BOUNDLESSNESS OF SOME INTEGRAL OPERATORS

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ABSTRACT. We apply the expression for the norm of a function in the weighted Lorentz space, with respect to the distribution function, to obtain as a simple consequence some weighted inequalities for integral operators.

1. Introduction. Given a measure space $\mathcal{M}$ and a function $k: \mathcal{M} \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$, we define the operator

$$T_k f(x) = \int_0^\infty k(x,t)f(t)dt.$$  

The boundedness of this operator

$$T_k: L^p(w_0) \rightarrow L^p(d\mu),$$  

(1)

for nonincreasing functions, where $w_0$ is a nonnegative locally integrable function (that is a weight) in $\mathbb{R}^+$ and $d\mu$ is a measure on $\mathcal{M}$, has been widely studied for particular choices of the kernel $k$ (see [1], [2], [6], [7], ...).

In particular, if $k(x,t) = x^{-a}(tx^{-a})$, the weak boundedness of

$$T_k: L^p(w_0) \rightarrow L^{p,\infty}(w_1),$$

with $w_1$ a weight in $\mathbb{R}^+$, has been completely solved by K. Andersen in [1]. If $a$ satisfies some extra condition, he also gets the strong boundedness of the operator $T_k: L^p(w) \rightarrow L^p(w)$. A related work can be found in [5] where the authors consider the boundedness of a particular case of the operator $T_k$, with $k(x,t) = \chi_{[0,1]}(t)\varphi(t/x)$ but with no monotone restriction on the functions $f$.

In [7], E. Sawyer solved question (1), for $1 < p_0, p_1$, via the study of $T_k^*$ whenever this operator can be easily identified and its boundedness easily studied. His argument is based upon a duality type result for nonincreasing functions (see Theorem 3.1). Results about particular cases of operators $T_k$ have many other proofs (see [2], [6], ...).

Our point of view consists mainly in studying this type of question as a consequence of the boundedness of an operator $T$ associated to $T_k$ in the weighted Lorentz spaces. To be precise, let $\mathcal{N}_c^\sigma(w)$ be the space of all measurable functions $f$ on a measure space $\mathcal{N}$ such that $\|f\|_{\mathcal{N}_c^\sigma(w)} = \left(\int_0^\infty \left(\left(\int f_\sigma^*(x)^p w(x)dx\right)^p\right)^{1/p} < +\infty, \right)$ where $\sigma$ is a $\sigma$-finite measure on $\mathcal{N}$ and $w$ is a locally integrable function (that is, a weight) and $f_\sigma^*$ denotes the rearrangement.

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function with respect to the measure $d\sigma$. Then, we try to characterize the measures $\sigma_j$ and the weights $w_j$ such that $T: \mathcal{N}_0^p(w_0) \rightarrow \mathcal{N}_{\sigma_1}^p(w_1)$, or $T: \mathcal{N}_0^p(w_0) \rightarrow \mathcal{N}_{\sigma_1}^{p,\infty}(w_1)$ are bounded, where the weak space $\mathcal{N}_\sigma^{p,\infty}(w)$ is defined as in [4], namely

$$\|f\|_{\mathcal{N}_\sigma^{p,\infty}(w)} = \sup_{y>0} \left( \int_0^{\lambda_y^p(y)} w(t) \, dt \right)^{1/p} < +\infty.$$  

Given a $\sigma$-finite measure $\sigma$ on $\mathcal{N}$, we shall denote by $\sigma(A) = \int_A d\sigma(x)$ and $\lambda_y^p(x) = \sigma\left\{ x : |f(x)| > y \right\}$. When $d\sigma(x) = u(x) \, dx$, we shall write $u(A)$, $\lambda_y^p$ and $f^*$ respectively. Finally, if we are working with the Lebesgue measure, $u$ is omitted and we simply write $|A|$, $\lambda_y^p$ or $f^*$. We shall write $L^p_{\mathrm{dec}}(w)$ to denote the set of all nonincreasing functions in $L^p(w)$.

The paper is organized as follows. In Section 2, the boundedness of the operator $T_k: L^p_{\mathrm{dec}}(w_0) \rightarrow L^p(\sigma_1)$ is completely solved in the range $0 < p_0 < 1$, $p_0 < p_1$ as a consequence of a more general result (see Theorem 2.4 and Proposition 2.5). In Section 3, we study the weak boundedness of $T_k: \mathcal{N}_0^p(w_0) \rightarrow \mathcal{N}_{\sigma_1}^{p,\infty}(w_1)$ whenever $T_k$ satisfies a weak monotone property condition. Also, if $T_kf$ is a nonincreasing function for $f$ nonincreasing, then we get a characterization of the boundedness of the operator $T_k: L^p_{\mathrm{dec}}(w_0) \rightarrow L^{p,\infty}(w_1)$ which gives another proof of the result of K. Andersen we mentioned above. In Section 4, we finish with a very simple proof of the boundedness of the generalized Hardy operator and its generalized conjugate operator, for the case $L_p(w)$. This proof is closely related to the (also very simple) proof of Neugebauer for the Hardy operator (see [6]).

2. Case $0 < p_0 \leq 1$. In [4], the following formula using the distribution function was proved.

**Theorem 2.1.** Let $(\mathcal{N}, \sigma)$ be a measure space and $w$ a weight in $\mathbb{R}^+$. Then, for $0 < p < \infty$, we get

$$\int_0^\infty (f^*(t))^p w(t) \, dt = p \int_0^\infty y^{p-1} \left( \int_0^{\lambda_y^p(y)} w(t) \, dt \right) dy.$$  

To prove it, it suffices to check it for simple functions.

It is trivial to show that for particular choices of $k$ we can obtain both the Hardy operator $\hat{S}f(x) = x^{-1} \int_0^x f(t) \, dt$ and its conjugate $\hat{T}f(x) = \int_0^x f(t) t^{-1} \, dt$. Then, a first application of Theorem 2.1 is given by the following result.

**Corollary 2.2.** (i) If $f$ is a nonincreasing function, $\int_0^\infty k(x, t)f(t) \, dt = \int_0^\infty \int_0^{\lambda_y^p(y)} k(x, t) \, dt \, dy$.

(ii) $S(f^*_x)(x) = \int_0^\infty \min\{1, \lambda_y^p(y)/x\} \, dx$.

(iii) $\hat{T}(f^*_x)(x) = \int_0^\infty \log^+\left(\lambda_y^p(y)/x\right) \, dy$.

By standard arguments using a dyadic decomposition, one can easily obtain the following discretization formula, (see [4]).
COROLLARY 2.3. For every measurable function \( f \) in \( \mathcal{N}_0(w) \), and \( 0 < p \leq \infty \),
\[
\|f\|_{\mathcal{N}_0(w)} \approx \left( \sum_{k=-\infty}^{\infty} 2^{kp} \left( \int_0^{\lambda_k(2^k)} w(t) \, dt \right) \right)^{1/p}.
\]

The main result of this section is the following:

THEOREM 2.4. Let \((\mathcal{N}, \sigma)\) and \((\mathcal{M}, d\mu)\) be two \( \sigma \)-finite measure spaces. Given a measurable function \( f \) in \( \mathcal{N} \), we can define \( Tf(x) = T_k(f^*(x)) \) for every \( x \in \mathcal{M} \). Let \( \sigma_0 \) be another \( \sigma \)-finite measure in \( \mathcal{N} \), and \( w_0 \) a weight in \( \mathbb{R}^+ \). Then, if \( 0 < p_0 \leq 1 \) and \( p_0 < p \), the operator \( T : \mathcal{N}_0(w_0) \to L^p(d\mu) \) is bounded if and only if, there exists a constant \( C > 0 \) such that
\[
\left( \int_{\mathcal{M}} \left( \int_0^{\lambda_0(y)} k(x, t) \, dt \right)^{p_1} \, d\mu(x) \right)^{1/p_1} \leq C \left( \int_0^{\lambda_0(y)} w_0(x) \, dx \right)^{1/p_0},
\]
for all measurable sets \( A \) in \( \mathcal{N} \).

PROOF. To prove the necessity condition, it is enough to apply the hypothesis to the characteristic function \( \chi_A \).

Conversely, condition (2) implies that
\[
\left( \int_{\mathcal{M}} \left( \int_0^{\lambda_0(y)} k(x, t) \, dt \right)^{p_1} \, d\mu(x) \right)^{1/p_1} \leq C \left( \int_0^{\lambda_0(y)} w_0(x) \, dx \right)^{1/p_0}.
\]

Then, if \( p_1 \leq 1 \), we get using Theorem 2.1 and Corollary 2.3, that
\[
\left( \int_{\mathcal{M}} (Tf(x))^{p_1} \, d\mu(x) \right)^{1/p_1} = \left( \int_{\mathcal{M}} \left( \int_0^{\lambda_0(y)} k(x, t) \, dt \right)^{p_1} \, d\mu(x) \right)^{1/p_1}
\leq \left( \int_{\mathcal{M}} \left( \int_0^{\lambda_0(y)} k(x, t) \, dt \right)^{p_1} \, d\mu(x) \right)^{1/p_1}
\leq C \left( \int_0^{\lambda_0(y)} w_0(x) \, dx \right)^{1/p_0}.
\]

Finally, since \( p_0 / p_1 \leq 1 \), using again Corollary 2.3, we get
\[
\left( \int_{\mathcal{M}} (Tf(x))^{p_1} \, d\mu(x) \right)^{1/p_1} \leq C \left( \int_0^{\lambda_0(y)} w_0(x) \, dx \, dy \right)^{1/p_0} \approx \|f\|_{\mathcal{N}_0(w_0)}.
\]

Now, for \( p_1 > 1 \),
\[
Tf(x) = T_k(f^*(x)) = \int_0^{\lambda_0(y)} k(x, t) \, dt \, dy.
\]

Hence, by Minkowski integral inequality and the hypothesis,
\[
\|Tf\|_{L^p(d\mu)} \leq \int_0^{\lambda_0(y)} \left( \int_0^{\lambda_0(y)} k(x, t) \, dt \right)^{1/p_1} \, dx \, dy.
\]

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Finally, since $p_0 < 1$, we get, by Corollary 2.3,

$$
\|Tf\|_{L^{p_1}(d\mu)} \leq C \int_0^\infty y^{p_0-1} \left( \int_0^\infty \left( \int_0^y w_0(x) \, dx \right) \, dy \right) \, dy = C \|f\|_{L_{\text{dec}}^{p_0}(w_0)}.
$$

**Proposition 2.5.** Let $w_0$ and $w_1$ be two weights in $\mathbb{R}^+$, $0 < p_0 \leq 1$ and $p_0 \leq p_1$. Then, the operator $T_k: L_{\text{dec}}^{p_0}(w_0) \to L^{p_1}(w_1)$ is bounded, if and only if, for every $r > 0$,

$$
\left( \int_0^\infty \left( \int_0^r k(x,t) \, dt \right)^{p_1} w_1(x) \, dx \right)^{1/p_1} \leq C \left( \int_0^\infty w_0(x) \, dx \right)^{1/p_0}.
$$

**Proof.** To prove the necessity condition, it is enough to apply the hypothesis to the characteristic function $f = \chi_{(0,r)}$. Conversely, by Theorem 2.4, with both $\sigma$ and $\sigma_0$ equals the Lebesgue measure, and $d\mu(x) = w_1(x) \, dx$, we obtain that $Tf = T_kf$ for every nonincreasing function and $T: \mathcal{A}^{p_0}(w_0) \to L^{p_1}(w_1)$. It now remains to observe that $L_{\text{dec}}^{p_0}(w_0)$ is a subspace of $\mathcal{A}^{p_0}(w_0)$.

**Proposition 2.6.** Let $u_0$, $u_1$ be two weights in $\mathbb{R}^n$ and $w_0$, $w_1$ two weights in $\mathbb{R}^+$. Then, if $0 < p_0 \leq 1$ and $p_0 \leq p_1$,

(a) 

$$
\left( \int_0^\infty \left( \frac{1}{x} \int_0^x f'_{u_0}(s) \, ds \right)^{p_1} w_1(x) \, dx \right)^{1/p_1} \leq C \left( \int_0^\infty f'_{u_0}(x)^{p_0} w_0(x) \, dx \right)^{1/p_0},
$$

if and only if,

$$
\left( \int_0^{u_1(A)} w_1(x) \, dx + u_1(A)^{p_1} \int_{u_1(A)}^\infty \frac{w_1(x)}{x^{p_1}} \, dx \right)^{1/p_1} \leq C \left( \int_0^{\sigma_0(A)} w_0(x) \, dx \right)^{1/p_0},
$$

for every measurable set $A \subset \mathbb{R}^n$.

(b) The Hardy operator $S$ is bounded from $L_{\text{dec}}^{p_0}(w_0)$ into $L^{p_1}(w_1)$ if and only if,

$$
\left( \int_0^r w_1(x) \, dx + r^{p_1} \int_r^\infty \frac{w_1(x)}{x^{p_1}} \, dx \right)^{1/p_1} \leq C \left( \int_0^r w_0(x) \, dx \right)^{1/p_0},
$$

for every $r > 0$.

**Proof.** It suffices to consider $M = \mathbb{R}^+$, $\mathcal{A} = \mathbb{R}^n$, $d\mu(x) = w_1(x) \, dx$, $d\sigma(x) = u_1(x) \, dx$, $d\sigma_0(x) = u_0(x) \, dx$ and $k(x,t) = x^{-1} \chi_{[0,1]}(t)$ in Theorem 2.4.

**Remark 2.7.** If $Mf$ is the Hardy-Littlewood maximal function of $f$ and using the fact that $(Mf)^*(x) \approx S(f^*)(x)$, the above proposition gives a characterization of the weights $u_0$, $w_0$ and $w_1$ for which $M$ is bounded from $\mathcal{A}^{p_0}(w_0)$ into $\mathcal{A}^{p_1}(w_1)$, for $0 < p_0 \leq 1$ and $p_0 \leq p_1$. If $p \geq 1$, the characterization of the boundedness of $M$ in $\mathcal{A}^{p}(w)$ was first given by Ariño and Muckenhoupt in [2]. In the case $1 < p_0, p_1$ and $u_0 = 1$, the boundedness of $M$ from $\mathcal{A}^{p_0}(w_0)$ into $\mathcal{A}^{p_1}(w_1)$ was proved by E. Sawyer in [7].
PROPOSITION 2.8. Let \( u_0, u_1 \) be two weights in \( \mathbb{R}^n \) and \( w_0, w_1 \) two weights in \( \mathbb{R}^+ \). Then, if \( 0 < p_0 \leq 1 \) and \( p_0 \leq p_1 \),

(a)

\[
\left( \int_0^\infty \left( \int_0^\infty f_{u_1}^*(s) \frac{ds}{s} \right)^{p_1} w_1(x) \, dx \right)^{1/p_1} \leq C \left( \int_0^\infty f_{u_0}^*(x)^{p_0} w_0(x) \, dx \right)^{1/p_0},
\]

if and only if,

\[
\left( \int_0^\infty \left( \log^+(\frac{u_1(A)}{x}) \right)^{p_1} w_1(x) \, dx \right)^{1/p_1} \leq C \left( \int_0^\infty w_0(x) \, dx \right)^{1/p_0},
\]

for every measurable set \( A \subset \mathbb{R}^n \).

(b) The conjugate Hardy operator \( S \) is bounded from \( L^p_{\text{dec}}(w_0) \) into \( L^p(w_1) \) if and only if,

\[
\left( \int_0^\infty \left( \log^+(\frac{r}{x}) \right)^{p_1} w_1(x) \, dx \right)^{1/p_1} \leq C \left( \int_0^\infty w_0(x) \, dx \right)^{1/p_0},
\]

for every \( r > 0 \).

PROOF. It suffices to consider \( \mathcal{M} = \mathbb{R}^+ \), \( \mathcal{N} = \mathbb{R}^n \), \( d\mu(x) = w_1(x) \, dx \), \( d\sigma(x) = u_1(x) \, dx \) and \( k(x, t) = t^{-1} \chi_{[0,\infty)}(t) \) in Theorem 2.4.

Another easy application of Theorem 2.4 is the boundedness of the Calderón operator. Recall that for \( 1 \leq r_0 < r_1 \leq \infty \), \( 1 \leq q_0, q_1 \leq \infty \), \( q_0 \neq q_1 \) and \( m = (1/q_0 - 1/q_1)/(1/r_0 - 1/r_1) \) the Calderón operator is defined by

\[
Sf(t) = t^{-1/q_0} \int_0^t s^{1/r_0} f(s) \frac{ds}{s} + t^{-1/q_1} \int_t^\infty s^{1/r_1} f(s) \frac{ds}{s}.
\]

This operator plays an important role in the theory of rearrangement invariant spaces (see [3]). Let us write \( S^1 \) for the first integral term and \( S^2 \) for the second one, so that \( S = S^1 + S^2 \).

PROPOSITION 2.9. Let \( (\mathcal{N}_0, \sigma_0) \) and \( (\mathcal{N}_1, \sigma_1) \) be two \( \sigma \)-finite measure spaces. Then, if \( 0 < p_0 \leq 1 \) and \( p_0 \leq p_1 \), we get that

(a)

\[
\|S^1(f_{\sigma_1}^*)\|_{L^p(w_1)} \leq C\|f\|_{\mathcal{N}_0^{w_0}(w_0)},
\]

if and only if,

\[
\left( \int_0^{\sigma_1(A)^{1/m}} x^{m/r_0 - 1/q_0} w_1(x) \, dx + \sigma_1(A)^{p_1/r_0} \int_{\sigma_1(A)^{1/m}}^{\infty} x^{-p_1/q_0} w_1(x) \, dx \right)^{1/p_1} \leq C \left( \int_0^{\sigma_0(A)} w_0(x) \, dx \right)^{1/p_0},
\]

for every measurable set \( A \subset \mathcal{N}_1 \).
(b) If and only if,

\[
\left( \int_0^{\sigma(A)} x^{-p_1/q_1} \sigma(A) - x^{m_1/p_1} w_1(x) \, dx \right)^{1/p_1} \leq C \left( \int_0^{\sigma(A)} w_0(x) \, dx \right)^{1/p_0},
\]

for every measurable set \( A \subset \mathcal{N} \).

**Proof.** (a) It suffices to consider \( M = \mathbb{R}^+, \mathcal{N} = \mathbb{N}, d\mu(x) = w_1(x) \, dx, d\sigma(x) = d\sigma_1(x) \) and \( k(x, t) = x^{-1/q_1 t^{1/n_0-1}} \chi_{[0, u]}(t) \) in Theorem 2.4.

(b) It suffices to consider \( k(x, t) = x^{-1/q_1 t^{1/n_0-1}} \chi_{[0, u]}(t) \) and \( M, \mathcal{N}, d\mu, d\sigma, d\sigma_0 \) as in (a), in Theorem 2.4.

3. Some results in the case \( p_0 > 0 \). The following result is due to E. Sawyer (see [7]) and it will be used very often in what follows.

**Theorem 3.1.** Suppose \( 1 < p < +\infty \) and that \( v(x) \) and \( g(x) \) are nonnegative measurable functions on \( \mathbb{R}^+ \), with \( v \) locally integrable. Then

\[
\int_{\mathbb{R}^+} f(x) g(x) \, dx \leq \left( \int_{\mathbb{R}^+} \left( \int_0^x g(t) \, dt \right)^{p'/p} \right)^{1/p'} \left( \int_{\mathbb{R}^+} v(x) \, dx \right)^{1-p'/p},
\]

where the supremum is taken over all nonnegative and nonincreasing functions \( f \).

Moreover, the right side of (3) can be replaced with the integral

\[
\left( \int_0^\infty \left( \int_0^x g(t) \, dt \right)^{p'-1} \left( \int_0^x v(t) \, dt \right)^{1-p'} g(x) \, dx \right)^{1/p'}. \]

Using the ideas developed in [1], we can give an easy proof for the \( \leq \) inequality.

**Proof.** For the \( \geq \) inequality we have to consider the function

\[
f(x) = \left( \int_x^{\infty} \frac{g(t)}{v(s) \, ds} \, dt \right)^{p'-1},
\]

(see [7]). Conversely, set

\[
h(t) = \left( \int_0^t \left( \int_0^x g(s) \, ds \right)^{p'-1} \left( \int_0^x v(s) \, ds \right)^{1-p'} v(x) \, dx + \left( \int_0^\infty g(s) \, ds \right)^{p'-1} \left( \int_0^\infty v(s) \, ds \right)^{1-p'} \right)^{1/p'}. \]

Then,

\[
\int_0^\infty f(x) g(x) \, dx = \int_0^\infty f(x) g(x) h(x) h^{-1}(x) \, dx \leq \left( \int_0^\infty f^p(x) h^{-p}(x) g(x) \, dx \right)^{1/p} \left( \int_0^\infty h^p(x) g(x) \, dx \right)^{1/p'}. \]
Applying Fubini to the second factor in the previous inequality we obtain the right hand side of (3). For the first factor, we observe that

$$h(x)^{-p} \leq \left( \int_0^x g(t) \, dt \right)^{-1} \left( \int_0^x v(t) \, dt \right)$$

and thus, by Theorem 2.1,

$$\int_0^\infty f^p(x) h^{-p}(x) g(x) \, dx = p \int_0^\infty y^{p-1} \left( \int_0^{\lambda(y)} h^{-p}(x) \, dx \right) \, dy.$$ 

Integrating by parts the inner integral and erasing the negative terms one has that the previous expression can be bounded, up to multiplicative constants, by

$$\int_0^\infty y^{p-1} \left( \int_0^{\lambda(y)} h^{-p}(x) \, dx \right) \, dy \approx \int_0^\infty f^p(x) v(x) \, dx.$$ 

Using this result, E. Sawyer proves that if $\int_0^\infty w(x) \, dx = +\infty$, then the dual space of $N^*_p(w)$ can be identified with the space $\Gamma^p_u(\tilde{w})$, defined by the norm

$$\|f\|_{\Gamma^p_u(\tilde{w})} = \left( \int_0^\infty \left( \frac{1}{x} \int_0^x f^*_u(s) \, ds \right)^{p'} \tilde{w}(x) \, dx \right)^{1/p'},$$

where $\tilde{w}(x) = x^{-1} \int_0^x w(t) \, dt$.

For $p > 1$, we also have (see [4]) the following result.

**THEOREM 3.2.** Suppose $p \leq 1$ and that $v(x)$ and $g(x)$ are nonnegative measurable functions on $\mathbb{R}^+$ with $v$ locally integrable. Then

$$\sup_{r > 0} \frac{\int_0^\infty f(x) g(x) \, dx}{\left( \int_0^\infty f^p(x) v(x) \, dx \right)^{1/p}} \approx \sup_{r > 0} \left( \int_0^r g(x) \, dx \left( \int_0^r v(x) \, dx \right)^{-1/p} \right),$$

where the supremum is taken over all nonnegative and nonincreasing functions $f$.

The first immediate consequence is the following.

**THEOREM 3.3.** Let $T_k f(x) = \int_0^\infty k(x, t) f(t) \, dt$ and let us assume that $T_k f$ is a nonincreasing function whenever $f$ is a nonincreasing function. Then, the operator $T_k : L^p_{\text{dec}}(w_0) \to N^{p, \infty}(w_1)$ is bounded if and only if,

(a) if $p_0 > 1$,

$$\sup_{t > 0} \left( \int_0^t \left( \int_0^\infty k(z, t) \, dt \right)^{p_0'} \left( \int_0^\infty w_0(t) \, dt \right)^{-p_0} w_0(y) \, dy \right)^{1/p_0'}$$

$$+ \int_0^\infty k(z, t) \, dt \left( \int_0^\infty w_0(s) \, ds \right)^{-1/p_0} \left( \int_0^\infty w_1(s) \, ds \right)^{1/p_1} < +\infty.$$
(b) if $p_0 \leq 1$,

$$\sup_{z>0} \left( \sup_{r>0} \left( \int_0^r k(z, x) \, dx \right) \left( \int_0^r w_0(x) \, dx \right)^{-1/p_0} \left( \int_0^r w_1(s) \, ds \right)^{1/p_1} \right) < +\infty$$

PROOF Observe that to show that $\sup_{y>0} y \left( \int_0^{\lambda y} w_1(x) \, dx \right)^{1/p_1} \leq \|f\|_{L^p(w_0)}$ it is enough to consider values of $y$ equals $T_k f(z)$ for all $z$, and thus, we have to see that

$$\sup_{z>0} T_k f(z) \left( \int_0^r w_1(x) \, dx \right)^{1/p_1} \leq C \|f\|_{L^p(w_0)},$$

for all nonincreasing $f$. This is equivalent to showing that

$$\sup_{z>0} \left( \sup_{f \in L^p_{\text{dec}}(w_0)} \frac{T_k f(z)}{\|f\|_{L^p(w_0)}} \left( \int_0^r w_1(x) \, dx \right)^{1/p_1} \right) \leq C < +\infty,$$

and Theorems 3.1 and 3.2, lead us to the conclusion.

In particular, if $k(x, t) = x^{-1}(tx^{-1})$ this was proved by K. Andersen (see [1]). The following two results give us the weak boundedness of an integral operator when it satisfies a monotone type condition.

**Theorem 3.4** Let $T_k f(x) = \int_0^\infty k(x, t) f(t) \, dt$ and let us assume that, for every $x$, there exists a measurable set $I_x$ of positive measure such that $T_k f(x) \leq T_k f(t)$, for every $t \in I_x$ and every $f$, and if $T_k f(x) < T_k f(t)$ for some $f$, then $t \in I_x$. Then, $T_k : \Lambda^p_0(w_0) \to \Lambda^p_0(\infty(w_1))$ is bounded if and only if,

(a) if $p_0 > 1$,

$$\sup_{z>0} \left( \|u_0^{-1}(z, \cdot)\|_{L^p_0(w_0)} + \int_0^\infty k(z, t) \, dt \left( \int_0^\infty w_0(s) \, ds \right)^{1/p_0} \left( \int_0^{u_0(t)} w_1(t) \, dt \right)^{1/p_1} \right) < +\infty,$$

(b) if $p_0 \leq 1$,

$$\sup_{z>0} \left( \sup_{r>0} \left( \int_0^r k(z, t) \, dt \right) \left( \int_0^r w_0(t) \, dt \right)^{-1/p_0} \left( \int_0^{u_0(t)} w_1(t) \, dt \right)^{1/p_1} \right) < +\infty$$

PROOF Let $f \in \Lambda^p_0(w_0)$ and assume that $f \geq 0$. Then, for every $t \in I_x$,

$$\int_0^\infty k(x, s)f(s) \, ds \leq T_k f(t)$$

and, hence, if we write $\xi = \int_0^\infty k(x, s)f(s) \, ds$, we get

$$u_1(I_x) \leq \int_{\{t, T_k f(t) > \xi\}} u_1(s) \, ds = \lambda^u_{T_k f} (\xi)$$
Then,
\[
\xi \left( \int_0^{w_1(t_x)} w_1(s) \, ds \right)^{1/p_1} \leq \xi \left( \int_0^{\frac{1}{2}} w_1(s) \, ds \right)^{1/p_1} \\
\leq \sup_{y > 0} \left( \int_0^1 w_1(s) \, ds \right)^{1/p_1} = \| T_kf \|_{\Lambda^{q_1}_{\infty}(w_1)} \leq C \| f \|_{\Lambda^{q_0}_{\infty}(w_0)}.
\]
Therefore, taking the supremum over all \( \xi < \left( \int_0^\infty k(x, s)f(s) \, ds \right) \), we get
\[
\sup_{x > 0} \left( \frac{\int_0^\infty k(x, s)f(s) \, ds}{\| f \|_{\Lambda^{q_0}_{\infty}(w_0)}} \right)^{1/p_1} \left( \int_0^{w_1(t_x)} w_1(s) \, ds \right)^{1/p_1} < \infty,
\]
and we get the conclusion by Theorems 3.1 and 3.2.

Conversely, we shall only prove (a) (the proof of (b) is entirely analogous). Let \( f \geq 0 \) in \( \Lambda^{q_0}_{\infty}(w_0) \) and set \( x_j \) such that if \( \sup_{x > 0} \int_0^\infty k(x, s)f(s) \, ds > 2^j \), then \( T_kf(x_j) = 2^j \). Then, since,
\[
\| T_kf \|_{\Lambda^{q_1}_{\infty}(w_1)} \leq C \sup_{x \in \mathbb{R}} \left( \int_0^\infty k(x, s)f(s) \, ds \right)^{1/p_1} \left( \int_0^{w_1(t_x)} w_1(s) \, ds \right)^{1/p_1} < \infty,
\]
we get,
\[
\| T_kf \|_{\Lambda^{q_1}_{\infty}(w_1)} \leq C \sup_{x \in \mathbb{R}} \left( \int_0^\infty k(x, s)f(s) \, ds \right)^{1/p_1} \left( \int_0^{w_1(t_x)} w_1(s) \, ds \right)^{1/p_1} < \infty.
\]

The following result will give us the strong boundedness of the operator \( T_k \) for a particular choice of \( k \). As a consequence, we partially obtain a result of E. Sawyer (see [7]).

**Theorem 3.5.** Let \( 1 < p_0 \leq p_1 \) and let \( k(x, t) = \chi_{[0,\infty)}(t)\phi(t) \), where \( \phi \) is a nonincreasing locally integrable function in \( \mathbb{R}^+ \). Then, if \( w_0 \) is a nondecreasing weight in \( \mathbb{R}^+ \), the operator \( T_k : L^{p_0}_{\text{dec}}(w_0) \rightarrow L^{p_1}(w_1) \) is bounded if and only if
\[
\sup_{z > 0} \left( \int_0^z \left( \int_0^{\min(z,\infty)} \phi(t) \, dt \right)^{1/p_0} \frac{w_0(x)}{\left( \int_0^{z} w_0(t) \, dt \right)^{1/p_0}} \, dx \right)^{1/p_0} < +\infty.
\]
PROOF. To prove the necessary condition, we observe that $k(x, t)$ satisfies the hypothesis of the previous theorem and since $L^{p_1, \infty}(w_1) = \Lambda_{w_1}^{p_1}(1)$ we get the result as in Theorem 3.4. Conversely, we proceed as in Theorem 3.4, but in this case we observe that

$$2^j \approx 2^j_{x_{j-1}} f(s)\phi(s) \, ds$$

and hence if we call $f_j = f\chi_{(0,x_{j-1},x_j)}$, we get

$$\|Tf\|_{L^{p_1}(w_1)} \leq C \sum_{j \in \mathbb{Z}} \left( \int_{x_{j-1}}^{2^j} f(s)\phi(s) \, ds \right)^{p_1} \left( \int_{x_j}^{2^j} w_1(s) \, ds \right)$$

$$\leq C \sum_{j \in \mathbb{Z}} \left( \int_{0}^{2^j-x_{j-1}} f(s+x_{j-1})\phi(s) \, ds \right)^{p_1} \left( \int_{x_j}^{2^j} w_1(s) \, ds \right)$$

$$\leq C \sum_{j \in \mathbb{Z}} \|f_j\|_{L^{p_1}(w_0)} \left( \int_{0}^{\infty} \left( \int_{0}^{|\min(x_j-x_{j-1})|} \phi(t) \, dt \right)^{p_0} \frac{w_0(x)}{\int_{0}^{\infty} w_0(t) \, dt} \right)^{1/p_0}$$

$$+ \int_{0}^{\infty} \phi(t) \, dt \left( \int_{0}^{\infty} w_0 \right)^{-1/p_0} \left( \int_{x_j}^{2^j} w_1(s) \, ds \right)$$

$$\leq C \sum_{j \in \mathbb{Z}} \left( \int_{x_{j-1}}^{2^j} f(s)w_0(s-x_{j-1}) \, ds \right)^{p_1/p_0}$$

$$\leq C \sum_{j \in \mathbb{Z}} \left( \int_{x_{j-1}}^{2^j} f(s)w_0(s) \, ds \right)^{p_1/p_0} \leq C \|f\|_{L^{p_1}(w_0)}^{p_1/p_0}.$$
Conversely, let us observe that
\[
\left( \int_0^t f(s) \phi(t) \, dt \right)^p = p \int_0^t \left( \int_0^s f(r) \phi(s) \, dr \right)^{p-1} f(s) \phi(t) \, dt
\]
\[= p \int_0^t \left( \frac{1}{\Phi(t)} \int_0^s f(r) \phi(s) \, dr \right)^{p-1} f(t) \Phi(t)^{-1} \phi(t) \, dt.
\]
Let us write \( g(t) = \left( \frac{1}{\Phi(t)} \int_0^t f(s) \phi(s) \, ds \right)^{p-1} f(t) \). Hence,
\[
\|S_0 f\|_{L^p(w)} = p^{1/p} \left( \int_0^\infty \left( \int_0^x g(t) \Phi(t)^{-p-1} \phi(t) \, dt \right) \frac{w(x)}{\Phi(x)^p} \, dx \right)^{1/p}.
\]
Now, since \( g \) is a nonincreasing function we get by Theorem 2.1,
\[
\int_0^x g(t) \Phi(t)^{-p-1} \phi(t) \, dt = \int_0^\infty \int_0^{\lambda_y(x)} \chi_{(0, x)}(t) \Phi(t)^{-p-1} \phi(t) \, dt \, dy
\]
\[= \frac{1}{p} \int_0^\infty \Phi \left( \min \left( \lambda_y(y), x \right) \right) \, dy.
\]
Therefore,
\[
\|S_0 f\|_{L^p(w)}^p = \int_0^\infty \int_0^\infty \Phi \left( \min \left( \lambda_y(y), x \right) \right) \frac{w(x)}{\Phi(x)^p} \, dx \, dy
\]
\[= \int_0^\infty \int_0^{\lambda_y(x)} w(x) \, dx + \Phi \left( \lambda_y(y) \right) \int_0^\infty \frac{w(x)}{\Phi(x)^p} \, dx \, dy
\]
\[\leq C \int_0^\infty \int_0^{\lambda_y(y)} w(x) \, dx \, dy = C \int_0^\infty g(x) w(x) \, dx
\]
\[= \int_0^\infty \left( \frac{1}{\Phi(x)} \int_0^x f(s) \phi(s) \, ds \right)^{p-1} f(x) w(x) \, dx
\]
\[\leq C \|f\|_{L^p(w)} \|S_0 f\|_{L^p(w)} = C \|f\|_{L^p(w)} \|S_0 f\|_{L^p(w)}^{-1},
\]
where the last inequality is obtained by using Hölder’s inequality. \( \square \)

**THEOREM 4.2.** Let \( p > 1 \) and \( \Phi(x) = \int_0^x \phi(t) \, dt \). Then, the generalized conjugate Hardy operator
\[
\tilde{S}_0 f(x) = \int_x^\infty f(t) \phi(t) \frac{dt}{\Phi(t)}
\]
satisfies that \( \|\tilde{S}_0 f\|_{L^p(w)} \leq C \|f\|_{L^p(w)} \) for all \( f \) nonincreasing, if and only if
\[
\int_0^\infty \left( \log \frac{\Phi(r)}{\Phi(x)} \right)^p w(x) \, dx \leq C \int_0^\infty w(x) \, dx.
\]

**PROOF.** One has to follow the same steps as in the previous proof but in this case we use the identity
\[
\left( \int_x^\infty f(t) \phi(t) \frac{dt}{\Phi(t)} \right)^p = p \int_x^\infty \left( \int_t^\infty f(s) \phi(s) \frac{ds}{\Phi(s)} \right)^{p-1} f(t) \phi(t) \frac{dt}{\Phi(t)}.
\]
and write $g(t) = \left( \int_t^{\infty} f(s) \phi(s) \frac{ds}{\Phi(s)} \right)^{p-1} f(t)$.

**Remark 4.3.** We observe that in Theorems 4.1 and 4.2 we can also prove, as a consequence of Proposition 2.5, the boundedness of these generalized Hardy operators in the case of $p \leq 1$.

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