REMARKS ON THE RANGE OF A VECTOR MEASURE

by JESÚS M. F. CASTILLO and FERNANDO SÁNCHEZ

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A long-standing problem is the characterization of subsets of the range of a vector measure. It is known that the range of a countably additive vector measure is relatively weakly compact and, in addition, possesses several interesting properties (see **[2]**). In **[6]** it is proved that if $m : \Sigma \to X$ is a countably additive vector measure, then the range of m has not only the Banach–Saks property, but even the alternate Banach–Saks property. A tantalizing conjecture, which we shall disprove in this article, is that the range of m has to have, for some $p > 1$, the p-Banach–Saks property. Another conjecture, which has been around for some time (see **[2]**) and is also disproved in this paper, is that weakly null sequences in the range of a vector measure admit weakly-2-summable sub-sequences. In fact, we shall show a weakly null sequence in the range of a countably additive vector measure having, for every $p < \infty$, no weakly-p-summable sub-sequences.

1. Preliminaries. In this paper, Σ is a σ-field of subsets of a set S, and (S, Σ, m) is a finite measure space; $B(\Sigma)$ denotes the space of all bounded Σ-measurable functions—a $C(K)$ space; if $1 \le p \le \infty$, p^* denotes the conjugate number of p; if $p = 1$, l_{p^*} plays the role of c_0.

DEFINITION 1.1. A sequence (x_n) in a Banach space X is said to be *weakly-p-summable* ($p \ge 1$) if there is a $C > 0$ such that

$$\sup_n \left\| \sum_{k=1}^{n} \xi_k x_k \right\| \le C \,.\, \|(\xi_n)\|_{l_{p^*}}$$

for any $(\xi_n) \in l_{p^*}$.

It is said to be *p-Banach-Saks* (see **[8]**), $1 < p < +\infty$, if

$$\left\| \sum_{k=1}^{n} x_k \right\| \le C \,.\, n^{1/p}$$

for some constant $C > 0$ and all $n \in \mathbb{N}$.

We shall say that the sequence (x_n) is weakly-p-convergent (resp. p-Banach-Saks convergent) to $x \in X$ if the sequence $(x_n - x)$ is weakly-p-summable (resp. p-Banach-Saks). Obviously weakly-p-summable sequences are p^*-Banach-Saks. The converse is, in general, false: the sequence $(n^{-1/2})$ is 2-Banach-Saks in $\mathbb{R}$, but it is not 2-summable.

DEFINITION 1.2. An operator $T \in \mathcal{L}(X, Y)$ is said to be *weakly-p-compact*, $1 \le p \le \infty$, if from the image of any bounded sequence in X it is possible to extract a weakly-p-convergent sub-sequence. We shall denote by W_p the ideal of weakly-p-compact operators.

DEFINITION 1.3. A Banach space X is said to belong to W_p if $id(X) \in W_p$; that is, if any bounded sequence admits a weakly p-convergent sub-sequence. It is said to have the p-Banach-Saks property (of Johnson) if any bounded sequence admits a p-Banach-Saks convergent sub-sequence. It is said to have the Banach-Saks property (case $p = 1$) if

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bounded sequences admit sub-sequences having norm convergent arithmetic means.

EXAMPLES 1.4. Parts a) and b) are not difficult to obtain; c) can be seen in [**4**] and d) in [**5**].

a) If $1<p<\infty$, $l_p \in W_r$ if and only if $r \geq p^*$.

b) If $1<p<\infty$, $L_p(\mu) \in W_r$ if and only if $r \geq \max(2, p^*)$.

c) Tsirelson's dual space T^* is such that $T^* \in W_p$ for all $p>1$.

d) Super-reflexive spaces belong to some class W_p.

2. Properties of the range of a vector measure. Concerning sequential properties of the range of a vector measure, a basic result, due to Anantharaman and Diestel, is that weakly-2-summable sequences always lie inside the range of a vector measure: one just has to check that the canonical basis of l_2 is in the range of a vector measure, since weakly-2-summable sequences are their continuous images. We show that this result is, in a sense, the best possible.

EXAMPLE 2.1. *A 2-Banach-Saks sequence which is not contained in the range of a countably additive vector measure.* Consider the Lorentz space $d(c, 1)$ defined by the sequence $\sum_{i \leq n} c_i = \sqrt{n}$. The canonical basis of $d(c, 1)$ is an unconditional basic sequence and a 2-Banach-Saks sequence. On the other hand, it is proved in [**8**] that it is not a weakly-2-summable sequence. The following Proposition of [**2**] settles the counter example.

PROPOSITION 2.2. *A normalized unconditional basic sequence in the range of a vector measure is weakly-2-summable.*

The proof runs as follows: a normalized basic sequence (x_n) in the range of a vector measure can be translated into a normalized weakly null sequence (f_n) in some space $L_1(\lambda)$. For a given $x^* \in X^*$, the sequence $\{x^*(x_n) . f_n\}$ is, together with (x_n), unconditionally summable; thus, from Orlicz' theorem, it is norm-2-summable, and therefore (x_n) is weakly-2-summable.

A positive result in the characterization of ranges of vector measures is:

THEOREM 2.3. *Let X be a Banach space of finite cotype. If $m : \Sigma \rightarrow X$ is a countably additive vector measure, then the range of m is a weakly-2-compact set.*

Proof. Consider the operator $T : B(\Sigma) \rightarrow X$, defined by $T(\chi_E) = m(E)$. Because X does not contain c_0 finitely represented, it follows from [**10**, p. 284] that there is a $p>1$ such that T is absolutely-p-summing, and therefore it sub-factorizes through an L_p-space. This and (1.4.b) imply $T \in W_2$.

All this leads one to ask whether the range of a vector measure might be a weakly-2-compact set. The answer is strongly negative:

EXAMPLE 2.4. Let Y be the following weakly compact set of $L_1[0, 1]$:

$$Y = \{f \circ \phi : \phi : [0, 1] \rightarrow [0, 1] \text{ is bijective and bi-measurable}\}$$

where the function $f \in L_1[0, 1]$ is chosen so that the sequence $(\langle r_n, f \rangle)_n$ does not belong to any l_p for $1 \leq p < \infty$. This is possible because the sequence (r_n) is equivalent to the canonical basis of l_1; since weakly-p-summable and weakly*-p-summable are equivalent notions, one gets, for every $p>1$, a function g_p in $L_1[0, 1]$ such that $(\langle r_n, g_p \rangle)_n$ does not belong to l_p. An easy consequence of Baire's theorem allows us to obtain the desired function f.

Let (χ_n) denote the following sequence of characteristic functions in $L_\infty[0, 1]$:

$$\chi_n(t) = \max\{r_{n+1}(t), 0\}$$

where $\{r_n\}$ denotes the sequence of Rademacher functions. This sequence (χ_n) is weak*-convergent to $\frac{1}{2}$. Since $\chi_n - \frac{1}{2} = \frac{1}{2} . r_n$, the sequence $(\chi_n - \frac{1}{2})$ is not weakly-p-summable in $L_\infty[0, 1]$ for every p.

By the Davis–Figiel–Johnson–Pelczynski factorization theorem, there exists a reflexive Banach space X and an operator $T : X \to L_1[0, 1]$ such that $Y \subset T(\mathbf{B}_X)$. The operator T^* gives us a vector measure μ whose range is not weakly-p-compact for any p; that is, the sequence $\{\mu(\chi_n)\}$ does not admit weakly-p-convergent sub-sequences, since the only possible accumulation point for $\{\mu(\chi_n)\}$ is $T^*(\frac{1}{2}\chi_{[0,1]})$. However this is not the case: let $A \subset \mathbb{N}$ infinite and $\mathbb{N} = N_0 \cup N_1$, both infinite and such that

$$\sum_{n \in N_1} |\langle r_n, f \rangle|^p = \infty.$$

Let us choose the representation $[0, 1] = \{-1, 1\}^{\mathbb{N}}$. It is well-known that they are Borel-equivalent, that is, there is a bijection bi-measurable $[0, 1] \to \{-1, 1\}^{\mathbb{N}}$. The Rademacher functions become canonical projections, and every permutation of $\mathbb{N}$ induces a bi-measurable bijection in $[0, 1]$. Let σ be a permutation of $\mathbb{N}$ such that $\sigma(N_1) = A$. Now, if ϕ is the induced function by σ then

$$\sum_{n \in A} |\langle r_n, f \circ \phi \rangle|^p = \sum_{n \in \sigma(N_1)} |\langle r_n \circ \phi^{-1}, f \rangle|^p = \sum_{n \in N_1} |\langle r_{\sigma(n)} \circ \phi^{-1}, f \rangle|^p = \infty.$$

COROLLARY 2.5. *The range of a countably additive vector measure need not have the p-Banach-Saks property for $p > 1$.*

Proof. It has been proved in **[5]** that the p-Banach-Saks property implies the W_r property for all $r > p^*$.

It is well-known that if the unit ball of a Banach space lies inside the range of a countably additive vector measure then the space is super-reflexive. To obtain the desired counter-examples we shall use a variation of (2.3).

THEOREM 2.6. *If the unit ball of X lies inside the range of a countably additive vector measure, then $X \in W_2$.*

Proof. The operator $T : B(\Sigma) \to X$ of the proof of (2.3) is surjective. From **[9**, Corollary 11] it follows that T, and consequently $id(X)$, belong to W_2.

COROLLARY 2.7. *The unit ball of Tsirelson's 2-convexified space T_2^* and, for $0 < \gamma < 10^{-6}$, of Tirilman's space $T_i(2, \gamma)$, do not lie inside the range of acountably additive vector measure.*

Proof. Let X be any of those spaces. We show that if $X \in W_2$ and $X^* \in W_2$ then X (and X^*) contain a copy of $l_2 : X \in W_2$ implies that X contains a semi-normalized weakly-2-summable basic sequence (x_n); since the sequence of associated biorthogonal functionals (x_n^*) contains a weakly-2-summable sub-sequence (x_k^*), it follows that the sequence (x_k) is equivalent to the canonical l_2-basis.

We only need to verify that $X \notin W_2$. Since X^* is a space of type 2 having an unconditional basis, and does not contain a copy of l_2 (see [3]), the following result implies that $X^* \in W_2$.

Proposition 2.8. *Let X be a reflexive Banach space with an unconditional basis. If X is of Rademacher type p, then* $X \in W_{p^*}$.

Proof. Since X is reflexive, we can assume that (x_n) is a weakly null sequence in X. If it is norm-null, then there is nothing to prove. If not, we apply the Bessaga–Pelczynski selection principle to obtain an unconditional basic sub-sequence (x_k), and so for any sequence (α_n) in the unit ball of l_p we have:

$$\left\| \sum_{k=1}^{n} \alpha_k x_k \right\| \leq K \, . \left\| \sum_{k=1}^{n} \alpha_k r_k(t) x_k \right\|,$$

where (r_n) is the Rademacher sequence.

On integrating this inequality, as we may, we get:

$$\left\| \sum_{k=1}^{n} \alpha_k x_k \right\| \leq K \, . \int_0^1 \left\| \sum_{k=1}^{n} \alpha_k r_k(t) x_k \right\| dt \leq K \, . \, C \, \|(\alpha_k \, \|x_k\|)\|_{l_p} \leq K \, . \, C \, \|(\alpha_k)\|_{l_p},$$

since X has Rademacher type p. This proves that $X \in W_{p^*}$ as required.

Since Lorentz function spaces $\Lambda(W, p)$ and Lorentz sequence spaces $d(a, p)$ contain a subspace isomorphic to l_p (see [7] and [1]), from this, (2.6) and (1.4.a) it follows:

Corollary 2.9. *If* $1 < p < 2$ *the unit ball of a Lorentz spaces* $\Lambda(W, p)$ *and* $d(a, p)$ *does not lie inside the range of a countably additive vector measure.*

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Note Added in Proof. Proposition 2.8 has been improved, omitting the hypothesis of an unconditional basis, by Farmer and Johnson, Polynomial Schur and polynomial Dunford-Pettis properties, *Contemporary Math. AMS.* **144** (1993), 95–105, and by Ito and Okado, Applications of spreading methods to regular methods of summability and growth rate of Cèsaro means, *J. Math. Soc. Japan* **44** (1992), 591–612.

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DEPARTAMENTO DE MATEMÁTICAS,
UNIVERSIDAD DE EXTREMADURA,
AVDA. DE ELVAS,
S/N. 06071 BADAJOZ,
ESPAÑA.