TOTALLY REAL PSEUDO-UMBILICAL SUBMANIFOLDS OF A QUATERNION SPACE FORM

by HUAFEI SUN†

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1. Introduction. Let $M(c)$ denote a $4n$-dimensional quaternion space form of quaternion sectional curvature $c$, and let $P(H)$ denote the $4n$-dimensional quaternion projective space of constant quaternion sectional curvature 4. Let $N$ be an $n$-dimensional Riemannian manifold isometrically immersed in $M(c)$. We call $N$ a totally real submanifold of $M(c)$ if each tangent 2-plane of $N$ is mapped into a totally real plane in $M(c)$. B. Y. Chen and C. S. Houh proved in [1].

**THEOREM A.** Let $M$ be an $n$-dimensional compact totally real minimal submanifold of the quaternion projective space $P(H)$. If

$$S \leq \frac{3n(n+1)}{(6n-1)},$$

then $N$ is totally geodesic. Here $S$ is the square of the length of the second fundamental form of $N$.

Let $h$ be the second fundamental form of the immersion, and $\xi$ the mean curvature vector. Let $\langle \cdot, \cdot \rangle$ denote the scalar product of $M(c)$. If there exists a function $\lambda$ on $N$ such that

$$\langle h(X, Y), \xi \rangle = \lambda \langle X, Y \rangle$$

for any tangent vector $X, Y$ on $N$, then $N$ is called a *pseudo-umbilical submanifold* of $M(c)$. It is clear that $\lambda \geq 0$. If the mean curvature vector $\xi = 0$ identically, then $N$ is called a minimal submanifold of $M(c)$. Every minimal submanifold of $M(c)$ is itself a pseudo-umbilical submanifold of $M(c)$. In this paper, we consider the case when $N$ is pseudo-umbilical and extend Theorem A. Our main results are

**THEOREM 1.** Let $N$ be an $n$-dimensional compact totally real pseudo-umbilical submanifold of $M(c)$. Then

$$\int_N \left\{ 6S^2 - [(n + 1)c + 16n^2H^2]S + 4n^2H^2c + 10n^2H^4 \right\} dN \geq 0,$$

where $H$ and $dN$ denote the mean curvature of $N$ and the volume element of $N$ respectively.

**THEOREM 2.** Let $N$ be an $n$-dimensional compact totally real submanifold of $M(c)$. If

$$6S^2 - [(n + 1)c + 16n^2H^2]S + 4n^2H^2c + 10n^2H^4 - 4nH^2H \leq 0,$$  \hspace{1cm} (1.1)

then the second fundamental form of $N$ is parallel. In particular, if the equality of (1.1) holds, then either $N$ is totally geodesic or $N$ is flat.

When $H = 0$, i.e. $N$ is minimal, from Theorem 1 we may get (cf. [4]).

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COROLLARY. Let $N$ be an $n$-dimensional compact totally real minimal submanifold of $P(H)$. If 
\[ S \leq \frac{3}{2}(n + 1), \]
then $N$ is totally geodesic or $S = \frac{3}{2}(n + 1)$.

2. Local formulas. We use the same notation and terminologies as in [1] unless otherwise stated. Let $M(c)$ denote a $4n$-dimensional quaternion space form of quaternion sectional curvature $c$, and let $N$ be an $n$-dimensional totally real submanifold of $M(c)$. We choose a local field of orthonormal frames, 
\[ e_1, \ldots, e_n, \ e_{l(1)} = le_1, \ldots, e_{l(n)} = le_n, \ e_{j(1)} = je_1, \ldots, e_{j(n)} = je_n, \ e_{k(1)} = ke_1, \ldots, e_{k(n)} = ke_n, \]
in such a way that when restricted to $N$, $e_1, \ldots, e_n$ are tangent to $N$. Here $I, J, K$ are the almost Hermitian structures and satisfy
\[ JJ = -JI = K, \ JK = -KJ = I, \ KI = -IK = J, \ I^2 = J^2 = K^2 = -1. \]

We shall use the following convention on the range of indices:
\[ A, B, \ldots = 1, \ldots, n, \ I(1), \ldots, I(n), \ J(1), \ldots, J(n), \ K(1), \ldots, K(n), \alpha, \beta, \ldots = 1(1), \ldots, I(1), \ldots, J(1), \ldots, J(n), \ K(1), \ldots, K(n), \]
\[ i, j, \ldots = 1, \ldots, n, \phi = I, J, K. \]

Let $\{\omega_A\}$ be the dual frame field. Then the structure equations of $M(c)$ are given by
\[ d\omega_A = -\sum_B \omega_{AB} \wedge \omega_B, \ \omega_{AB} + \omega_{BA} = 0, \]
\[ d\omega_{AB} = -\sum_C \omega_{AC} \wedge \omega_{CB} + \frac{1}{2} \sum_{CD} K_{ABCD} \omega_C \wedge \omega_D, \]
\[ K_{ABCD} = \frac{c}{4} (\delta_{AC} \delta_{BD} - \delta_{AD} \delta_{BC} + IAClBD - IADlBC + 2IABlCD + JACJBD - JADJBC + 2IABKCD). \]

Restricting these forms to $N$, we get the following structure equations of the immersion:
\[ d\omega_\alpha = 0, \ \omega_{\alpha i} = \sum_j h_i^\alpha_j \omega_j, \ h_i^\alpha = h_j^\alpha, \ h_k^{\phi(i)} = h_k^{\phi(j)} = h_k^{\phi(k)}, \]
\[ d\omega_{ij} = -\sum_k \omega_{jk} \wedge \omega_{ki} + \frac{1}{2} \sum_{kl} R_{ijkl} \omega_k \wedge \omega_l, \]
\[ R_{ijkl} = K_{ijkl} + \frac{1}{2} \sum_\alpha (h_i^\alpha_j h_j^\alpha_k - h_i^\alpha_k h_j^\alpha_j), \]
\[ d\omega_{\alpha\beta} = -\sum_\gamma \omega_{\alpha\gamma} \wedge \omega_{\gamma\beta} + \frac{1}{2} \sum_{ij} R_{\alpha\beta ij} \omega_i \wedge \omega_j, \]
\[ R_{\alpha\beta ij} = K_{\alpha\beta ij} + \frac{1}{2} \sum_k (h_i^\alpha_k h_j^\beta_k - h_i^\beta_k h_j^\alpha_k). \]
We call \( h = \sum_{ij\alpha} h_{ij\alpha} \omega_i \omega_j \epsilon_\alpha \) the second fundamental form of the immersed manifold \( N \). We denote by \( S = \sum_{ij\alpha} (h_{ij\alpha}^2) \) the square of the length of \( h \). \( \xi = \frac{1}{n} \sum_{\alpha} \text{tr} \ H_{\alpha} \epsilon_\alpha \) and \( H = \frac{1}{n} \sqrt{\sum_{\alpha} (\text{tr} \ H_{\alpha})^2} \) denote the mean curvature vector and the mean curvature of \( N \), respectively. Here \( \text{tr} \) is the trace of the matrix \( H_{\alpha} \). Now, let \( e_{k(n)} \) be parallel to \( \xi \). Then we have
\[
\text{tr} \ H_{k(n)} = nH, \quad \text{tr} \ H_{\alpha} = 0, \quad \alpha \neq k(n). \tag{2.4}
\]

We define \( h^\alpha_{ijk} \) and \( h^\alpha_{ijkl} \) by
\[
\sum_k h^\alpha_{ijk} \omega_k = dh^\alpha_{ij} - \sum_l h^\alpha_{ijl} \omega_l - \sum_l h^\alpha_{ijl} \omega_l + \sum_\beta h^\beta_{ijl} \omega_\beta, \tag{2.5}
\]
and
\[
\sum_l h^\alpha_{ijlk} \omega_l = dh^\alpha_{ijk} - \sum_l h^\alpha_{ijlk} \omega_l - \sum_l h^\alpha_{ijlk} \omega_l - \sum_l h^\alpha_{ijlk} \omega_l + \sum_\beta h^\beta_{ijlk} \omega_\beta,
\]
respectively. Where
\[
h^\alpha_{ijk} = h^\alpha_{ikj}.
\]
and
\[
h^\alpha_{ijkl} - h^\alpha_{ijkl} = \sum_m h^\alpha_{im} R_{mkj} + \sum_m h^\alpha_{mj} R_{mkl} - \sum_\beta h^\beta_{ij} R_{\alpha kl}. \tag{3.1}
\]

The Laplacian \( h^\alpha_{ij} \) of the second fundamental form \( h^\alpha_{ij} \) is defined by \( \Delta h^\alpha_{ij} = \sum_k h^\alpha_{ijk k} \). By a direct calculation we have (cf. [1, 2, 3])
\[
\frac{1}{2} \Delta S = \sum_{ij\alpha} (h_{ij\alpha}^2)^2 + \sum_{ij\alpha} h_{ij\alpha}^2 \Delta h_{ij\alpha}^2
\]
\[
= \sum_{ij\alpha} (h_{ij\alpha}^2)^2 + \sum_{ij\alpha} h_{ij\alpha}^2 h_{ij\alpha k}^2 + \sum_{ij\alpha} h_{ij\alpha h}^2 \epsilon_{h jk} - \sum_{ij\alpha} h_{ij\alpha}^2 h_{ij\alpha k} R_{ij\alpha k} + \sum_{ij\alpha} h_{ij\alpha}^2 R_{ij\alpha k} + \sum_{ij\alpha} h_{ij\alpha}^2 R_{ij\alpha k}. \tag{2.6}
\]

3. Proofs of Theorems. From (*) and (2.4) we get \( \sum_{\alpha} \text{tr} \ H_{\alpha} h_{ij\alpha} = n\lambda \delta_{ij} \), \( H^2 = \lambda \) and
\[
h_{ij}^{k(n)} = H \delta_{ij}. \tag{3.1}
\]
Using (3.1) we have
\[
\sum_{ij\alpha} h_{ij\alpha}^2 h_{ij\alpha k} = nH \Delta H. \tag{3.2}
\]
Using (2.1)–(2.4) and (3.1), we derive (cf. [1, 2, 3])

\[
\begin{align*}
\sum_{ijkl} h_{ij}^\alpha h_{kl}^\beta R_{ij} + \sum_{ijkl} h_{ij}^\alpha h_{kl}^\beta R_{ikj} + \sum_{ijkl} h_{ij}^\alpha h_{jk}^\beta R_{\alpha jk} \\
= \frac{c}{4} (n + 1) S - n^2 H^2 c + \sum_{ijkl} h_{ij}^\alpha h_{kl}^\beta h_{\alpha jk} + \sum_{\alpha \beta} \text{tr} (H_\alpha H_\beta - H_\beta H_\alpha)^2 - \sum_{\alpha \beta} (\text{tr} H_\alpha H_\beta)^2 \\
= \frac{c}{4} (n + 1) S - n^2 H^2 c + \sum_{\alpha \beta} \text{tr} (H_\alpha H_\beta - H_\beta H_\alpha)^2 - \sum_{\alpha \beta} (\text{tr} H_\alpha H_\beta)^2.
\end{align*}
\]

Substituting (3.2) and (3.3) into (2.6), we obtain

\[
\Delta S = \sum_{ijkl} (h_{ijk}^\alpha)^2 + n H^2 S + \sum_{ijkl} (h_{ijkl}^\alpha)^2 + \sum_{\alpha \beta} \text{tr} (H_\alpha H_\beta - H_\beta H_\alpha)^2 - \sum_{\alpha \beta} (\text{tr} H_\alpha H_\beta)^2.
\]

In order to prove our Theorems, we need the following Lemmas.

**Lemma 1** [4]. Let \( H_i, i \geq 2 \) be symmetric \( n \times n \)-matrices, \( S_i = \text{tr} H_i^2, S = \sum_i S_i \). Then

\[
\sum_{ij} \text{tr} (H_i H_j - H_j H_i)^2 - \sum_{ij} (\text{tr} H_i H_j)^2 \geq -\frac{3}{2} S^2,
\]

and the equality holds if and only if all \( H_i = 0 \) or there exist two of \( H_i \) different from zero. Moreover, if \( H_1 \neq 0, H_2 \neq 0, H_i = 0, i \neq 1, 2 \), then \( S_1 = S_2 \) and there exists an orthogonal \( (n \times n) \)-matrix \( T \) such that \( TH_i T = \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & 0 \end{pmatrix} \), \( TH_2 T = \begin{pmatrix} 0 & \alpha & 0 \\ \alpha & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \), where \( a = \frac{S_1}{2} \).

**Lemma 2.**

\[
\sum_{\alpha \beta} (H_\alpha H_\beta - H_\beta H_\alpha)^2 - \sum_{\alpha \beta} (\text{tr} H_\alpha H_\beta)^2 \geq \frac{3}{2} s^2 + 3n H^2 S - \frac{s^2}{2}.
\]

In fact, using (2.4), (3.1) and noting that \( \alpha \) runs up to \( 3n > 2 \), we have

\[
\sum_{\alpha \beta} (H_\alpha H_\beta - H_\beta H_\alpha)^2 - \sum_{\alpha \beta} (\text{tr} H_\alpha H_\beta)^2 = \sum_{\alpha \beta \neq k(n)} (\text{tr} H_\alpha H_\beta)^2 - (\text{tr} H_{k(n)})^2.
\]
Applying Lemma 1 to (3.5), we get

$$\sum_{\alpha \beta} (H_{\alpha} H_{\beta} - H_{\beta} H_{\alpha})^2 - \sum_{\alpha \beta} (\text{tr} H_{\alpha} H_{\beta})^2 \geq -\frac{3}{2} \left( \sum_{\alpha \neq k(n)} \text{tr} H_{\alpha}^2 \right) - (\text{tr} H_{k(n)}^2)^2$$

$$= -\frac{3}{2}(S - \text{tr} H_{k(n)}^2)^2 - (\text{tr} H_{k(n)}^2)^2 = -\frac{3}{2}(S - nH^2)^2 - n^2 H^4 = -\frac{3}{2}S^2 + 3nH^2S - \frac{3}{2}n^2 H^4.$$

On the other hand, by (3.1) we have

$$\sum_{i,j,k} (h_{ijk}^\alpha)^2 \geq \sum_{i,k} (h_{ik}^{k(n)})^2 = n \sum_{i} (\nabla_i H)^2 = n |\nabla H|^2.$$  

(3.6)

It is obvious that

$$\frac{1}{2} \Delta H^2 = H \Delta H + |\nabla H|^2.$$  

(3.7)

Therefore, using Lemma 2, (2.6) and (2.7) by (3.4) we get

$$\frac{1}{2} \Delta S = \sum_{i,j,k} (h_{ijk}^\alpha)^2 + \sum_{i,j} h_{ij}^\alpha \Delta h_{ij}^\alpha$$

$$= \sum_{i,j,k} (h_{ijk}^\alpha)^2 + nH \Delta H + \frac{c}{4} (n + 1)S + nH^2S - n^2 H^2c$$

$$+ \sum_{\alpha \beta} (H_{\alpha} H_{\beta} - H_{\beta} H_{\alpha})^2 - \sum_{\alpha \beta} (\text{tr} H_{\alpha} H_{\beta})^2$$

$$\geq \sum_{i,j,k} (h_{ijk}^\alpha)^2 + nH \Delta H + \frac{c}{4} (n + 1)S + 4nH^2S - n^2 H^2c - \frac{3}{2}S^2 - \frac{3}{2}n^2 H^4$$

$$\geq n |\nabla H|^2 + nH \Delta H + \frac{c}{4} (n + 1)S + 4nH^2S - n^2 H^2c - \frac{3}{2}S^2 - \frac{3}{2}n^2 H^4$$

$$= \frac{1}{2} nH^2 + \frac{c}{4} (n + 1)S + 4nH^2S - \frac{3}{2}S^2 - n^2 H^2c - \frac{3}{2}n^2 H^4.$$  

(3.8)

Since $N$ is compact, we obtain from (3.8)

$$\int_N \{6S^2 - [(n + 1)c + 16nH^2]S + 4n^2 H^2c + 10n^2 H^4\} dN \geq 0.$$

From the first inequality of (3.8) we know that if $N$ is compact and

$$6S^2 - [(n + 1)c + 16nH^2]S + 4n^2 H^2c + 10n^2 H^4 - 4nH \Delta H \leq 0,$$

(3.9)

then $\sum_{i,j,k} (h_{ijk}^\alpha)^2 = 0$, that is, the second fundamental form $h_{ij}^\alpha$ is parallel. In particular, when the equality of (3.9) holds, we see from (3.8) that the equality

$$\sum_{\alpha \beta \neq k(n)} (H_{\alpha} H_{\beta} - H_{\beta} H_{\alpha})^2 - \sum_{\alpha \beta \neq k(n)} (\text{tr} H_{\alpha} H_{\beta})^2 = -\frac{3}{2} \left( \sum_{\alpha \neq k(n)} \text{tr} H_{\alpha}^2 \right)^2$$
holds. Thus, by Lemma 1 we see that (i) \( H_\alpha = 0 \) \((\alpha \neq k(n))\) or (ii) there exist two non-zero \( H_\alpha \). In the case (i), we get \( S = nH^2 \). Hence noting \( H = \text{constant} \) and substituting it into the equality of (3.9), we obtain

\[
(3n - 1)cnH^2 = 0.
\]

This implies \( H = 0 \), so that \( N \) is totally geodesic or \( c = 0 \) so that \( N \) is flat. Now, we will prove that the case (ii) can not occur. Otherwise, using the same method as in [3]), we may see \( n = 2 \). Thus we may assume

\[
H_{(1)} = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}, \quad H_{(2)} = \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix}, \quad H_{K(2)} = \begin{pmatrix} H & 0 \\ 0 & H \end{pmatrix}, \quad H_\alpha = 0. \tag{3.10}
\]

Here \( a \neq 0 \), \( \alpha \neq I(1), I(2), K(2) \).

Let the codimension of \( N \) be \( p = 3n \). Put

\[
S_\alpha = \sum_{ij} (h^\alpha_{ij})^2,
\]

\[
p\sigma_1 = \sum_{\alpha} S_\alpha = S,
\]

\[
p(p - 1)\sigma_2 = 2 \sum_{\alpha < \beta} S_\alpha S_\beta.
\]

It can be easily seen (cf. [3])

\[
p^2(p - 1)(\sigma_1 - \sigma_2) = \sum_{\alpha < \beta} (S_\alpha - S_\beta)^2. \tag{3.11}
\]

By a direct calculation using (3.10), we get

\[
p^2(p - 1)\sigma_1^2 = (p - 1)(4a^2 + 2H^2)^2, \tag{3.12}
\]

\[
p^2(p - 1)\sigma_2 = p(8a^4 + 16a^2H^2), \tag{3.13}
\]

and

\[
\sum_{\alpha < \beta} (S_\alpha - S_\beta)^2 = 8(a^2 - H^2)^2. \tag{3.14}
\]

Substituting (3.12)–(3.14) into (3.11), we obtain

\[
(p - 1)(4a^2 + 2H^2)^2 - p(8a^4 + 16a^2H^2) = 8(a^2 - H^2)^2. \tag{3.15}
\]

From (3.15) we get

\[
(p - 3)(2a^4 + H^4) = 0,
\]

namely

\[
(3n - 3)(2a^4 + H^4) = 0,
\]

implying \( n = 1 \), because \( 2a^4 + H^4 \neq 0 \). This is a contradiction, since \( n = 2 \).
REFERENCES


DEPARTMENT OF MATHEMATICS

TOKYO METROPOLITAN UNIVERSITY

HACHIOJI, TOKYO, 192-03

JAPAN