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ON THE SCATTERING OF WAVES BY NEARLY HARD OR SOFT INCOMPLETE VERTICAL BARRIERS IN WATER OF INFINITE DEPTH

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Abstract

In this paper the scattered progressive waves are determined due to progressive waves incident normally on certain types of partially immersed and completely submerged vertical porous barriers in water of infinite depth. The forms are approximate only, and are obtained using perturbation theory for nearly hard or soft barriers having high and low porosities respectively. The results for arbitrary porosity are difficult to obtain, in contrast to the well known hard limit of impermeable barriers.

1. Introduction

Two problems that have received considerable attention in the theory of surface waves involve progressive waves incident normally on impermeable incomplete vertical barriers in water of infinite depth and extent, the barriers being either partially immersed or completely submerged with a single tip in the water at a specified depth. The effect of the barriers is to partly reflect and partly transmit the incident waves, without loss of energy if surface tension is ignored. The linearized solutions for the velocity potentials in these two transmission problems were obtained long ago by Ursell [6], using Havelock's [3] classical wave-maker theory to set up integral equations for the unknown horizontal velocity in the gap below or above the barrier; they may also be solved by complex variable techniques. The scattered waves (only) were obtained by an integral equation method in Williams [8] after a reformulation.

If the barriers are no longer impermeable but porous, comparable results with loss of energy are difficult to obtain by any of these methods. A number of results involving porous walls or barriers extending throughout the depth of water have been obtained recently by Chakrabarti and Sahoo [1] and Rhodes-Robinson [5] to extend known

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impermeable results in simpler problems, but this does not seem possible here.

To make some progress, we consider herein the two asymptotic situations when the barriers are *nearly* impermeable ('hard') or completely porous ('soft') and set up suitable perturbation solutions involving certain hard and soft limit solutions. The former are known from Ursell [6], Evans [2] and Rhodes-Robinson [4] to an extent that enables the scattered waves (only) to be found to the first order; the latter are classical and the full first-order solution is obtainable. The corresponding scattered amplitude and energy ratios are calculated, and tabulated numerical values given for the expansion coefficients. Two new Bessel function integrals arise for the nearly hard barriers, and these are fully investigated also.

2. General formulation

Water occupies the region of infinite depth $y > 0$ and contains a single fixed vertical barrier along part of $x = 0$ that has its tip $(0, c)$ at depth c below the equilibrium free surface $y = 0$; the barrier is either partially immersed or completely submerged, and the remainder of $x = 0$ forms a gap either below or above the barrier. The barrier is porous—in fact is assumed to have fine pores—and has porosity constant $k > 0$; in the familiar case of an impermeable (hard) barrier $k = 0$, and for a completely porous (soft) barrier $k \to \infty$ (this barrier has no effect on waves so is removable). The effect of surface tension is omitted in this investigation so that the wave motion is under the action of gravity alone with acceleration g. The usual tip singularity is allowed for and there is no motion at infinite depth.

The infinitesimal motion is harmonic in time t with angular frequency σ and may be described by a velocity potential of the form $\mathrm{Re}[\phi(x, y)e^{-i\sigma t}]$, where ϕ is complex-valued. The scattered motion to be investigated is due to incident progressive waves with potential $e^{-Ky-iKx}$, where the wave number is $K = \sigma^2/g$. If $\phi = \phi_1$ $(x > 0)$, $\phi = \phi_2$ $(x < 0)$, the potentials ϕ_1, ϕ_2 are given by the linearized coupled boundary-value problem in the region of water

$$\nabla^2\phi_1 = 0 = \nabla^2\phi_2,$$
$$K\phi_1 + \phi_{1y} = 0 = K\phi_2 + \phi_{2y} \quad \text{on} \quad y = 0,$$
$$\phi_1, \phi_2 \to 0 \quad \text{as} \quad y \to \infty,$$
$$\phi_{1x} = -ik(\phi_1 - \phi_2) = \phi_{2x} \quad \text{on} \quad \text{barrier},$$
$$\phi_1 = \phi_2 \quad \text{in} \quad \text{gap},$$
$$r\left[|\phi_{1x}|^2 + |\phi_{1y}|^2\right],\ r\left[|\phi_{2x}|^2 + |\phi_{2y}|^2\right] \quad \text{are bounded as} \quad r \to 0,$$
$$\phi_1 \to e^{-Ky-iKx} + Re^{-Ky+iKx} \quad \text{as} \quad x \to \infty, \qquad \phi_2 \to Te^{-Ky-iKx} \quad \text{as} \quad x \to -\infty,$$

where $r = [x^2 + (y - c)^2]^{1/2}$ is the distance from the tip and the reflected and transmitted amplitude constants R, T are part of the solution that involves two non-dimensional parameters Kc, kc. More details on the barrier conditions are given in Rhodes-Robinson [5].

A reformulation reduces this transmission problem if the incident waves are subtracted out by putting $\phi = \Phi + e^{-Ky-iKx}$; then the potential Φ is antisymmetric about the gap and in $x > 0$ satisfies the boundary-value problem

$$\nabla^2\Phi = 0,$$
$$K\Phi + \Phi_y = 0 \quad \text{on} \quad y = 0, \qquad \Phi \to 0 \quad \text{as} \quad y \to \infty,$$
$$\Phi_x + 2ik\Phi = iKe^{-Ky} \quad \text{on} \quad \text{barrier}, \qquad \Phi = 0 \quad \text{in} \quad \text{gap},$$
$$r\left[|\Phi_x|^2 + |\Phi_y|^2\right] \quad \text{is bounded as} \quad r \to 0,$$
$$\Phi \to Re^{-Ky+iKx} \quad \text{as} \quad x \to \infty,$$

where R is part of the solution and it should be noted that the barrier condition is linear in both k and k^{-1}.

Note also due to the antisymmetry that ϕ_1, ϕ_2 are related by

$$\phi_1(x, y) + \phi_2(-x, y) = 2e^{-Ky}\cos Kx$$

and in particular

$$R + T = 1. \tag{2.1}$$

The above problem has been solved in full by Ursell [6] when $k = 0$ for the two barriers envisaged, but now such an achievement seems difficult and is not attempted herein. Instead nearly hard or soft perturbation solutions are sought corresponding to small or large values respectively of the parameter kc, with coefficients depending on the parameter Kc; the linear form that these expansions should have is indicated by that of the barrier condition noted above. Emphasis is placed on the determination of the scattered amplitude constants R, T in order to calculate the scattered amplitude ratios $|R|$, $|T|$ and energy ratios $|R|^2$, $|T|^2$. Note that $|R|^2 + |T|^2 < 1$, since energy is lost with a porous barrier.

3. Perturbation formulation for nearly hard barriers

First suppose that $\epsilon = kc$ is *small* and look for a perturbation solution to the above problem of the linear form

$$\Phi = \Phi_0 + \epsilon\Phi_1 \tag{3.1}$$

to the first order in ϵ, where Φ_0, Φ_1 are hard limit ($k \to 0$) potentials that involve Kc; also let

$$R = R_0 + \epsilon R_1, \qquad T = T_0 + \epsilon T_1 \tag{3.2}$$

and note that

$$R_0 + T_0 = 1, \qquad R_1 + T_1 = 0 \tag{3.3}$$

from (2.1).

The perturbation potential Φ_0 in $x > 0$ satisfies the boundary-value problem

$$\begin{gathered}
\nabla^2\Phi_0 = 0, \\
K\Phi_0 + \Phi_{0y} = 0 \quad \text{on} \quad y = 0, \qquad \Phi_0 \to 0 \quad \text{as} \quad y \to \infty, \\
\Phi_{0x} = iKe^{-Ky} \quad \text{on} \quad \text{barrier}, \qquad \Phi_0 = 0 \quad \text{in} \quad \text{gap}, \\
r\left[|\Phi_{0x}|^2 + |\Phi_{0y}|^2\right] \quad \text{is bounded as} \quad r \to 0, \\
\Phi_0 \to R_0 e^{-Ky+iKx} \quad \text{as} \quad x \to \infty
\end{gathered}$$

(unperturbed problem), where R_0 is part of the solution; this is of course the familiar transmission problem referred to above for a hard barrier with the incident waves subtracted out. Note that $|R_0|^2 + |T_0|^2 = 1$ as no energy is lost here.

The perturbation potential Φ_1 in $x > 0$ satisfies the boundary-value problem

$$\begin{gathered}
\nabla^2\Phi_1 = 0, \\
K\Phi_1 + \Phi_{1y} = 0 \quad \text{on} \quad y = 0, \qquad \Phi_1 \to 0 \quad \text{as} \quad y \to \infty, \\
\Phi_{1x} = -(2i/c)\Phi_0(0, y) \quad \text{on} \quad \text{barrier}, \qquad \Phi_1 = 0 \quad \text{in} \quad \text{gap}, \\
r\left[|\Phi_{1x}|^2 + |\Phi_{1y}|^2\right] \quad \text{is bounded as} \quad r \to 0, \\
\Phi_1 \to R_1 e^{-Ky+iKx} \quad \text{as} \quad x \to \infty
\end{gathered}$$

(first-order correction problem), where R_1 is part of the solution; this is a familiar hard wave-maker problem for a special normal velocity depending on the previous solution. Once R_0, R_1 and therefore $T_0 = 1 - R_0$, $T_1 = -R_1$ are obtained from (3.3), the scattered amplitude ratios are calculated using (3.2) as

$$|R| = a_0 - \epsilon a_1, \qquad |T| = b_0 - \epsilon b_1 \tag{3.4}$$

to the first order in ϵ, where

$$a_0 = |R_0|, \qquad b_0 = |T_0|, \qquad a_1 = -\frac{\mathrm{re}[R_0\bar{R}_1]}{|R_0|}, \qquad b_1 = -\frac{\mathrm{re}[T_0\bar{T}_1]}{|T_0|};$$

and then the scattered energy ratios as

$$|R|^2 = c_0 - \epsilon c_1, \qquad |T|^2 = d_0 - \epsilon d_1 \tag{3.5}$$

likewise, where

$$c_0 = a_0^2, \qquad d_0 = b_0^2, \qquad c_1 = 2a_0a_1, \qquad d_1 = 2b_0b_1.$$

Note also from (3.5) that $|R|^2 + |T|^2 = 1 - \epsilon(c_1 + d_1)$ as $c_0 + d_0 = 1$, and the loss of energy is $\epsilon(c_1 + d_1)$.

The expansion coefficients $a_0, a_1, b_0, b_1, c_0, c_1, d_0, d_1$ in (3.4), (3.5) all depend on Kc and we now obtain these for partially immersed and completely submerged barriers, for which Φ_0, R_0 and R_1 are known or may be found.

4. Solution for nearly hard partially immersed barrier

This barrier occupies $x = 0$, $0 \le y \le a$ so that $c = a$ and the parameters are Ka, $\epsilon = ka$.

The full unperturbed solution was obtained in Ursell [6] as

$$\Phi_0 = \frac{1}{B_1(Ka)}\left[\pi I_1(Ka)e^{-Ky+iKx} + \int_0^\infty \frac{J_1(ua)e^{-ux}}{u^2+K^2}(u\cos uy - K\sin uy)\,du\right] \tag{4.1}$$

($x > 0$) so that

$$R_0 = \frac{\pi I_1(Ka)}{B_1(Ka)}, \qquad T_0 = \frac{iK_1(Ka)}{B_1(Ka)} \tag{4.2}$$

from (3.3), where I_1, J_1, $K_1(z)$ are Bessel functions and $B_1(z) = \pi I_1(z) + iK_1(z)$.

The first-order correction outgoing waves are obtained using the formula determined in Evans [2] for the amplitude constant as

$$\begin{aligned} R_1 &= \frac{(-2i)^2}{a^2B_1(Ka)}\int_0^a \frac{Ye^{KY}}{(a^2-Y^2)^{1/2}}\int_0^Y \Phi_0(0,s)e^{-Ks}\,ds\,dY \\ &= \frac{-4}{a^2B_1^2(Ka)}\int_0^a \frac{Y}{(a^2-Y^2)^{1/2}}\left[\pi I_1(Ka)\frac{\sinh KY}{K} + \int_0^\infty \frac{J_1(ua)}{u^2+K^2}\sin uY\,du\right]dY \end{aligned}$$

from (4.1)

$$= \frac{-2\pi}{aB_1^2(Ka)}\left[\frac{\pi I_1^2(Ka)}{K} + \int_0^\infty \frac{J_1^2(ua)}{u^2+K^2}\,du\right]$$

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on interchanging the order of integration and noting the integral representations

$$I_1(z) = \frac{2}{\pi}\int_0^1 \frac{v\sinh zv}{(1-v^2)^{1/2}}\,dv, \qquad J_1(z) = \frac{2}{\pi}\int_0^1 \frac{v\sin zv}{(1-v^2)^{1/2}}\,dv;$$

thus

$$R_1 = -\frac{\alpha_1(Ka)}{B_1^2(Ka)}, \qquad T_1 = \frac{\alpha_1(Ka)}{B_1^2(Ka)} \tag{4.3}$$

from (3.3), where $\alpha_1(z) = 2\pi[\pi I_1^2(z) + S_1(z)]/z$ and

$$S_1(z) = \int_0^\infty \frac{J_1^2(zv)}{v^2+1}\,dv. \tag{4.4}$$

Hence the expansion coefficients in (3.4) are

$$a_0 = \frac{\pi I_1(Ka)}{D_1(Ka)}, \qquad b_0 = \frac{K_1(Ka)}{D_1(Ka)},$$
$$a_1 = \frac{\pi I_1(Ka)\alpha_1(Ka)}{D_1^3(Ka)}, \qquad b_1 = \frac{K_1(Ka)\alpha_1(Ka)}{D_1^3(Ka)}$$

and in (3.5)

$$c_0 = \frac{\pi^2 I_1^2(Ka)}{D_1^2(Ka)}, \qquad d_0 = \frac{K_1^2(Ka)}{D_1^2(Ka)},$$
$$c_1 = \frac{2\pi^2 I_1^2(Ka)\alpha_1(Ka)}{D_1^4(Ka)}, \qquad d_1 = \frac{2K_1^2(Ka)\alpha_1(Ka)}{D_1^4(Ka)}$$

from (4.2), (4.3), where $D_1(z) = |B_1(z)| = [\pi^2 I_1^2(z) + K_1^2(z)]^{1/2}$.

Numerical values of the expansion coefficients for $0 < K_a < 2.5$ calculated to 4 decimal place accuracy are given in Appendix 1 (Table 1), together with their limits as $Ka \to 0, \infty$; this accuracy is not sufficient for $Ka > 2.5$ to produce non-zero values, but higher accuracy can be achieved to extend the range if necessary. The expansions obtained using these values in (3.4), (3.5) are suitable for all $Ka > 0$ due to the finite limits of all coefficients.

Calculations of the integral S_1 in (4.4) involved in these are given in Appendix 2 (Table 3); the integral cannot be evaluated explicitly, although asymptotic forms can be derived.

5. Solution for nearly hard completely submerged barrier

This barrier occupies $x = 0$, $y \geq b$ so that $c = b$ and the parameters are Kb, $\epsilon = kb$.

The full unperturbed solution was also given in Ursell [6] as

$$\Phi_0 = \frac{i}{B_0(Kb)}\left[-K_0(Kb)e^{-Ky+iKx} + \int_0^\infty \frac{J_0(ub)e^{-ux}}{u^2+K^2}(u\cos uy - K\sin uy)\,du\right] \tag{5.1}$$

$(x > 0)$ so that

$$R_0 = -\frac{iK_0(Kb)}{B_0(Kb)}, \qquad T_0 = \frac{\pi I_0(Kb)}{B_0(Kb)} \tag{5.2}$$

from (3.3), where I_0, J_0, $K_0(z)$ are Bessel functions and $B_0(z) = \pi I_0(z) - iK_0(z)$.

The first-order correction outgoing waves are obtained using the formula given in Rhodes-Robinson [4], Section 5 for the amplitude constant as

$$\begin{aligned} R_1 &= \frac{-4i}{bB_0(Kb)}\int_b^\infty \frac{e^{KY}}{(Y^2-b^2)^{1/2}}\int_Y^\infty \Phi_0(0,s)e^{-Ks}\,ds\,dY \\ &= \frac{4}{bB_0^2(Kb)}\int_b^\infty \frac{1}{(Y^2-b^2)^{1/2}}\left[K_0(Kb)\frac{e^{-KY}}{2K} + \int_0^\infty \frac{J_0(ub)}{u^2+K^2}\sin uY\,du\right]dY \end{aligned}$$

from (5.1)

$$= \frac{2}{bB_0^2(Kb)}\left[\frac{K_0^2(Kb)}{K} + \pi\int_0^\infty \frac{J_0^2(ub)}{u^2+K^2}\,du\right]$$

on interchanging the order of integration and noting the integral representations

$$K_0(z) = \int_1^\infty \frac{e^{-zv}}{(v^2-1)^{1/2}}\,dv, \qquad J_0(z) = \frac{2}{\pi}\int_1^\infty \frac{\sin zv}{(v^2-1)^{1/2}}\,dv \quad (z > 0);$$

thus

$$R_1 = \frac{\alpha_0(Kb)}{B_0^2(Kb)}, \qquad T_1 = -\frac{\alpha_0(Kb)}{B_0^2(Kb)} \tag{5.3}$$

from (3.3), where $\alpha_0(z) = 2[K_0^2(z) + \pi S_0(z)]/z$ and

$$S_0(z) = \int_0^\infty \frac{J_0^2(zv)}{v^2+1}\,dv. \tag{5.4}$$

Hence the expansion coefficients in (3.4) are

$$a_0 = \frac{K_0(Kb)}{D_0(Kb)}, \qquad b_0 = \frac{\pi I_0(Kb)}{D_0(Kb)},$$
$$a_1 = \frac{K_0(Kb)\alpha_0(Kb)}{D_0^3(Kb)}, \qquad b_1 = \frac{\pi I_0(Kb)\alpha_0(Kb)}{D_0^3(Kb)}$$

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and in (3.5)

$$c_0 = \frac{K_0^2(Kb)}{D_0^2(Kb)}, \qquad d_0 = \frac{\pi^2 I_0^2(Kb)}{D_0^2(Kb)},$$
$$c_1 = \frac{2K_0^2(Kb)\alpha_0(Kb)}{D_0^4(Kb)}, \qquad d_1 = \frac{2\pi^2 I_0^2(Kb)\alpha_0(Kb)}{D_0^4(Kb)}$$

from (5.2), (5.3), where $D_0(z) = |B_0(z)| = [\pi^2 I_0^2(z) + K_0^2(z)]^{1/2}$.

Numerical values of the expansion coefficients for $0 < Kb \leq 2.5$ calculated to 4 decimal place accuracy are given in Appendix 1 (Table 2), together with their limits as $Kb \to 0, \infty$; similar comments pertain on accuracy here for $Kb > 2.5$ as before. The expansions obtained using these values in (5,6) becomes less suitable for smaller Kb due to the infinite limits of some coefficients, being then valid only for smaller $\epsilon = kb$.

Calculations of the integral S_0 in (5.4) involved in these are also given in Appendix 2 (Table 3); again the integral cannot be evaluated explicitly, although asymptotic forms can be derived.

6. Perturbation formulation for nearly soft barriers

Now suppose that kc is *large* so that $\delta = (kc)^{-1}$ is small and look for a perturbation solution to the problem in Section 2 of the linear form

$$\Phi = \Phi_0 + \delta\Phi_1 \tag{6.1}$$

to the first order in δ, where Φ_0, Φ_1 are soft limit ($k \to \infty$) potentials that involve Kc again; also let

$$R = R_0 + \delta R_1, \qquad T = T_0 + \delta T_1 \tag{6.2}$$

and note again that

$$R_0 + T_0 = 1, \qquad R_1 + T_1 = 0 \tag{6.3}$$

from (3.1).

The perturbation potential Φ_0 is now trivially obtained as

$$\Phi_0 = 0 \tag{6.4}$$

(unperturbed solution) for any soft (removable) barrier so that

$$R_0 = 0, \qquad T_0 = 1 - R_0 = 1 \tag{6.5}$$

from (6.3).

The perturbation potential Φ_1 in $x > 0$ satisfies the boundary-value problem

$$\nabla^2\Phi_1 = 0,$$
$$K\Phi_1 + \Phi_{1y} = 0 \quad \text{on} \quad y = 0, \qquad \Phi_1 \to 0 \quad \text{as} \quad y \to \infty,$$
$$\Phi_1 = \frac{1}{2}Kce^{-Ky} \quad \text{on} \quad \text{barrier}, \quad \Phi_1 = 0 \quad \text{in} \quad \text{gap},$$
$$r\left[|\Phi_{1x}|^2 + |\Phi_{1y}|^2\right] \quad \text{is bounded as} \quad r \to 0,$$
$$\Phi_1 \to R_1 e^{-Ky+iKx} \quad \text{as} \quad x \to \infty$$

(first-order correction problem), where R_1 is part of the solution; the full solution is obtained for any barrier similarly as in Havelock's [3] classical wave-maker problem. Note that R_1 is always found to be real and positive.

Once R_1 and therefore $T_1 = -R_1$ are obtained from (6.3b), the scattered amplitude and energy ratios are easily calculated using (6.2) after noting (6.5) as

$$|R| = \delta R_1, \qquad |T| = 1 - \delta R_1 \tag{6.6}$$

and

$$|R|^2 = 0, \qquad |T|^2 = 1 - 2\delta R_1 \tag{6.7}$$

in terms of R_1 to the first order in δ; the loss of energy is $2\delta R_1$ from (6.7). The expansion coefficients in (6.6), (6.7) depend on Kc again.

To conclude we obtain Φ_1 and R_1 for the two particular barriers described earlier.

7. Solutions for nearly soft barriers

For the *partially immersed* barrier $\delta = (ka)^{-1}$. The full first-order solution is

$$\Phi_1 = \frac{Ka}{\pi}e^{-Ka}\int_0^\infty \frac{e^{-ux}\sin ua}{u^2 + K^2}(u\cos uy - K\sin uy)\,du + \frac{1}{2}Ka(1 - e^{-2Ka})e^{-Ky+iKx} \tag{7.1}$$

($x > 0$) so that

$$R_1 = \frac{1}{2}Ka(1 - e^{-2Ka}). \tag{7.2}$$

Numerical values of R_1 for $0 \le Ka \le 2.5$ (again, say) can easily be calculated from this exact formula and the limit is infinite as $Ka \to \infty$. The expansions (6.6), (6.7) obtained using these become less suitable for larger Ka due to the infinite limit of some coefficients, being then valid only for smaller $\delta = (ka)^{-1}$.

For the *completely submerged* barrier $\delta = (kb)^{-1}$. The full first-order solution is

$$\Phi_1 = -\frac{Kb}{\pi}e^{-Kb}\int_0^\infty \frac{e^{-ux}\sin ub}{u^2+K^2}(u\cos uy - K\sin uy)\,du + \frac{1}{2}Kbe^{-2Kb}e^{-Ky+iKx} \tag{7.3}$$

so that

$$R_1 = \frac{1}{2}Kbe^{-2Kb}. \tag{7.4}$$

Numerical values of R_l for $0 \le Kb \le 2.5$ can again be calculated from this exact formula and the limit is zero as $Kb \to \infty$. The expansions (6.6), (6.7) obtained using these are now suitable for all $Kb \ge 0$ due to the finite limits of the coefficients.

Acknowledgement

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Appendix 1: Expansion coefficients for nearly hard barriers

Numerical values calculated to 4 decimal place accuracy are given in Tables 1, 2, together with their limits.

TABLE 1. Values of expansion coefficients for a nearly hard partially immersed barrier.

Ka	a_0	a_1	b_0	b_1	c_0	c_1	d_0	d_1
0	0	0	1	0	0	0	1	0
0.1	0.0160	0.0005	0.9999	0.0301	0.0003	0.0000	0.9997	0.0603
0.2	0.0660	0.0093	0.9978	0.1403	0.0044	0.0012	0.9956	0.2801
0.3	0.1541	0.0572	0.9881	0.3670	0.0237	0.0176	0.9763	0.7253
0.4	0.2816	0.2134	0.9595	0.7271	0.0793	0.1201	0.9207	1.3954
0.5	0.4394	0.5620	0.8983	1.1489	0.1931	0.4938	0.8069	2.0642
0.6	0.6033	1.0911	0.7975	1.4423	0.3640	1.3164	0.6360	2.3006
0.7	0.7437	1.6173	0.6685	1.4540	0.5530	2.4055	0.4470	1.9441
0.8	0.8447	1.9493	0.5353	1.2353	0.7135	3.2931	0.2865	1.3225
0.9	0.9089	2.0564	0.4170	0.9435	0.8261	3.7381	0.1739	0.7869
1.0	0.9471	2.0143	0.3211	0.6829	0.8969	3.8153	0.1031	0.4385
1.1	0.9691	1.9020	0.2467	0.4841	0.9392	3.6865	0.0608	0.2388
1.2	0.9818	1.7679	0.1900	0.3422	0.9639	3.4714	0.0361	0.1301
1.3	0.9891	1.6351	0.1471	0.2432	0.9784	3.2347	0.0216	0.0716
1.4	0.9934	1.5129	0.1145	0.1744	0.9869	3.0059	0.0131	0.0399
1.5	0.9960	1.4038	0.0896	0.1263	0.9920	2.7962	0.0080	0.0226
1.6	0.9975	1.3075	0.0704	0.0923	0.9950	2.6084	0.0050	0.0130
1.7	0.9985	1.2228	0.0556	0.0681	0.9969	2.4418	0.0031	0.0076
1.8	0.9990	1.1481	0.0441	0.0507	0.9981	2.2941	0.0019	0.0045
1.9	0.9994	1.0821	0.0351	0.0380	0.9988	2.1629	0.0012	0.0027
2.0	0.9996	1.0235	0.0280	0.0286	0.9992	2.0461	0.0008	0.0016
2.1	0.9997	0.9710	0.0224	0.0217	0.9995	1.9416	0.0005	0.0010
2.2	0.9998	0.9239	0.0179	0.0166	0.9997	1.8475	0.0003	0.0006
2.3	0.9999	0.8813	0.0144	0.0127	0.9998	1.7625	0.0002	0.0004
2.4	0.9999	0.8427	0.0116	0.0098	0.9999	1.6853	0.0001	0.0002
2.5	1.0000	0.8074	0.0093	0.0075	0.9999	1.6148	0.0001	0.0001
∞	1	0	0	0	1	0	0	0

TABLE 2. Values of expansion coefficients for a nearly hard completely submerged barrier.

Kb	a_0	a_1	b_0	b_1	c_0	c_1	d_0	d_1
0	1	∞	0	∞	1	∞	0	∞
0.1	0.6104	8.0687	0.7921	10.4702	0.3726	9.8503	0.6274	16.5866
0.2	0.4835	2.6848	0.8753	4.8606	0.2338	2.5963	0.7662	8.5094
0.3	0.3929	1.2472	0.9196	2.9194	0.1543	0.9799	0.8457	5.3693
0.4	0.3227	0.6639	0.9465	1.9470	0.1042	0.4285	0.8958	3.6857
0.5	0.2667	0.3802	0.9638	1.3740	0.0711	0.2028	0.9289	2.6484
0.6	0.2210	0.2275	0.9753	1.0039	0.0489	0.1006	0.9511	1.9582
0.7	0.1835	0.1402	0.9830	0.7508	0.0337	0.0514	0.9663	1.4761
0.8	0.1525	0.0880	0.9883	0.5707	0.0232	0.0268	0.9768	1.1280
0.9	0.1267	0.0561	0.9919	0.4389	0.0161	0.0142	0.9839	0.8708
1.0	0.1053	0.0361	0.9944	0.3406	0.0111	0.0076	0.9889	0.6774
1.1	0.0874	0.0233	0.9962	0.2660	0.0076	0.0041	0.9924	0.5300
1.2	0.0726	0.0152	0.9974	0.2088	0.0053	0.0022	0.9947	0.4165
1.3	0.0602	0.0099	0.9982	0.1646	0.0036	0.0012	0.9964	0.3286
1.4	0.0499	0.0065	0.9988	0.1301	0.0025	0.0006	0.9975	0.2599
1.5	0.0413	0.0043	0.9991	0.1031	0.0017	0.0004	0.9983	0.2060
1.6	0.0342	0.0028	0.9994	0.0819	0.0012	0.0002	0.9988	0.1636
1.7	0.0283	0.0018	0.9996	0.0651	0.0008	0.0001	0.9992	0.1302
1.8	0.0233	0.0012	0.9997	0.0518	0.0005	0.0001	0.9995	0.1037
1.9	0.0193	0.0008	0.9998	0.0413	0.0004	0.0000	0.9996	0.0826
2.0	0.0159	0.0005	0.9999	0.0330	0.0003	0.0000	0.9997	0.0660
2.1	0.0131	0.0003	0.9999	0.0263	0.0002	0.0000	0.9998	0.0527
2.2	0.0108	0.0002	0.9999	0.0210	0.0001	0.0000	0.9999	0.0421
2.3	0.0089	0.0001	1.0000	0.0168	0.0001	0.0000	0.9999	0.0337
2.4	0.0073	0.0001	1.0000	0.0135	0.0001	0.0000	0.9999	0.0269
2.5	0.0060	0.0001	1.0000	0.0108	0.0000	0.0000	1.0000	0.0216
∞	0	0	1	0	0	0	1	0

Appendix 2: Bessel function integrals

The integrals

$$S_n(z) = \int_0^\infty \frac{J_n^2(zv)}{v^2+1}\,dv \quad (n = 0, 1) \tag{A2.1}$$

are even in z (real) and positive in value, but cannot be evaluated analytically except for $z = 0$.

The asymptotic forms are

$$S_0(z) \sim \frac{\pi}{2} - z\int_0^\infty \frac{1 - J_0^2(w)}{w^2}\,dw = \frac{\pi}{2} - \frac{4z}{\pi},$$
$$S_1(z) \sim z\int_0^\infty \frac{J_1^2(w)}{w^2}\,dw = \frac{4z}{3\pi}$$

as $z \to 0$ from Watson [7]; and

$$S_0(z) \sim \frac{1}{\pi z}(\ln 8z + \gamma), \qquad S_1(z) \sim \frac{1}{\pi z}(\ln 8z + \gamma - 2)$$

as $z \to \infty$, where $\gamma = 0.577216\ldots$ is Euler's constant. The latter forms are obtained after converting (A2.1) to the alternative integral

$$S_n(z) = \frac{2}{\pi}\int_0^1 e^{-2zw} Q_{n-\frac{1}{2}}(1 - 2w^2)\,dw \quad (n = 0, 1) \tag{A2.2}$$

in terms of Legendre functions, and using Watson's lemma.

Numerical values of the integrals for $0 \le z \le 20$ calculated to 6 decimal place accuracy from either (A2.1) or (A2.2) are given in Table 3; only those for $0 < z \le 2.5$ are used herein, however.

TABLE 3. Values of Bessel function integrals.

z	$S_0(z)$	$S_1(z)$
0.0	1.570796	0.000000
0.1	1.450963	0.038731
0.2	1.344767	0.070839
0.3	1.250409	0.097374
0.4	1.166346	0.119222
0.5	1.091253	0.137127
0.6	1.023992	0.151715
0.7	0.963580	0.163515
0.8	0.909173	0.172971
0.9	0.860040	0.180458
1.0	0.815549	0.186292
1.1	0.775153	0.190740
1.2	0.738376	0.194027
1.3	0.704806	0.196341
1.4	0.674084	0.197843
1.5	0.645895	0.198669
1.6	0.619967	0.198931
1.7	0.596059	0.198727
1.8	0.573962	0.198137
1.9	0.553490	0.197229
2.0	0.534481	0.196062
2.1	0.516792	0.194684
2.2	0.500297	0.193137
2.3	0.484883	0.191454
2.4	0.470450	0.189665
2.5	0.456911	0.187794
2.6	0.444187	0.185862
2.7	0.432208	0.183887
2.8	0.420910	0.181881
2.9	0.410239	0.179859
3.0	0.400142	0.177828
3.1	0.390576	0.175799
3.2	0.381499	0.173777
3.3	0.372874	0.171769
3.4	0.364668	0.169779
3.5	0.356850	0.167810
3.6	0.349394	0.165866
3.7	0.342274	0.163949
3.8	0.335468	0.162061
3.9	0.328955	0.160204
4.0	0.322715	0.158377
4.1	0.316732	0.156583
4.2	0.310989	0.154822
4.3	0.305473	0.153093
4.4	0.300169	0.151397
4.5	0.295065	0.149734
4.6	0.290149	0.148103
4.7	0.285411	0.146505
4.8	0.280842	0.144938
4.9	0.276431	0.143403
5.0	0.272172	0.141899
5.0	0.272172	0.141899
5.1	0.268055	0.140426
5.2	0.264073	0.138982
5.3	0.260220	0.137567
5.4	0.256489	0.136181
5.5	0.252875	0.134822
5.6	0.249371	0.133491
5.7	0.245973	0.132187
5.8	0.242676	0.130908
5.9	0.239475	0.129655
6.0	0.236366	0.128427
6.1	0.233344	0.127223
6.2	0.230407	0.126042
6.3	0.227550	0.124884
6.4	0.224770	0.123749
6.5	0.222064	0.122635
6.6	0.219429	0.121542
6.7	0.216862	0.120470
6.8	0.214360	0.119418
6.9	0.211920	0.118385
7.0	0.209542	0.117372
7.1	0.207221	0.116377
7.2	0.204956	0.115400
7.3	0.202745	0.114441
7.4	0.200586	0.113499
7.5	0.198476	0.112574
7.6	0.196415	0.111665
7.7	0.194401	0.110772
7.8	0.192431	0.109894
7.9	0.190505	0.109031
8.0	0.188620	0.108183
8.1	0.186777	0.107350
8.2	0.184972	0.106530
8.3	0.183205	0.105724
8.4	0.181475	0.104932
8.5	0.179780	0.104152
8.6	0.178120	0.103385
8.7	0.176493	0.102630
8.8	0.174898	0.101888
8.9	0.173335	0.101157
9.0	0.171801	0.100437
9.1	0.170298	0.099729
9.2	0.168823	0.099032
9.3	0.167375	0.098346
9.4	0.165955	0.097670
9.5	0.164561	0.097004
9.6	0.163192	0.096348
9.7	0.161848	0.095702
9.8	0.160527	0.095066
9.9	0.159231	0.094438
10.0	0.157957	0.093820

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z	$S_0(z)$	$S_1(z)$
10.0	0.157957	0.093820
10.1	0.156705	0.093211
10.2	0.155474	0.092611
10.3	0.154265	0.092019
10.4	0.153076	0.091436
10.5	0.151907	0.090860
10.6	0.150757	0.090293
10.7	0.149626	0.089733
10.8	0.148514	0.089181
10.9	0.147419	0.088637
11.0	0.146342	0.088100
11.1	0.145282	0.087570
11.2	0.144239	0.087048
11.3	0.143212	0.086532
11.4	0.142200	0.086023
11.5	0.141205	0.085520
11.6	0.140224	0.085024
11.7	0.139258	0.084535
11.8	0.138307	0.084052
11.9	0.137369	0.083574
12.0	0.136446	0.083103
12.1	0.135535	0.082638
12.2	0.134638	0.082178
12.3	0.133754	0.081725
12.4	0.132883	0.081276
12.5	0.132024	0.080833
12.6	0.131176	0.080396
12.7	0.130341	0.079964
12.8	0.129517	0.079537
12.9	0.128704	0.079115
13.0	0.127903	0.078698
13.1	0.127112	0.078285
13.2	0.126332	0.077878
13.3	0.125562	0.077475
13.4	0.124802	0.077077
13.5	0.124053	0.076684
13.6	0.123313	0.076295
13.7	0.122582	0.075910
13.8	0.121861	0.075530
13.9	0.121149	0.075154
14.0	0.120446	0.074782
14.1	0.119752	0.074414
14.2	0.119067	0.074050
14.3	0.118390	0.073690
14.4	0.117722	0.073334
14.5	0.117061	0.072982
14.6	0.116409	0.072633
14.7	0.115764	0.072289
14.8	0.115128	0.071948
14.9	0.114498	0.071610
15.0	0.113877	0.071276

z	$S_0(z)$	$S_1(z)$
15.0	0.113877	0.071276
15.1	0.113262	0.070946
15.2	0.112655	0.070618
15.3	0.112055	0.070295
15.4	0.111461	0.069974
15.5	0.110875	0.069657
15.6	0.110295	0.069343
15.7	0.109722	0.069032
15.8	0.109155	0.068724
15.9	0.108594	0.068419
16.0	0.108040	0.068118
16.1	0.107492	0.067819
16.2	0.106950	0.067523
16.3	0.106414	0.067230
16.4	0.105883	0.066940
16.5	0.105358	0.066652
16.6	0.104839	0.066368
16.7	0.104326	0.066086
16.8	0.103818	0.065807
16.9	0.103315	0.065530
17.0	0.102817	0.065256
17.1	0.102325	0.064984
17.2	0.101838	0.064715
17.3	0.101356	0.064449
17.4	0.100878	0.064184
17.5	0.100406	0.063923
17.6	0.099938	0.063663
17.7	0.099475	0.063406
17.8	0.099017	0.063152
17.9	0.098563	0.062899
18.0	0.098114	0.062649
18.1	0.097669	0.062401
18.2	0.097229	0.062155
18.3	0.096792	0.061912
18.4	0.096361	0.061670
18.5	0.095933	0.061431
18.6	0.095509	0.061193
18.7	0.095089	0.060958
18.8	0.094674	0.060725
18.9	0.094262	0.060493
19.0	0.093854	0.060264
19.1	0.093450	0.060037
19.2	0.093050	0.059811
19.3	0.092653	0.059587
19.4	0.092260	0.059366
19.5	0.091871	0.059146
19.6	0.091485	0.058928
19.7	0.091103	0.058711
19.8	0.090724	0.058497
19.9	0.090348	0.058284
20.0	0.089976	0.058073