x)| > \alpha/2\} \leq \frac{c}{\alpha} \int |f(x)| \, dx + \frac{c}{\alpha} \sum_k \int_{B^*_k} |b_k(x)| \, dx$$

$$\leq \frac{c}{\alpha} \int |f(x)| \, dx.$$

This completes the proof of Theorem 2.2. $\square$

For the $(L^1, L^1_{\text{weak}})$-boundedness of $T_2$, we need the following lemma.
Lemma 2.4 [3, Lemma 3]. Suppose that $V \in B_q$ for some $q > n/2$. Then there exists $\delta > 0$ such that for any integer $K > 0$, $0 < h < |x - y|/16$,

$$|K_2(x, y)| \leq \frac{C_K}{(1 + m(y, V)|x - y|)^K} \frac{1}{|x - y|^{n-1}} V^{1/2}(x), \tag{2.3}$$

$$|K_2(x, y + h) - K_2(x, y)| \leq \frac{C_K}{(1 + m(y, V)|x - y|)^K} \frac{|h|^\delta}{|x - y|^{n-1+\delta}} V^{1/2}(x). \tag{2.4}$$

We now prove the $(L^1, L^1_{\text{weak}})$-boundedness of $T_2$.

Theorem 2.5. Suppose $V \in B_q$ for some $q > n/2$. If $T_2 = V^{1/2}(x)(-\Delta + V)^{-1/2}$, then $T_2$ is bounded from $L^1(R^n)$ into $L^1_{\text{weak}}(R^n)$.

Proof. By the Calderón–Zygmund decomposition,

$$|[x : |T_2 f(x)| > \alpha]| \leq |[x : |T_2 g(x)| > \alpha/2]| + |[x : |T_2 b(x)| > \alpha/2]|.$$

Similarly, we only need to estimate $|[x \in (\cup 16B_k^*)^c : |T_2 b(x)| > \alpha/2]|$. Set $B_k^* = B(x_k, r_k)$. Then by the cancelling of $b_k(x)$,

$$|[x \in (\cup 16B_k^*)^c : |T_2 b(x)| > \alpha/2]|$$

$$\leq \frac{C}{\alpha} \sum_k \int_{(\cup 16B_k^*)^c} \left| \int_{B_k^*} [K_2(x, y) - K_2(x, x_k)] b(y) \, dy \right| \, dx$$

$$\leq \frac{C}{\alpha} \sum_k \int_{B_k^*} |b(y)| \, dy \int_{(\cup 16B_k^*)^c} |K_2(x, y) - K_2(x, x_k)| \, dx.$$

Since $y \in B_k^*$ and $x \in (\cup 16B_k^*)^c$, then $|y - x_k| < r_k < |x - x_k|/16$. Let $h = |y - x_k|$, by Lemma 2.4 and Hölder’s inequality,

$$\int_{(\cup 16B_k^*)^c} |K_2(x, y) - K_2(x, x_k)| \, dx$$

$$\leq \sum_{j=4}^\infty \int_{2^jr_k < |x - x_k| \leq 2^{j+1}r_k} \frac{C_K}{(1 + m(x_k, V)|x - x_k|)^K} \frac{|y - x_k|^\delta}{|x - x_k|^{n-1+\delta}} V^{1/2}(x) \, dx$$

$$\leq \sum_{j=4}^\infty \frac{C_K}{(1 + m(x_k, V)2^jr_k)^K} \frac{r_k^\delta}{(2^jr_k)^{n-1+\delta}} (2^{2j+1}r_k)^{n/(2q)' - 1/2}$$

$$\times \left( \int_{|x - x_k| < 2^{j+1}r_k} V^q(x) \, dx \right)^{1/q}$$

$$\leq \sum_{j=4}^\infty \frac{C_K}{(1 + m(x_k, V)2^jr_k)^K} \frac{r_k^\delta}{(2^jr_k)^{n-1+\delta}} (2^{2j+1}r_k)^{n/(2q)' + n/2q - n/2}$$

$$\times \left( \int_{|x - x_k| < 2^{j+1}r_k} V(x) \, dx \right)^{1/2}.$$
Finally, we obtain
\[
\{|x : |T_2 b(x)| > \alpha/2|\} \leq \frac{C}{\alpha} \|f\|_1 + \frac{C}{\alpha} \sum_k \int_{B_k} |b_k(x)| \, dx \leq \frac{C}{\alpha} \|f\|_1.
\]

This completes the proof of Theorem 2.5. \(\square\)

In a similar manner to the two previous theorems, and using the following lemma, we can prove the \((L^1, L^1_{\text{weak}})\)-boundedness of \(T_3\).

**Lemma 2.6** [3, Lemma 4]. Suppose that \(V \in B_q\) for some \(n/2 < q < n\). Then there exists \(\delta > 0\) and for any integer \(K > 0\), \(0 < h < |x - y|/16\),

\[
|K_3(x, y)| \leq \frac{C_K}{\{1 + m(y, V)|x - y|\}^K |x - y|^n |x - y|^n - 1}
\times \left( \int_{B(x, |x - y|)} \frac{V(\xi)}{|x - \xi|^n} \, d\xi + \frac{1}{|x - y|} \right),
\]

(2.5)

\[
|K_3(x, y + h) - K_3(x, y)| \leq \frac{C_K}{\{1 + m(y, V)|x - y|\}^K |x - y|^n |x - y|^n + \delta}
\times \left( \int_{B(x, |x - y|)} \frac{V(\xi)}{|x - \xi|^n} \, d\xi + \frac{1}{|x - y|} \right).
\]

(2.6)

**Theorem 2.7.** Suppose that \(V \in B_q\), \(n/2 < q < n\). Letting \(T_3 = \nabla(-\Delta + V)^{-1/2}\), then \(T_3\) is bounded from \(L^1(R^n)\) into \(L^1_{\text{weak}}(R^n)\).

3. Failure for \((H^1_L, L^1)\)-boundedness of \([b, T_3]\)

In [2] Dziubanski and Zienkiewicz studied the Hardy space \(H^1_L\) associated with a Schrödinger operator \(L\). In that paper they constructed the atomic Hardy space as follows.

**Definition 3.1** \((H^1_L)\)-atom. For \(n \in \mathbb{Z}\), define the set \(\mathfrak{B}_n\) by

\[
\mathfrak{B}_n = \{x : 2^n/2 \leq m(x, V) < 2^{(n+1)/2}\}.
\]

Since \(0 < m(x, V) < \infty\), then \(R^n = \bigcup_n \mathfrak{B}_n\).

A function \(a(x)\) is an atom for the Hardy space \(H^1_L(R^n)\) associated with a ball \(B(x_0, r)\), if the following conditions hold:

(i) \(\supp a(x) \subset B(x_0, r)\);
(ii) \(\|a\|_{L^\infty} \leq 1/|B(x_0, r)|\);
(iii) if \(x_0 \in \mathfrak{B}_n\), then \(r \leq 2^{1-n/2}\);
(iv) if \(x_0 \in \mathfrak{B}_n\) and \(r \leq 2^{-1-n/2}\), then \(\int a(x) \, dx = 0\).
The atomic norm in $H^1_L(R^n)$ is defined by $\|f\|_{L-\text{atom}} = \inf(\sum_j |\lambda_j|)$, where the infimum is taken over all decompositions $f = \sum_j \lambda_j a_j$ where $a_j$ are $H^1_L$-atoms.

In [2] the authors obtained the atomic decomposition of $H^1_L$ as follows.

**Theorem 3.2 [2, Theorem 1.5].** Assuming that $V$ is a nonnegative potential such that $V \in B_{n/2}$, then the norms $\|f\|_{H^1_L}$ and $\|f\|_{L-\text{atom}}$ are equivalent, that is, there exists a constant $C > 0$ such that

$$C^{-1} \|f\|_{H^1_L} \leq \|f\|_{L-\text{atom}} \leq C \|f\|_{H^1_L}.$$  

Using atomic decomposition, the authors obtained the following result.

**Theorem 3.3 [2, Theorem 1.7].** If $V \in B_{n/2}$ is a nonnegative potential, then there is a constant $C > 0$ such that

$$C^{-1} \|f\|_{H^1_L} \leq \|f\|_{L^1} + \sum_{j=1}^d \|R^L_j f\|_{L^1} \leq C \|f\|_{H^1_L}.$$ 

where $R^L_j$ denotes the $j$th component of the operator $T_3 = \nabla(\Delta + V)^{-1/2}$.

Theorem 3.3 implies that the Riesz transform $R^L_j$ is bounded from $H^1_L(R^n)$ into $L^1(R^n)$. A natural question is whether the commutator $[b, R^L_j]$ is bounded from $H^1_L(R^n)$ into $L^1(R^n)$ for $b \in \text{BMO}(R^n)$. For Calderón–Zygmund operators, the answer is negative. In [4], Harboure et al. proved that for a singular integral operator $T$, if $[b, T]$ is bounded from $H^1(R^n)$ into $L^1(R^n)$, then $b$ must be a constant. In this section we prove in a similar manner that for $T_3 = \nabla(\Delta + V)^{-1/2}$, the commutator $[b, T_3]$ may not be bounded from $H^1_L(R^n)$ into $L^1(R^n)$.

First we state the definition of the dual space of $H^1_L(R^n)$ which was introduced in [1].

**Definition 3.4.** We shall say that a locally integrable function $f$ belongs to $\text{BMO}_L(R^n)$ whenever there is a constant $C > 0$ such that

$$\frac{1}{|B_s|} \int_{B_s} |f(y) - f_{B_s}| \, dy \leq C \quad \text{and} \quad \frac{1}{|B_r|} \int_{B_r} |f(y)| \, dy \leq C,$$

for all balls $B_s = B_s(x), B_r = B_r(x)$ such that $s \leq \rho(x) \leq r$. We let $\|f\|_{\text{BMO}_L}$ denote the smallest $C$ in the above inequalities. Here and subsequently, we set $f_B = (1/|B|) \int_B f(x) \, dx$.

**Theorem 3.5.** Let $T_3 = \nabla(\Delta + V)^{-1/2}$ be the Riesz transform associated with the Schrödinger operator and let $b \in \text{BMO}_L(R^n)$. Then the following two statements are equivalent.

(i) The commutator $[b, T_3]$ is bounded from $H^1_L(R^n)$ into $L^1(R^n)$.

(ii) For any atom $a$ supported in a ball with center $x_0$ and radius $r < \rho(x_0)$, for $u \in B$,

$$\int_{(33B)^c} |K_3(x, u)| \left| \int_B b(y) a(y) \, dy \right| \, dx \leq C.$$
**PROOF.** Because $a(x)$ is an $H^1_L$-atom, we assume that the support of $a(x)$ is $B(x_0, r)$. In order to estimate the $L^1$ norm of $T_3a(x)$, we divide the discussion into two cases as follows.

**Case I.** For $\rho(x_0) \leq r \leq 4\rho(x_0)$,

$$[b, T_3]a(x) = \chi_{2B}(x)[b, T_3]a(x) + \chi_{(2B)^c}(x)[b, T_3]a(x)$$

$$= \chi_{2B}(x)[b, T_3]a(x) + \chi_{(2B)^c}(x)b(x)T_3a(x) - \chi_{(2B)^c}(x)T_3(ba)(x)$$

$$=: M_1 + M_2 + M_3.$$

For $M_1$, by the $L^p$-boundedness $[b, T_3]$, we get

$$\|M_1\|_{L^1} = \int_{2B} |[b, T_3]a(x)| \, dx$$

$$\leq C \left( \int_{2B} |[b, T_3]a(x)|^p \, dx \right)^{1/p} |B|^{1-1/p}$$

$$\leq C \|a\|_p |B|^{1-1/p} \|b\|_{BMO_L}$$

For $M_2$, we have

$$\|M_2\|_{L^1} = \int_{(2B)^c} |b(x)||T_3a(x)| \, dx \leq \int_B |a(y)| \, dy \int_{(2B)^c} |b(x)||K_3(x, y)| \, dx.$$  

Using Lemma 2.6,

$$\int_{(2B)^c} |b(x)||K_3(x, y)| \, dx$$

$$\leq \int_{(2B)^c} |b(x)| \frac{C_K}{[1 + m(y, V)|x - y|]K} \frac{1}{|x - y|^{n-1}}$$

$$\times \left( \int_{B(x,|x-y|)} \frac{V(z)}{|x - z|^{n-1}} \, dz \right) \, dx$$

$$+ \int_{(2B)^c} |b(x)| \frac{1}{[1 + m(y, V)|x - y|]K} \frac{1}{|x - y|^n} \, dx$$

$$=: M_{21} + M_{22}.$$  

For $M_{22}$, because $y \in B$ and $|x - x_0| > 2^k r$ imply $|x - y| > |x - x_0| - |y - x_0| > 2^k r - r > 2^{k-1} r$,

$$M_{22} \leq \sum_{k=1}^{\infty} \int_{2^k < |x-x_0| \leq 2^{k+1} r} |b(x)| \frac{1}{|x - y|^n} \frac{C_K}{[1 + m(y, V)|x - y|]K} \, dx$$

$$\leq \sum_{k=1}^{\infty} \frac{C_K}{[1 + m(y, V)2^{k-1} r]^K} \frac{1}{(2^{k-1} r)^n} \int_{|x-x_0| \leq 2^{k+1} r} |b(x)| \, dx.$$
Therefore, choosing $K$ large enough, we have

$$
M_{22} \leq \sum_{k=1}^{\infty} \frac{C_K}{1 + 2^{k-1} K} \|b\|_{\text{BMO}_L}.
$$

Because $|x - z| < |x - y|$ implies that $|z - x_0| \leq |z - x| + |x - x_0| \leq |x - y| + |x - x_0| \leq 2|x - x_0| + |y - x_0| < 2^{k+2}r + r < 2^{k+3}r$, then

$$
M_{21} \leq \sum_{k=1}^{\infty} \int_{2^r < |x - x_0| \leq 2^{k+1}r} \frac{C_K |y - x|^{|1-n|}}{(1 + m(y, V)|x - y|)^K} \left( \int_{B(x, |x - y|)} \frac{V(z)}{|x - z|^{|n-1|}} dz \right) |b(x)| dx
$$

$$
\leq \sum_{k=1}^{\infty} \frac{C_K}{1 + 2^{k-1} K} \frac{1}{(2^{k+1}r)^{n-1}} (2^{k+1}r)^{n/p'} \|b\|_{\text{BMO}_L}
$$

$$
\times \left\| \int_{B(x_0, 2^{k+3}r)} V(z) \left| x_{B(x_0, 2^{k+3}r)}(z) \right| |z - x|^{|n-1|} dz \right\|_{L^{p_1}(dx)}
$$

$$
\leq \sum_{k=1}^{\infty} \frac{C_K}{1 + 2^{k-1} K} \frac{1}{(2^{k+1}r)^{n-1}} \int_{B(x_0, 2^{k+3}r)} V(z) dz
$$

Because $2^{k+3}r > r \geq \rho(x_0)$ for $k \geq 1$, $2^{k+3}r m(x_0, V) > 1$. Then by Lemma 1.4, the double property of $V(x, dx)$ and $rm(x_0, V) \leq 4$ for $r \leq 4 \rho(x_0)$,

$$
\frac{1}{(2^{k+1}r)^{n-2}} \int_{B(x_0, 2^{k+3}r)} V(z) dz \leq C(2^{k+3}r m(x_0, V))^{k_0} \leq C 2^{k_0}.
$$

Therefore, choosing $K$ large enough, we obtain

$$
M_{21} \leq C \|b\|_{\text{BMO}_L} \sum_{k=1}^{\infty} \frac{C_K}{(1 + 2^{k-1} K)^{K}} \cdot 2^{k_0} \leq C \|b\|_{\text{BMO}_L}.
$$

This implies that $\|M_2\|_{L^1} \leq C \|b\|_{\text{BMO}_L}$.

Finally, we estimate $M_3$:

$$
\|M_3\|_{L^1} = \int_{(2B)^c} \left| \int_B K_3(x, y) b(y) a(y) dy \right| dx
$$

$$
\leq \int_B |b(y)| |a(y)| \int_{|x - x_0| > 2r} \frac{C_K |x - y|^{|1-n|}}{(1 + m(y, V)|x - y|)^K} dy
$$
Then by Lemma 1.4, choosing $K$ large enough,

\[
M_{31} \leq \sum_{k=1}^{\infty} \frac{C_K}{(1 + 2^{k-1})^K} \left( \int_{B(x, |x-y|)} \frac{1}{n-1}\left( \int_{B(x, |x-y|)} \frac{V(z)}{|z-x|^{n-1}} d\nu(z) \right) \right) d\nu(x).
\]

For $y \in B$, $|x - x_0| > 2^{k}r$, we have $|x - y| > |x - x_0| - |y - x_0| > 2^{k}r - r > 2^{k-1}r$, where $k \geq 1$. Then

\[
M_{32} = \int_{(2B)^c} \frac{C_K}{1 + m(y, V)[|x-y|]^{K}} \frac{1}{|x-y|^n} d\nu(x)
\]

\[
\leq \sum_{k=1}^{\infty} \int_{2^{k}r < |x-x_0| \leq 2^{k+1}r} \frac{C_K}{1 + m(y, V)[|x-y|]^{K}} \frac{1}{|x-y|^n} d\nu(x)
\]

\[
\leq \sum_{k=1}^{\infty} \frac{C_K}{1 + 2^{k-1}m(y, V)r} \left( \int_{|x-x_0| \leq 2^{k+1}r} \frac{1}{(2^{k-1}r)^n} d\nu(x) \right)
\]

Here we have used the fact that, for $4\rho(x_0) \geq r > \rho(x_0)$ and any $|y - x_0| < r < 4\rho(x_0)$, we have $m(y, V)r \geq r \rho(x_0) \sim 1$. For $M_{31}$, since $|y - x_0| < r$, $|x - x_0| > 2^{k}r$, then $|x - y| > |x - x_0| - |y - x_0| \geq 2^{k-1}r$. Then

\[
M_{31} = \int_{(2B)^c} \frac{C_K}{1 + m(y, V)[|x-y|]^{K}} \frac{1}{|x-y|^n} \left( \int_{B(x, |x-y|)} \frac{V(z)}{|z-x|^{n-1}} d\nu(z) \right) d\nu(x)
\]

\[
\leq \sum_{k=1}^{\infty} \frac{C_K}{1 + 2^{k-1}m(y, V)2^{k-1}r} \left( \int_{B(x, |x-y|)} \frac{V(z)}{|z-x|^{n-1}} d\nu(z) \right) d\nu(x)
\]

For $z \in B(x, |x-y|)$, $|z - x| \leq |x - y|$. So for every $y \in B(x_0, r)$ and $|x - x_0| \leq 2^{k+1}r$,

\[
|z - x_0| \leq |z - x| + |x - x_0|
\]

\[
\leq |x - y| + |x - x_0| \leq 2|x - x_0| + |y - x_0|
\]

\[
\leq 2^{k+2}r + r \leq 2^{k+3}r.
\]

Then by Lemma 1.4, choosing $K$ large enough,

\[
M_{31} \leq \sum_{k=1}^{\infty} \frac{C_K}{(1 + 2^{k-1})^K} \left( \int_{B(x, |x-y|)} \frac{1}{n-1}\left( \int_{B(x, |x-y|)} \frac{V(z)}{|z-x|^{n-1}} d\nu(z) \right) \right) d\nu(x)
\]

\[
\leq \left\| \int_{B(x, 2^{k+1}r)} \frac{V(z)\chi_{B(x, 2^{k+1}r)}(z)}{|z-x_0|^{n-1}} d\nu(z) \right\|_{L^p(dx)}
\]
In fact, we have proved that for an

\[ \rho(x_0) \leq r \leq 4\rho(x_0), \]  

then \( 1 \leq rm(x_0, V) \leq 4. \) Then for \( y \in B, \) \( |y - x_0| \leq r \leq 4\rho(x_0). \) Therefore we have \( m(x_0, V) \sim m(y, V) \) and \( 1 \leq rm(y, V) \leq 4. \) Finally, using (ii) of Definition 3.1, we obtain

\[ \| M_3 \|_{L^1} \leq \int_B |b(y)||a(y)|(M_{31} + M_{32}) \, dy \leq C \frac{1}{|B|} \int_B |b(y)| \, dy \leq C \| b \|_{BMO_L}. \]

In fact, we have proved that for an \( H^1 \)-atom \( a(x) \) with support \( B(x_0, r) \) with \( \rho(x_0) \leq r \leq 4\rho(x_0), \) if \( b \in BMO_L(R^n), \) then \( \| [b, T_3]a \|_{L^1} \leq C \| b \|_{BMO_L}. \)

**Case II.** For \( r < \rho(x_0), \) the atom \( a(x) \) has the cancelling condition \( \int_B a(x) \, dx = 0. \) For any \( u \in B, \)

\[
[b, T_3]a(x) = \chi_{33B}(x)[b, T_3]a(x) + \chi_{(33B)^c}(x)[b, T_3]a(x)
\]

\[
= \chi_{33B}(x)[b, T_3]a(x) + \chi_{(33B)^c}(x)(b(x) - b_B)T_3a(x)
\]

\[
- \chi_{(33B)^c}(x)T_3((b - b_B))a(x)
\]

\[
= \chi_{33B}(x)[b, T_3]a(x) + \chi_{(33B)^c}(x)(b(x) - b_B)T_3a(x)
\]

\[
- \chi_{(33B)^c}(x) \int [K_3(x, y) - K_3(x, u)](b(y) - b_B)a(y) \, dy
\]

\[
- \chi_{(33B)^c}(x) \int K_3(x, u)[b(y) - b_B]a(y) \, dy
\]

\[
=: I_1 + I_2 + I_3 + I_4.
\]

Clearly we can see that \( I_4 \) is the term in the integral of (ii) of Theorem 3.5. So we need only estimate \( I_i \) (i = 1, 2, 3) separately.

Because \([b, T_3] \) is bounded on \( L^p \) for \( 1 < p < p_1, \) then we have

\[
\| I_1 \|_{L^1} \leq \int_{33B} \| [b, T_3]a(x) \| \, dx
\]

\[
\leq C|B|^{1-1/p} \left( \int_{33B} \| [b, T_3]a(x) \|^p \, dx \right)^{1/p}
\]

\[
\leq C|B|^{1-1/p} \| a \|^p \| b \|_{BMO_L}
\]

\[
\leq C \| b \|_{BMO_L}.
\]
By the cancelling property of $a(x)$,

$$\|I_2\|_{L^1} \leq \int_{(33B)^c} |b(x) - b_B| |T_3a(x)| \, dx$$

$$\leq \int_{(33B)^c} |b(x) - b_B| \int_B |K_3(x, y) - K_3(x, x_0)||a(y)| \, dy$$

$$\leq \int_B |a(y)| \, dy \int_{(33B)^c} |b(x) - b_B||K_3(x, y) - K_3(x, x_0)| \, dx.$$

Because $y \in B(x_0, r)$ and $x \in (33B)^c$, we have $|y - x_0| < |x - x_0|/16$. By Lemma 2.6, setting $h = |y - x_0|$,

$$|K_3(x, y) - K_3(x, x_0)| \leq \frac{C_K}{\{1 + m(x_0, V)|x - x_0|\}^K} \frac{|y - x_0|^\delta}{|x - x_0|^{n-1+\delta}}$$

$$\times \left( \int_{B(x, |x-x_0|)} \frac{V(\xi)}{|x - \xi|^{n-1}} \, d\xi + \frac{1}{|x - x_0|} \right).$$

Naturally we divide the integral into two parts,

$$\int_{(33B)^c} |b(x) - b_B||K_3(x, y) - K_3(x, x_0)| \, dx$$

$$\leq \int_{(33B)^c} \frac{C_K |b(x) - b_B|}{\{1 + m(x_0, V)|x - x_0|\}^K} \frac{|y - x_0|^\delta}{|x - x_0|^{n-1+\delta}}$$

$$\times \left( \int_{B(x, |x-x_0|)} \frac{V(\xi)}{|x - \xi|^{n-1}} \, d\xi \right) \, dx$$

$$+ \int_{(33B)^c} |b(x) - b_B| \frac{C_K}{\{1 + m(x_0, V)|x - x_0|\}^K} \frac{|y - x_0|^\delta}{|x - x_0|^{n+\delta}} \, dx$$

$$=: I_{22} + I_{22}.$$

For $I_{22}$, because $\text{BMO}_L(R^n)$ is a subspace of $\text{BMO}(R^n)$, then $\|b\|_{\text{BMO}} \leq \|b\|_{\text{BMO}_L}$.

We have

$$I_{22} \leq \sum_{k=5}^{\infty} \int_{2^k r < |x-x_0| \leq 2^{k+1} r} |b(x) - b_B| \frac{C_K}{\{1 + m(x_0, V)|x - x_0|\}^K} \frac{|y - x_0|^\delta}{|x - x_0|^{n+\delta}} \, dx$$

$$\leq \sum_{k=5}^{\infty} \frac{C_K}{\{1 + m(x_0, V)2^k r\}^K (2^k r)^{n+\delta}} (2^{k+1} r)^n (k + 2) \|b\|_{\text{BMO}}$$

$$\leq C\|b\|_{\text{BMO}_L} \sum_{k=5}^{\infty} \frac{(k + 2)}{2^{k\delta}}$$

$$\leq C\|b\|_{\text{BMO}_L}.$$
For \( I_{21} \), by Hölder’s inequality and Lemma 1.5,

\[
I_{21} \leq \sum_{k=5}^{\infty} \frac{C_K}{(1+2^k r_m(x_0, V))^K} \int_{|x-x_0| \leq 2^k r} \frac{r^\delta |b(x) - b_B|}{(2^k r)^{n-1+\delta}} \times \left( \int_{B(x,|x-x_0|)} \frac{V(\xi)}{|x - \xi|^{n-1}} \, d\xi \right) \, dx
\]

\[
\leq \sum_{k=5}^{\infty} \frac{C_K \|b\|_{BMO}}{(1+2^k r_m(x_0, V))^K} \frac{(k+2)r^\delta}{(2^k r)^{n-1+\delta}} \frac{(2^k r)^n}{p'_1}
\times \left( \int_{B(x_0,2^k r)} \frac{V^q(\xi)}{|x - \xi|^{n-1}} \, d\xi \right)^{1/q}
\leq \sum_{k=5}^{\infty} \frac{C_K \|b\|_{BMO_L}}{(1+2^k r_m(x_0, V))^K} \frac{(k+2)r^\delta}{(2^k r)^{n-1+\delta}} \frac{(2^k r)^n}{p'_1+n/q-n}
\times \int_{B(x_0,r)} V(\xi) \, d\xi
\leq C \|b\|_{BMO_L} \sum_{k=5}^{\infty} \frac{(k+2)r^\delta}{(2^k r)^{n-1+\delta}} \frac{(2^k r)^n}{p'_1+n/q-n-2}
\leq C \|b\|_{BMO_L}.
\]

Finally, for \( \|I_3\|_{L^1} \), we get

\[
\|I_3\|_{L^1} \leq \int_{(33B)^c} \int_B |K_3(x, y) - K_3(x, u)||b(y) - b_B||a(y)| \, dy \, dx
= \int_B |b(y) - b_B||a(y)| \, dy \int_{(16B)^c} |K_3(x, y) - K_3(x, u)| \, dx.
\]

On the one hand, because \( u \in B \), we have \(|y-u| \leq |y-x_0| + |x_0-u| \leq 2r \). On the other hand, for \( x \in (33B)^c \), we have \(|x-u| > |x-x_0| - |u-x_0| > 32r \). Therefore \(|y-u| \leq 2r \leq |x-u|/16 \). By Lemma 2.6, setting \( h = |y-u| \),

\[
|K_3(x, y) - K_3(x, u)| \leq \frac{C_K}{(1+m(u, V)|x-u|)K} \frac{|y-u|^\delta}{|x-u|^{n-1+\delta}} \times \left( \int_{B(x,|x-u|)} \frac{V(\xi)}{|x - \xi|^{n-1}} \, d\xi + \frac{1}{|x-u|} \right).
\]
Similarly, we divide the integral of the above inequality into
\[ \int_{(33B)^c} |K_3(x, y) - K_3(x, u)| \, dx = I_{31} + I_{32}. \]

For \( I_{32} \), we have
\[
I_{32} \leq \sum_{k=5}^{\infty} \int_{2^k r \leq |x-u| \leq 2^{k+1} r} \frac{C_K}{(1 + m(u, V)|x-u|)^K} \frac{|y-u|^{\delta}}{|x-u|^{n+\delta}} \, dx \\
\leq C \sum_{k=5}^{\infty} \frac{r^{\delta}}{(2^k r)^{n+\delta}} (2^{k+1} r)^n \\
\leq C.
\]

For \( I_{31} \), notice that every \( \xi \in B(x, |x-u|) \), and \( |\xi - u| \leq 2|x-u| \). If \( |x-u| \leq 2^k r \), then \( |\xi - u| \leq 2^{k+2} r \). So we have
\[
I_{31} \leq \sum_{k=5}^{\infty} \frac{C_K}{(1 + m(u, V)|2^k r|)^K} \frac{r^{\delta}}{(2^k r)^{n-1+\delta}} (2^{k+1} r)^{n/p_1'} \\
\times \left\| \int V(\xi) \chi_{B(u, 2^k r)}(\xi) \frac{|x-\xi|^{n-1}}{|x-u|^{n-1}} \, d\xi \right\|_{L^{p_1}(dx)} \\
\leq \sum_{k=5}^{\infty} \frac{C_K}{(1 + m(u, V)|2^k r|)^K} \frac{r^{\delta}}{(2^k r)^{n-1+\delta}} (2^{k+1} r)^{n/p_1'} \left( \int_{B(u, 2^k r)} V^q(\xi) \, d\xi \right)^{1/q} \\
\leq \sum_{k=5}^{\infty} \frac{C_K}{(2^k r)^{\delta}} (2^{k+1} r)^{n/p_1'+n/q-n-2} \leq C.
\]

Then we have \( \|I_3\|_{L^1} \leq \int_B |b(y) - b_B| |a(y)| \, dy \leq (1/|B|) \int_B |b(y) - b_B| \, dy \leq \|b\|_{BMO_L} \). Finally, the estimate of \( \|I_i\|_{L^1} \) \( (i = 1, 2, 3) \) implies that, for an \( H^1_L \)-atom \( a(x) \), \( \|T_3a(x)\|_{L^1} \leq C \) if and only if \( \|I_4\|_{L^1} \leq C \). This completes the proof of Theorem 3.5.

**Counterexample 3.6.** From Theorem 3.5, we find that the commutator \([b, T_3]\) may not be bounded from \( H^1_L(R^n) \) into \( L^1(R^n) \). We use a simple example to imply this conclusion. If we choose \( r \) small enough such that \( 33r < \rho(x_0) \),
\[
\int_{|x-x_0|>33r} |K_3(x, x_0)| \, dx \\
\geq \int_{|x-x_0|>33r} |R(x, x_0)| \, dx - \int_{|x-x_0|>33r} |K_3(x, x_0) - R(x, x_0)| \, dx
\]
\[
\begin{align*}
\geq & \int_{|x-x_0|>33r} |R(x, x_0)| \, dx - \int_{|x-x_0|>\rho(x_0)} |K_3(x, x_0)| \, dx \\
& - \int_{|x-x_0|>\rho(x_0)} |R(x, x_0)| \, dx \\
& - \int_{33r<|x-x_0|\leq\rho(x_0)} |K_3(x, x_0) - R(x, x_0)| \, dx \\
\geq & \int_{33r<|x-x_0|<\rho(x_0)} |R(x, x_0)| \, dx - \int_{|x-x_0|>\rho(x_0)} |K_3(x, x_0)| \, dx \\
& - \int_{33r<|x-x_0|<\rho(x_0)} |K_3(x, x_0) - R(x, x_0)| \, dx \\
=: & \ M_1 - M_2 - M_3.
\end{align*}
\]

Shen [7] proved that there exist constants \( C_1, C_2 \) such that \( M_2 \leq C_1 \) and \( M_3 \leq C_2 \). Then by Theorem 3.5, if \( [b, T_3] \) is bounded from \( H^1_L \) to \( L^1 \), then
\[
\left( \int_{33r<|x-x_0|<\rho(x_0)} |R(x, x_0)| \, dx - C_1 - C_2 \right) \left| \int b(y)a(y) \, dy \right| \leq C
\]
where \( |R(x, x_0)| = 1/|x-x_0|^n \). If we set \( V(x) = 1 \) for convenience, then by Definition 3.1, it is easy to see that \( \rho(x_0) = 1 \). By Definition 3.1, because \( r \) is the radius of the atom \( a(x) \), then \( r \leq 2^{1-n/2} \). This means that if \( n \) is large enough,
\[
\left( C \frac{n}{2} - C_1 - C_2 \right) \left| \int b(y)a(y) \, dy \right| \leq \left( \ln \frac{1}{33r} - C_1 - C_2 \right) \left| \int b(y)a(y) \, dy \right| \leq C,
\]
that is,
\[
\left| \int b(y)a(y) \, dy \right| \to 0 \quad \text{when} \ r \to 0 \ (n \to \infty). \quad (\ast)
\]

Unfortunately the conclusion \((\ast)\) is not true for a general atom \( a(x) \). For example, we set
\[
b(x) = \log |x|, \quad \text{when} \ |x| \leq 1, \quad b(x) = 0, \quad \text{otherwise};
a_k(x) = \begin{cases} -2^k, & \text{when} \ x \in \left( 0, \frac{1}{2^{k+1}} \right), \\ 2^k, & \text{when} \ x \in \left( \frac{1}{2^{k+1}}, \frac{1}{2^k} \right). \end{cases}
\]

It can be proved that \( b(x) \in \text{BMO}_L(R^n) \) and \( a_k(x), k \in \mathbb{Z}^+ \) are \( H^1_L \)-atoms. We have, for every \( k \in \mathbb{Z}^+, \left| \int b(y)a_k(y) \, dy \right| = \ln 2 \), which is contrary to the conclusion \((\ast)\).

4. \((H^1_L, L^1_{\text{weak}})\)-boundedness of \([b, T_i], i = 1, 2, 3\)

The counterexample in Section 3 implies that, if \( b \in \text{BMO}_L(R^n) \) and \( b \) is nonzero in the \( \text{BMO}_L \) norm, we cannot guarantee that the commutators \([b, T_i] (i = 1, 2, 3)\) are bounded from \( H^1_L(R^n) \) into \( L^1(R^n) \). In this section we prove that if \( L^1 \) is replaced by a larger space, namely \( L^1_{\text{weak}}(R^n) \), then the \([b, T_i] (i = 1, 2, 3)\) are bounded on \( H^1_L(R^n) \).
THEOREM 4.1. Suppose that $V \in B_q$, $q > n/2$. Let $T_1 = V(x)(-\Delta + V)^{-1}$, $T_2 = V^{1/2}(-\Delta + V)^{-1/2}$ and $T_3 = \nabla(-\Delta + V)^{-1/2}$. For $b \in \text{BMO}$, the commutators $[b, T_i]$ ($i = 1, 2, 3$) are bounded from $H^1_L(R^n)$ into $L^1_{\text{weak}}(R^n)$.

PROOF. For convenience, we prove the $(H^1_L, L^1_{\text{weak}})$-boundedness of $[b, T_3]$. The proofs for $[b, T_1]$ and $[b, T_2]$ are similar. From Theorem 3.2, we know that for every $f \in H^1_L$, there exist a sequence of $H^1_L$-atoms $\{a_j(x)\}$ and a sequence of $\{\lambda_j\}$ for $j \in \mathbb{Z}$ such that $f = \sum_j \lambda_j a_j(x)$ and $\sum_j |\lambda_j| \leq \|f\|_{H^1_L}$. If we set the support of $a_j(x)$ as $B_j = B(x_j, r_j)$, then $r_j \leq 4\rho(x_0)$ by Definition 3.1. Therefore,

$$[b, T_3]f(x) = \sum_j \lambda_j [b, T_3]a_j(x)$$

$$= \sum_{r_j < \rho(x_j)} \lambda_j [b, T_3]a_j(x) + \sum_{\rho(x_j) \leq r_j < 4\rho(x_j)} \lambda_j [b, T_3]a_j(x)$$

$$=: \sum_1 \lambda_j [b, T_3]a_j(x) + \sum_2 \lambda_j [b, T_3]a_j(x),$$

where we denote

$$\sum_{r_j < \rho(x_j)} \lambda_j [b, T_3]a_j(x) \quad \text{by} \quad \sum_1 \lambda_j [b, T_3]a_j(x)$$

and

$$\sum_{\rho(x_j) \leq r_j < 4\rho(x_j)} \lambda_j [b, T_3]a_j(x) \quad \text{by} \quad \sum_2 \lambda_j [b, T_3]a_j(x).$$

Then

$$|\{x : |[b, T_3]f(x)| > \lambda\}| \leq \left| \left\{ x : \sum_1 \lambda_j [b, T_3]a_j(x) > \lambda/2 \right\} \right| + \left| \left\{ x : \sum_2 \lambda_j [b, T_3]a_j(x) > \lambda/2 \right\} \right|. $$

Hence we need to estimate $|\{x : \sum_i \lambda_j [b, T_3]a_j(x) > \lambda/2\}|$, $i = 1, 2$, separately.

Step I. First, we estimate $\left| \left\{ x : \sum_1 \lambda_j [b, T_3]a_j(x) > \lambda/2 \right\} \right|$. We have

$$\left| \left\{ x : \sum_1 \lambda_j [b, T_3]a_j(x) > \lambda/2 \right\} \right| \leq \left| \left\{ x : \sum_1 \lambda_j (b(x) - b_{B_j})T_3a_j(x) \chi_{(16B_j)}(x) > \lambda/6 \right\} \right| + \left| \left\{ x : \sum_1 \lambda_j (b(x) - b_{B_j})T_3a_j(x) \chi_{(16B_j)^c}(x) > \lambda/6 \right\} \right| + \left| \left\{ x : \sum_1 \lambda_j T_3((b - b_{B_j})a_j)(x) > \lambda/6 \right\} \right|$$

$$=: I_1 + I_2 + I_3.$$
For $I_1$, because $T_3$ is bounded on $L^p(\mathbb{R}^n)$, $1 < p < p_1$, $1/p_1 = 1/q - 1/n$,
\[
I_1 = \left| \left\{ x : \sum_{j=1}^\infty \lambda_j (b(x) - b_{B_j}) T_3 a_j(x) \chi_{(16B_j)^c}(x) > \lambda / 6 \right\} \right| 
\leq \frac{C}{\lambda} \sum_{j=1}^\infty |\lambda_j| \int_{16B_j} |b(x) - b_{B_j}| |T_3 a_j(x)| \, dx 
\leq \frac{C}{\lambda} \sum_{j=1}^\infty |\lambda_j| \left( \int_{16B_j} |b(x) - b_{B_j}|^2 \, dx \right)^{1/2} \left( \int_{16B_j} |T_3 a_j(x)|^2 \, dx \right)^{1/2} 
\leq \frac{C}{\lambda} \sum_{j=1}^\infty |\lambda_j| \|B_j\|^{1/2} \|b\|_{\text{BMO}} \|a_j\|_2 
\leq \frac{C}{\lambda} \sum_{j=1}^\infty |\lambda_j| \|b\|_{\text{BMO}}.
\]

For $I_3$, by Theorem 2.7, $T_3$ is of weak-type $(1, 1)$. Using Hölder’s inequality,
\[
\left| \left\{ x : \sum_{j=1}^\infty \lambda_j T_3((b - b_{B_j})a_j)(x) > \lambda / 6 \right\} \right| 
\leq \frac{C}{\lambda} \sum_{j=1}^\infty \int_{B_j} |b(x) - b_{B_j}| |a_j(x)| \, dx 
\leq \frac{C}{\lambda} \sum_{j=1}^\infty |\lambda_j| \|b\|_{\text{BMO}}.
\]

For $I_2$, the atom $a_j$ has the cancelling property when $r_j \leq \rho(x_j)$. We have
\[
\left| \left\{ x : \sum_{j=1}^\infty \lambda_j (b(x) - b_{B_j}) T_3 a_j(x) \chi_{(16B_j)^c}(x) > \lambda / 6 \right\} \right| 
\leq \frac{C}{\lambda} \sum_{j=1}^\infty |\lambda_j| \int_{(16B_j)^c} |b(x) - b_{B_j}| \times |T_3 a_j(x)| \, dx 
\leq \frac{C}{\lambda} \sum_{j=1}^\infty |\lambda_j| \int_{(16B_j)^c} |b(x) - b_{B_j}| \left| \int_{B_j} |K_3(x, y) - K_3(x, x_j)| a_j(y) \, dy \right| \, dx 
\leq \frac{C}{\lambda} \sum_{j=1}^\infty |\lambda_j| \int_{B_j} |a_j(y)| \, dy \int_{(16B_j)^c} |b(x) - b_{B_j}| |K_3(x, y) - K_3(x, x_j)| \, dx.
\]

We set $I_{2, y} = \int_{(16B_j)^c} |b(x) - b_{B_j}| |K_3(x, y) - K_3(x, x_j)| \, dx$. Because $y \in B_j$, $|y - x_j| < r_j$ and $x \in (16B_j)^c$, $|x - x_j| > 16r_j$, then $|y - x_j| \leq |x - x_j|/16$. By (2.6) of Lemma 2.6,
\[
|K_3(x, y) - K_3(x, x_j)| \leq \frac{C_K}{\left(1 + m(x_j, V)|x - x_j|\right)^K} \frac{|y - x_j|^\delta}{|x - x_j|^{n-1+\delta}} \times \left( \int_{B(x, |x - x_j|)} \frac{V(u)}{|x - u|^{n-1}} \, du + \frac{1}{|x - x_j|} \right).
\]
Then
\[ I_{2,y} = \int_{(16B)\gamma} |b(x) - b_{B_j}| |K_3(x, y) - K_3(x, x_j)| \, dx \]
\[ \leq \int_{(16B)\gamma} \frac{C_K |b(x) - b_{B_j}| |y - x_j|^\delta}{\{1 + m(x, V)|x - x_j|\}^K |x - x_j|^{n-1+\delta}} \times \left( \int_{B(x, |x-x_j|)} \frac{V(u)}{|x-u|^n} \, du \right) \, dx \]
\[ + \int_{(16B)\gamma} |b(x) - b_{B_j}| \frac{|y - x_j|^\delta}{|x - x_j|^{n+\delta}} \, dx \]
\[ =: I_{14,y}^1 + I_{14,y}^2. \]

For \(I_{2,y}^2\), we have
\[ I_{2,y}^2 = \int_{(16B)\gamma} |b(x) - b_{B_j}| \frac{|y - x_j|^\delta}{|x - x_j|^{n+\delta}} \, dx \]
\[ \leq \sum_{k=4}^\infty \int_{2^k r_j \leq |x-x_j|<2^{k+1}r_j} |b(x) - b_{B_j}| \frac{|y - x_j|^\delta}{|x - x_j|^{n+\delta}} \, dx \]
\[ \leq \sum_{k=4}^\infty \frac{r_j^\delta}{(2^k r_j)^{n+\delta}} \int_{|x-x_j|<2^{k+1}r_j} |b(x) - b_{B_j}| \, dx \]
\[ \leq C\|b\|_{BMO} \sum_{k=4}^\infty \frac{(k + 2)r_j^\delta}{(2^k r_j)^{n+\delta}} (2^{k+1} r_j)^n \]
\[ \leq C\|b\|_{BMO}. \]

For \(I_{2,y}^1\), we have
\[ I_{2,y}^1 \leq \sum_{k=4}^\infty \int_{2^k r_j \leq |x-x_j|<2^{k+1}r_j} \frac{C_K |b(x) - b_{B_j}| |y - x_j|^\delta}{\{1 + m(x, V)|x - x_j|\}^K |x - x_j|^{n-1+\delta}} \times \left( \int_{B(x, |x-x_j|)} \frac{V(u)}{|x-u|^n} \, du \right) \, dx \]
\[ \leq \sum_{k=4}^\infty \frac{C_K}{\{1 + m(x, V)2^k r_j\}^K} \frac{r_j^\delta}{(2^k r_j)^{n-1+\delta}} \int_{2^k r_j \leq |x-x_j|<2^{k+1}r_j} |b(x) - b_{B_j}| \times \left( \int_{B(x, |x-x_j|)} \frac{V(u)}{|x-u|^n} \, du \right) \, dx. \]

Because every \(u \in B(x, |x-x_j|)\) implies that \(|u - x_j| \leq 2|x - x_j| \leq 2^{k+2}r_j\) for \(|x - x_j| < 2^{k+1}r_j\), then by Hölder’s inequality and Lemma 1.5.
We estimate Step II. Similar to the proof of step I, using the

\[
I_2 = \sum_{k=4}^{\infty} \frac{C_K}{\{1 + m(x_j, V)^{2k}r_j\}^K} \frac{r_j^\delta}{(2k r_j)^{n-1+\delta}}
\]

\[
\times \left( \int_{|x-x_j| < 2^{k+1}r_j} |b(x) - b_{B_j}|^{p_1'} dx \right)^{1/p_1'}
\]

\[
\times \left| \int \frac{V(u) \chi_{B(x_j, 2^{k+2}r_j)}(u)}{|x - u|^{n-1}} du \right|_{L^{p_1}(dx)}
\]

\[
\leq C \sum_{k=4}^{\infty} \frac{\||b||_{BMO}}{\{1 + m(x_j, V)^{2k}r_j\}^K} \frac{(k + 2)r_j^\delta}{(2k r_j)^{n-1+\delta}}
\]

\[
\times \left( \int_{B(x_j, 2^{k+2}r_j)} V(u) du \right)^{1/q}
\]

\[
\leq C \sum_{k=4}^{\infty} \frac{(k + 2)r_j^\delta}{(2k r_j)^{n-1+\delta}} \frac{(2k+1) r_j^{n/p_1'+n/q-n}}{\{1 + m(x_j, V)^{2k}r_j\}^K}
\]

\[
\times \left[ \int_{B(x_j, 2^{k+2}r_j)} V(u) du \right]^{1/q}
\]

\[
\leq C \|b\|_{BMO} \sum_{k=4}^{\infty} \frac{(k + 2)r_j^\delta}{(2k r_j)^{n-1+\delta}} \frac{(2k+1) r_j^{n/p_1'+n/q-n-n-2}}{
\}

\] where we have used the fact that, for \(1/q = 1/p - 1/n, n/p_1' + n/q - n + n - 2 = n - 1\). Then

\[
I_2 \leq \frac{C}{\lambda} \sum_{1} |\lambda_j| \int_{B_j} |a_j(y)|(I_{14, y}) dy \leq \frac{C}{\lambda} \|b\|_{BMO} \sum_{1} |\lambda_j|.
\]

**Step II.** We estimate \(|\{x : |\sum_2 \lambda_j [b, T_3]a_j(x) > \lambda/2\}|. Notice that in this case, \(\rho(x_j) \leq r_j \leq \rho(x_0)\), the atom \(a_j(x)\) has no cancelling property. Similarly,

\[
\left| \left\{ x : \sum_2 \lambda_j [b, T_3]a_j(x) > \lambda/2 \right\} \right|
\]

\[
\leq \left| \left\{ x : \sum_2 \lambda_j (b(x) - b_{B_j})T_3a_j(x) \chi_{(2B_j)}(x) > \lambda/6 \right\} \right|
\]

\[
+ \left| \left\{ x : \sum_2 \lambda_j (b(x) - b_{B_j})T_3a_j(x) \chi_{(2B_j)\complement}(x) > \lambda/6 \right\} \right|
\]

\[
+ \left| \left\{ x : \sum_2 \lambda_j T_3((b - b_{B_j})a_j)(x) > \lambda/6 \right\} \right|
\]

\[
=: I_4 + I_5 + I_6.
\]

Similar to the proof of step I, using the \(L^p\)- and \((L^1, L^1)\)-boundedness of \(T_3\),

\[
I_4 \leq \frac{C}{\lambda} \|b\|_{BMO} \sum_2 |\lambda_j| \quad \text{and} \quad I_6 \leq \frac{C}{\lambda} \|b\|_{BMO} \sum_2 |\lambda_j|.
\]
For $I_5$, we have
\[
I_5 = \left\{ x : \left| \sum_j \lambda_j (b(x) - b_{B_j}) T_3 a_j(x) (x_{2B_j}) \right| > \lambda / 6 \right\}
\]
\[
\leq \frac{C}{\lambda} \sum_j |\lambda_j| \int_{(2B_j)^c} |b(x) - b_{B_j}| |T a_j(x)| \, dx
\]
\[
\leq \frac{C}{\lambda} \int_{B_j} |a_j(y)| \, dy \int_{(2B_j)^c} |b(x) - b_{B_j}| |K_3(x, y)| \, dx.
\]
We set $I_{5,y} = \int_{(2B_j)^c} |b(x) - b_{B_j}| |K_3(x, y)| \, dx$. By (2.5) of Lemma 2.6,
\[
|K_3(x, y)| \leq \frac{C_K}{\{1 + m(y, V)|x - y|\}^K} \frac{1}{|x - y|^{n-1}}
\times \left( \int_{B(x,|x-y|)} \frac{V(u)}{|x - u|^{n-1}} \, du + \frac{1}{|x - y|} \right).
\]
Then
\[
I_{5,y} = \int_{(2B_j)^c} |b(x) - b_{B_j}| |K_3(x, y)| \, dx
\]
\[
\leq \int_{(2B_j)^c} \frac{C_K |b(x) - b_{B_j}|}{\{1 + m(y, V)|x - y|\}^K} \frac{1}{|x - y|^{n-1}} \left( \int_{B(x,|x-y|)} \frac{V(u)}{|x - u|^{n-1}} \, du + \frac{1}{|x - y|} \right) \, dx
\]
\[
+ \int_{(2B_j)^c} \frac{C_K |b(x) - b_{B_j}|}{\{1 + m(y, V)|x - y|\}^K} \frac{1}{|x - y|^n} \, dx
\]
\[=: I_{5,y}^1 + I_{5,y}^2.
\]
For $I_{5,y}^2$, we have
\[
I_{5,y}^2 = \int_{(2B_j)^c} \frac{C_K |b(x) - b_{B_j}|}{\{1 + m(y, V)|x - y|\}^K} \frac{1}{|x - y|^n} \, dx
\]
\[
\leq \sum_{k=1}^{\infty} \frac{C_K}{\{1 + m(y, V)2^{k-1}r_j\}^K} \frac{1}{(2^{k-1}r_j)^n} \int_{2^k r_j \leq |x-x_j| < 2^{k+1} r_j} |b(x) - b_{B_j}| \, dx
\]
\[
\leq \sum_{k=1}^{\infty} (k+2) \frac{C_K \|b\|_{\text{BMO}}}{\{1 + m(y, V)2^{k-1}r_j\}^K}
\]
\[
\leq C \|b\|_{\text{BMO}} \sum_{k=1}^{\infty} \frac{C_K (k+2)}{(1 + 2^{k-1})^K}
\]
\[
\leq C \|b\|_{\text{BMO}}.
\]
Here, in the second inequality, we used the fact that because $y \in B_j$, $|y - x_j| < r_j$, then $|x - y| > |x - x_j| - |y - x_j| > 2^{k-1} r_j$ for $2^k r_j \leq |x-x_j| < 2^{k+1} r_j$. In the fourth inequality, we used the fact that because $\rho(x_j) \leq r_j \leq \rho(x_0)$, then $|y - x_j| < r_j < 4 \rho(x_j)$, $m(y, V) \sim m(x, V)$ and $1 \leq r_j m(x_j, V) \leq 4$. 
Finally, we estimate $I_{5,y}^1$. For every $u \in B(x, |x - y|)$, $|u - x| < |y - x_j| + |x - x_j|$, then for $2^kr_j \leq |x - x_j| < 2^k r_j$, we have $|x - y| > 2^{k-1}r_j$ and $|u - x_j| < |x - u| + |x - x_j| < |y - x_j| + 2|x - x_j| < 2^{k+3}r_j$. Using Hölder’s inequality,

$$I_{5,y}^1 = \int_{(2B_j)^c} \frac{C_K |b(x) - b_{B_j}|}{\{1 + m(y, V)|x - y|\}^{K}} \frac{1}{|x - y|^{n-1}} \left( \int_{B(x, |x - y|)} V(u) \frac{1}{|x - u|^{n-1}} \, du \right) \, dx \leq \sum_{k=1}^{\infty} \frac{C_K \|b\|_{\text{BMO}}}{\{1 + m(y, V)2^{k-1}r_j\}^{K}} \frac{(k + 2)(2^{k+1}r_j)^{n/\rho_1'}}{(2^{k-1}r_j)^{n-1}} \left( \int_{|x - x_j| < 2^{k+3}r_j} V^q(x) \, dx \right)^{1/q} \times \left( \int_{|x - x_j| < 2^{k+1}r_j} V^{q'}(x) \, dx \right)^{1/q'} \times \left( \int_{|x - x_j| < 2^{k+1}r_j} \frac{1}{|x - u|^{n-1}} \, du \right)^{1/p_1}.$$

Because $y \in B(x_j, r)$, we have $|y - x_j| < 4 \rho(x_j)$ and $m(x_j, V) \sim m(y, V)$. For $\rho(x_j) \leq r_j \leq 4 \rho(x_j)$, we have $1 \leq m(x_j, V)r_j \leq 4$. By Lemma 1.4 and the fractional integral,

$$I_{5,y}^1 \leq C \sum_{k=1}^{\infty} \frac{C_K \|b\|_{\text{BMO}}}{\{1 + m(y, V)2^{k-1}r_j\}^{K}} \frac{(k + 2)(2^{k+1}r_j)^{n/\rho_1'}}{(2^{k-1}r_j)^{n-1}} \left( \int_{|x - x_j| < 2^{k+3}r_j} V^{q}(x) \, dx \right)^{1/q} \times \left( \int_{|x - x_j| < 2^{k+3}r_j} V^{q'}(x) \, dx \right)^{1/q'} \times \left( \int_{|x - x_j| < 2^{k+1}r_j} \frac{1}{|x - u|^{n-1}} \, du \right)^{1/p_1} \leq C \sum_{k=1}^{\infty} \frac{C_K \|b\|_{\text{BMO}}}{\{1 + m(y, V)2^{k-1}r_j\}^{K}} \frac{(k + 2)(2^{k+1}r_j)^{n/\rho_1'+n/q-n}}{(2^{k-1}r_j)^{n-1}} \left( \int_{|x - x_j| < 2^{k+3}r_j} V(x) \, dx \right) \leq C \sum_{k=1}^{\infty} \frac{C_K \|b\|_{\text{BMO}}}{\{1 + m(y, V)2^{k-1}r_j\}^{K}} \frac{(k + 2)(2^{k+1}r_j)^{n/\rho_1'+n/q-n}}{(2^{k-1}r_j)^{n-1}} \left( \int_{|x - x_j| < 2^{k+3}r_j} V(x) \, dx \right) \leq C \sum_{k=1}^{\infty} \frac{C_K \|b\|_{\text{BMO}}}{\{1 + m(y, V)2^{k-1}r_j\}^{K}} \frac{(k + 2)(2^{k+1}r_j)^{n/\rho_1'+n/q-n}}{(2^{k-1}r_j)^{n-1}} \left( \int_{|x - x_j| < 2^{k+3}r_j} V(x) \, dx \right) \leq C \|b\|_{\text{BMO}} \sum_{k=1}^{\infty} \frac{C_K}{\{1 + 2^{k-1}r_j\}^{K}} \frac{(k + 2)(2^{k-1}r_j)^{n/\rho_1'+n/q-n}}{(2^{k-1}r_j)^{n-1}} \leq C \|b\|_{\text{BMO}}.$$

Finally, we obtain

$$I_{5} \leq \frac{C}{\lambda} \sum_{j} |\lambda_j| \int_{B_j} |a_j(y)| (I_{5,y}) \, dy \leq \frac{C}{\lambda} \|b\|_{\text{BMO}} \sum_{j} |\lambda_j|.$$

This completes the proof of Theorem 4.1. \qed
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References


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