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CARTAN–EILENBERG FP-INJECTIVE COMPLEXES

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Abstract

In this article, we extend the notion of FP-injective modules to that of Cartan–Eilenberg complexes. We show that a complex C is Cartan–Eilenberg FP-injective if and only if C and $\mathrm{Z}(C)$ are complexes consisting of FP-injective modules over right coherent rings. As an application, coherent rings are characterized in various ways, using Cartan–Eilenberg FP-injective and Cartan–Eilenberg flat complexes.

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1. Introduction

In classical homological algebra, the projective and injective modules play important and fundamental roles. In Chapter XVII of *Homological Algebra*, Cartan and Eilenberg [3] gave the definitions of projective and injective resolutions of a complex of modules. Subsequently, Verdier considered these resolutions and called them Cartan–Eilenberg projective and injective resolutions of a complex. Also, the definitions of Cartan–Eilenberg injective, projective and flat complexes were introduced [23].

In [2], Beligiannis developed a homological algebra in a triangulated category which parallels the homological algebra in an exact category in the sense of Quillen. In particular, he defined projective and injective objects in triangulated categories, and called them ξ-projective objects and ξ-injective objects, respectively, where ξ denotes the proper class of triangles. However, in general it is not so easy to find a proper class ξ of triangles in a triangulated category that has enough ξ-projective objects or ξ-injective objects.

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As we know, the homotopy category of complexes of R-modules is a triangulated category and so-called homotopically projective complexes form the relative projective objects for a proper class of triangles in the homotopy category; see [2, Sections 12.4 and 12.5]. It is easy to see that C is homotopically projective in the homotopy category [2] if and only if C is homotopy equivalent to a Cartan–Eilenberg projective complex in the category of complexes [9].

Therefore, Cartan–Eilenberg complexes play an important role in the category of complexes and the homotopy category. In [9, 13, 14, 26], the authors also considered Cartan–Eilenberg complexes and obtained some important results. For instance, Enochs proved that every complex has a Cartan–Eilenberg injective envelope, every complex has a Cartan–Eilenberg projective precover and a complex is Cartan–Eilenberg flat if and only if it is the direct limit of finitely generated Cartan–Eilenberg projective complexes [9]. In [13, 14, 26], the authors investigated the Cartan–Eilenberg Gorenstein complexes, the stability of Cartan–Eilenberg Gorenstein categories and established some relationships between Cartan–Eilenberg complexes and DG complexes.

A left R-module M is called FP-injective if $\mathrm{Ext}^1(P, M) = 0$ for any finitely presented module P. General background material on FP-injective modules can be found in [1, 12, 15, 18–21].

Motivated by these, our purpose in this article is to introduce and investigate a Cartan–Eilenberg version of FP-injective modules. We call them Cartan–Eilenberg FP-injective complexes.

The paper is organized as follows.

In Section 2, we give some notation and some fundamental facts about the Cartan–Eilenberg complexes, which will be important later on. Section 3 defines Cartan–Eilenberg finitely generated and Cartan–Eilenberg finitely presented complexes and gives some properties. In Section 4, we introduce the concept of Cartan–Eilenberg FP-injective complexes and characterize such complexes. Finally, in Section 5, we interpret coherent rings in terms of Cartan–Eilenberg FP-injective complexes and Cartan–Eilenberg flat complexes.

For the rest of this paper, we will use the abbreviation C-E for Cartan–Eilenberg.

2. Preliminaries

Throughout this paper, R denotes a ring with unity. A complex

$$\cdots \xrightarrow{\delta_2} C_1 \xrightarrow{\delta_1} C_0 \xrightarrow{\delta_0} C_{-1} \xrightarrow{\delta_{-1}} \cdots$$

of R-modules will be denoted by (C, δ) or C. For a ring R, R-Mod denotes the category of left R-modules and $\mathscr{C}(R)$ denotes the abelian category of complexes of left R-modules.

We will use superscripts to distinguish complexes. So, if $\{C^i\}_{i\in I}$ is a family of complexes, C^i will be

$$\cdots \xrightarrow{\delta_2} C^i_1 \xrightarrow{\delta_1} C^i_0 \xrightarrow{\delta_0} C^i_{-1} \xrightarrow{\delta_{-1}} \cdots .$$

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Given an R-module M, we use $\overline{M}$ to denote the complex

$$\cdots \longrightarrow 0 \longrightarrow M \xrightarrow{id} M \longrightarrow 0 \longrightarrow \cdots$$

with M's in the 1st and 0th positions. We also use $\underline{M}$ to denote the complex with M in the 0th place and 0 in the other places.

Given a complex C and an integer m, $\sum^m C$ denotes the complex such that $(\sum^m C)_l = C_{l-m}$ and whose boundary operators are $(-1)^m \delta_{l-m}$. The lth homology module of C is the module $\mathrm{H}_l(C) = \mathrm{Z}_l(C)/\mathrm{B}_l(C)$, where $\mathrm{Z}_l(C) = \mathrm{Ker}(\delta_l^C)$ and $\mathrm{B}_l(C) = \mathrm{Im}(\delta_{l+1}^C)$.

Let C be a complex of left R-modules (respectively, of right R-modules) and let D be a complex of left R-modules. We will denote by $\mathrm{Hom}_R(C, D)$ (respectively, $C \otimes_R D$) the usual homomorphism complex (respectively, tensor product) of the complexes C and D.

Given two complexes C and D, $\mathrm{Hom}(C, D)$ is the abelian group of morphisms from C to D and Ext^i for $i \geq 0$ will denote the groups we get from the right derived functor of Hom. Let $\underline{\mathrm{Hom}}(C, D) = \mathrm{Z}(\mathrm{Hom}_R(C, D))$. Then $\underline{\mathrm{Hom}}(C, D)$ can be made into a complex with $\mathrm{Hom}(C, D)_m$, the abelian group of morphisms from C to $\sum^{-m} D$, and with boundary operator given by $f \in \mathrm{Hom}(C, D)_m$; then $\delta_m(f) : C \to \sum^{-(m-1)} D$ with $\delta_m(f)_l = (-1)^m \delta^D f_l$ for any $l \in \mathbb{Z}$ and we put $C^+ = \underline{\mathrm{Hom}}(C, \overline{\mathbb{Q}/\mathbb{Z}})$. Let C be a complex of right R-modules and D be a complex of left R-modules. We define $C \overline{\otimes} D$ to be $(C \otimes_R D)/\mathrm{B}(C \otimes_R D)$. Then, with the maps

$$\frac{(C \otimes_R D)_m}{\mathrm{B}_m(C \otimes_R D)} \to \frac{(C \otimes_R D)_{m-1}}{\mathrm{B}_{m-1}(C \otimes_R D)}, \quad x \otimes y \longmapsto \delta^C(x) \otimes y,$$

where $x \otimes y$ is used to denote the coset in $(C \otimes_R D)_m/\mathrm{B}_m(C \otimes_R D)$, we get a complex. We note that the new functor $\underline{\mathrm{Hom}}(C, D)$ will have right derived functors whose values will be complexes. These values should certainly be denoted by $\underline{\mathrm{Ext}}^i(C, D)$. It is not hard to see that $\underline{\mathrm{Ext}}^i(C, D)$ is the complex

$$\cdots \to \mathrm{Ext}^i\Big(C, \sum^{-(m+1)} D\Big) \to \mathrm{Ext}^i\Big(C, \sum^{-m} D\Big) \to \mathrm{Ext}^i\Big(C, \sum^{-(m-1)} D\Big) \to \cdots$$

with boundary operator induced by the boundary operator of D. For a complex C of left R-modules, since $-\overline{\otimes} C$ is a right exact functor, we can construct right derived functors, which we denote by $\overline{\mathrm{Tor}}_i(-, C)$.

We will use $\mathcal{P}$ to denote the category of projective left R-modules. Then we will use the obvious modifications, for example $\mathcal{I}$, $\mathcal{F}$ and $\mathcal{FP}$, of this notation.

We recall some notions and facts needed in the sequel.

Definition 2.1 [9]. A complex P is said to be C-E projective if $P, \mathrm{Z}(P), \mathrm{B}(P)$ and $\mathrm{H}(P)$ are complexes consisting of projective modules.

A complex I is said to be C-E injective if $I, \mathrm{Z}(I), \mathrm{B}(I)$ and $\mathrm{H}(I)$ are complexes consisting of injective modules.

A complex F is said to be C-E flat if $F, \mathrm{Z}(F), \mathrm{B}(F)$ and $\mathrm{H}(F)$ are complexes consisting of flat modules.

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More generally, for any class $\mathscr{X}$ of R-modules, we will let CE($\mathscr{X}$) consist of all complexes C with $C_n, \mathrm{Z}_n(C), \mathrm{B}_n(C), \mathrm{H}_n(C) \in \mathscr{X}$. So then CE ($\mathcal{P}$) is the class of C-E projective complexes. In particular, we take $\mathscr{X}$ be the class of all free R-modules. Then we call them C-E free complexes.

Definition 2.2 [9]. A complex of complexes

$$\cdots \to C^{-1} \to C^0 \to C^1 \to \cdots$$

is said to be C-E exact if:

(1) $\cdots \to C^{-1} \to C^0 \to C^1 \to \cdots;$
(2) $\cdots \to \mathrm{Z}(C^{-1}) \to \mathrm{Z}(C^0) \to \mathrm{Z}(C^1) \to \cdots;$
(3) $\cdots \to \mathrm{B}(C^{-1}) \to \mathrm{B}(C^0) \to \mathrm{B}(C^1) \to \cdots;$
(4) $\cdots \to C^{-1}/\mathrm{Z}(C^{-1}) \to C^0/\mathrm{Z}(C^0) \to C^1/\mathrm{Z}(C^1) \to \cdots;$
(5) $\cdots \to C^{-1}/\mathrm{B}(C^{-1}) \to C^0/\mathrm{B}(C^0) \to C^1/\mathrm{B}(C^1) \to \cdots;$
(6) $\cdots \to \mathrm{H}(C^{-1}) \to \mathrm{H}(C^0) \to \mathrm{H}(C^1) \to \cdots$

are all exact.

Lemma 2.3 [9]. *The functor* $\mathrm{Hom}(-,-)$ *on* $\mathscr{C}(R)\times\mathscr{C}(R)$ *is right balanced by* $\mathrm{CE}(\mathcal{P}) \times \mathrm{CE}(\mathcal{I})$.

This result says that we can compute derived functors of $\mathrm{Hom}(-,-)$ using either of the two resolutions (that is, C-E projective resolution and C-E injective resolution). For given C and D, we will denote these derived functors applied to (C, D) as $\overline{\mathrm{Ext}}^n(C, D)$. It is obvious that $\overline{\mathrm{Ext}}^n(C, D) \subseteq \mathrm{Ext}^n(C, D)$.

The proof of the following results is routine.

Lemma 2.4.

(1) *The functor* $\underline{\mathrm{Hom}}(-,-)$ *on* $\mathscr{C}(R) \times \mathscr{C}(R)$ *is right balanced by* $\mathrm{CE}(\mathcal{P}) \times \mathrm{CE}(\mathcal{I})$.
(2) *The functor* $-\overline{\otimes}-$ *on* $\mathscr{C}(R) \times \mathscr{C}(R)$ *is left balanced by* $\mathrm{CE}(\mathcal{F}) \times \mathrm{CE}(\mathcal{F})$.

So, we can compute derived functors of $\underline{\mathrm{Hom}}(-,-)$ using either of the two resolutions. For given C and D, we will denote these derived functors applied to (C, D) as $\overline{\underline{\mathrm{Ext}}}^n(C, D)$. It is obvious that $\overline{\underline{\mathrm{Ext}}}^n(C, D) \subseteq \underline{\mathrm{Ext}}^n(C, D)$. We also can compute derived functors of $-\overline{\otimes}-$ using the C-E flat resolutions. For given C and D, we will denote the derived functors applied to (C, D) as $\overline{\underline{\mathrm{Tor}}}_n(C, D)$. It is clear that $\overline{\underline{\mathrm{Tor}}}_n(C, D) \subseteq \overline{\mathrm{Tor}}_n(C, D)$.

3. Cartan–Eilenberg finitely generated and Cartan–Eilenberg finitely presented complexes

Definition 3.1. A complex C is said to be C-E finitely generated if C is bounded and $C_m, \mathrm{Z}_m(C), \mathrm{B}_m(C), \mathrm{H}_m(C)$ are finitely generated in R-Mod for all $m \in \mathbb{Z}$. (Equivalently, C is bounded and $C_m, \mathrm{Z}_m(C)$ are finitely generated in R-Mod for all $m \in \mathbb{Z}$.)

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A complex C is said to be C-E finitely presented if C is bounded and $C_m, \mathrm{Z}_m(C), \mathrm{B}_m(C), \mathrm{H}_m(C)$ are finitely presented in R-Mod for all $m \in \mathbb{Z}$. (Equivalently, C is bounded and $C_m, \mathrm{Z}_m(C)$ are finitely presented in R-Mod for all $m \in \mathbb{Z}$.)

Example 3.2. Let M be a finitely presented R-module. Then $\overline{M}$ and $\underline{M}$ are C-E finitely presented complexes.

Definition 3.3. A C-E exact sequence of complexes $0 \to A \to B \to C \to 0$ is said to be C-E pure if $0 \to \mathrm{Hom}(P, A) \to \mathrm{Hom}(P, B) \to \mathrm{Hom}(P, C) \to 0$ is exact for any C-E finitely presented complex P.

The following observations are useful, whose proofs are routine.

Lemma 3.4. *Let $0 \to A \to B \to C \to 0$ be a short* C-E *exact sequence of complexes. Then the following statements hold*:

(1) *if A is* C-E *finitely generated and B is* C-E *finitely presented, then C is* C-E *finitely presented;*
(2) *if A and C are* C-E *finitely presented, then so is B;*
(3) *if R is a left coherent ring and B and C are* C-E *finitely presented, then so is A.*

Lemma 3.5. *Let C be a complex. Then the following statements are equivalent:*

(1) *C is* C-E *finitely presented;*
(2) *there exists a* C-E *exact sequence $0 \to L \to P \to C \to 0$ of complexes, where P is* C-E *finitely generated and* C-E *projective, and L is* C-E *finitely generated;*
(3) *there exists a* C-E *exact sequence $P^1 \to P^0 \to C \to 0$ of complexes, where P^0, P^1 are* C-E *finitely generated and* C-E *free.*

Lemma 3.6. *Any complex is the direct limits of* C-E *finitely presented complexes.*

Proof. Let C be any complex. Then C is a direct union of bounded complexes. Hence, we can suppose that C has the following form:

$$C =: \cdots \to 0 \to C_0 \to \cdots \to C_n \to 0 \to \cdots .$$

Assume that $F_i' \to \mathrm{H}_i(C) \to 0$ and $F_i'' \to \mathrm{B}_i(C) \to 0$ are free presentations of $\mathrm{H}_i(C)$ and $\mathrm{B}_i(C)$ for $i = 0, 1, \ldots, n$, respectively. Then we can construct a C-E free presentation of C: $F \to C \to 0$ in $\mathscr{C}(R)$.

We consider the pairs (G, S), where $G \subseteq F$ is a C-E finitely generated subcomplex with G C-E free and $S \subseteq G$ a C-E finitely generated subcomplex of G. We order the family $\{(G, S)\}$ by $(G, S) \leq (G', S') \Leftrightarrow G \subseteq G',\ S \subseteq S'$. Then G/S is C-E finitely presented in $\mathscr{C}(R)$ and $\lim_{\to} G/S = C$. □

Lemma 3.7. *Let R and S be rings, L a complex of right S-modules, K a complex of (R, S)-bimodules and P a complex of left R-modules. Suppose that P is* C-E *finitely presented and L is* C-E *injective as complexes of right S-modules. Then*

$$\underline{\mathrm{Hom}}(K, L) \overline{\otimes} P \cong \underline{\mathrm{Hom}}(\underline{\mathrm{Hom}}(P, K), L)$$

as complexes. This isomorphism is functorial in P, K and L.

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Proof. We define

$$\lambda^P : \underline{\mathrm{Hom}}(K, L) \overline{\otimes} P \to \underline{\mathrm{Hom}}(\underline{\mathrm{Hom}}(P, K), L)$$

$$f \otimes p \mapsto \lambda^P(f \otimes p)$$

for $f \in \underline{\mathrm{Hom}}(K, L)$ and $p \in P$ in the following way. For $m \in \mathbb{Z}$, we consider

$$\lambda^P_m : (\underline{\mathrm{Hom}}(K, L) \overline{\otimes} P)_m \to (\underline{\mathrm{Hom}}(\underline{\mathrm{Hom}}(P, K), L))_m$$

$$f \otimes p \mapsto \lambda^P_m(f \otimes p) : \underline{\mathrm{Hom}}(P, K) \to \sum^{-m} L.$$

Suppose that $f \in \underline{\mathrm{Hom}}(K, L)_d,\ p \in P_t$ with $d + t = m$. Take $n \in \mathbb{Z}$. Then

$$\lambda^P_m(f \otimes p)_n : \underline{\mathrm{Hom}}(P, K)_n \to L_{m+n}$$

$$g \mapsto (-1)^{\beta(d,t,n)}(f_{n+t}g_t)(p),$$

where $\beta(d, t, n) = dt + \binom{n+t+1}{2}$, $g \in \underline{\mathrm{Hom}}(P, K)_n$. By the proof of [10, Lemma 4.2.2], we have that λ^P_m is well defined for all $m \in \mathbb{Z}$ and λ^P is a map of complexes.

Therefore, we have a map of complexes

$$\lambda^P : \underline{\mathrm{Hom}}(K, L) \otimes P \to \underline{\mathrm{Hom}}(\underline{\mathrm{Hom}}(P, K), L).$$

If we take $P = \overline{R}$ or $P = \underline{R}$, it is easy to see that $\lambda^{\overline{R}}$ and $\lambda^{\underline{R}}$ are isomorphisms. On the other hand, any C-E finitely generated and C-E free complex

$$F = \bigoplus_{n \in \mathbb{Z}} \Big(\sum^n \overline{F^1_n} \bigoplus \sum^n \underline{F^2_n} \Big),$$

where F^1_n and F^2_n are finitely generated free modules for all $n \in \mathbb{Z}$. Hence, if F is C-E finitely generated and C-E free, then λ^F is an isomorphism.

Since our original P is C-E finitely presented, we can find a C-E exact sequence

$$H \to F \to P \to 0$$

with H and F C-E finitely generated and C-E free complexes. Since λ^F and λ^H are isomorphisms, standard arguments show that λ^P is also an isomorphism. □

Lemma 3.8. *The following conditions are equivalent for a* C-E *exact sequence* $0 \to S \to C \to C/S \to 0$ *in* $\mathscr{C}(R)$*:*

(1) $0 \to S \to C \to C/S \to 0$ *is* C-E *pure;*

(2) $\underline{\mathrm{Hom}}(P, C) \to \underline{\mathrm{Hom}}(P, C/S) \to 0$ *is exact for every* C-E *finitely presented complex* P*;*

(3) $0 \to D \overline{\otimes} S \to D \overline{\otimes} C$ *is exact for every complex* D *or every* C-E *finitely presented complex* D*;*

(4) $0 \to (C/S)^+ \overline{\otimes} P \to C^+ \overline{\otimes} P \to S^+ \overline{\otimes} P$ *is exact for every* C-E *finitely presented complex* P *or the sequence splits;*

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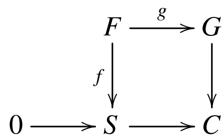
(5) $0 \to S \to C \to C/S \to 0$ *is a direct limit of splitting short* C-E *exact sequences;*

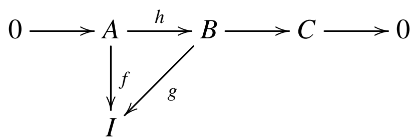
(6) *for any commutative diagram*



where F, G are C-E *finitely generated and* C-E *free complexes, there exists $h : G \to S$ with $hg = f$.*

Proof. Using Lemmas 3.6 and 3.7 and the adjoint isomorphism of complexes, the proof follows by the same argument as in the case of modules (see for example [25, page 287]). □

Definition 3.9. We will say that a complex I is C-E pure injective if it is injective relative to every C-E pure exact sequence. That is to say, for any C-E pure exact sequence of complexes $0 \to A \xrightarrow{h} B \to C \to 0$ and any morphism of complexes $f : A \to I$, there exists $g : B \to I$ such that $gh = f$, that is, the following diagram:



is commutative.

Proposition 3.10. *Let R be a ring. Then the following statements are true*:

(1) *if $N \to M$ is a pure monomorphism in R-*Mod*, then $\overline{N} \to \overline{M}$ and $\underline{N} \to \underline{M}$ are* C-E *pure monomorphisms in $\mathscr{C}(R)$;*

(2) *C^+ is* C-E *pure injective for any complex C;*

(3) *$C \to C^{++}$ is a* C-E *pure monomorphism for any complex C.*

Proof. (1) is clear by Lemma 3.8.

(2) and (3) are proved as in the case of modules. □

Remark 3.11. From Lemma 3.8, we have that C-E pure exact sequences coincide with pure exact sequences.

4. Cartan–Eilenberg FP-injective complexes

In this section, we will introduce and investigate the concept of C-E FP-injective complexes and give some equivalent characterizations of C-E FP-injective complexes.

Definition 4.1. A complex C is said to be C-E FP-injective if $\overline{\mathrm{Ext}}^1(P, C) = 0$ for any C-E finitely presented complex P.

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REMARK 4.2. (1) Recall that a complex C is FP-injective if $\mathrm{Ext}^1(P, C) = 0$ for any finitely presented complex P [24]. Any FP-injective complex is C-E FP-injective; however, the converse is not true (see Example 4.3).

(2) The class of C-E FP-injective complexes is closed under direct products and summands.

(3) A complex C is C-E FP-injective if and only if $\overline{\mathrm{Ext}}^1(P, C) = 0$ for any C-E finitely presented complex P.

EXAMPLE 4.3. Let M be an FP-injective module. Then $\underline{M}$ is C-E FP-injective; however, $\underline{M}$ is not FP-injective.

PROOF. Let P be any C-E finitely presented complex. Then

$$\overline{\mathrm{Ext}}^1(P, \underline{M}) = \mathrm{Ext}^1(P_0/\mathrm{B}_0(P), M) = 0,$$

and so $\underline{M}$ is C-E FP-injective. It is clear that $\underline{M}$ is not FP-injective by [24, Theorem 2.10]. □

As we know, a complex I is said to be C-E injective if $I, \mathrm{Z}(I), \mathrm{B}(I)$ and $\mathrm{H}(I)$ are complexes consisting of injective modules. In [24], the authors proved that a complex C is FP-injective if and only if C is exact and $\mathrm{Z}_n(C)$ is FP-injective in R-Mod for each $n \in \mathbb{Z}$. In the present article, the following result can be obtained.

THEOREM 4.4. *Let R be a right coherent ring. Then a complex C is* C-E *FP-injective if and only if C and $\mathrm{Z}(C)$ are complexes consisting of FP-injective modules.*

To prove Theorem 4.4, we first establish the following lemmas.

LEMMA 4.5. *If C is a* C-E *FP-injective complex, then C and $\mathrm{Z}(C)$ are complexes consisting of FP-injective modules.*

PROOF. It follows from the isomorphisms $\overline{\mathrm{Ext}}^1(\sum^k(\underline{M}), D) = \mathrm{Ext}^1(M, \mathrm{Z}_k(D))$ and $\overline{\mathrm{Ext}}^1(\sum^k(\overline{M}), D) = \mathrm{Ext}^1(M, D_k)$, where M is a module, D is a complex and k is an integer [9, Lemmas 9.1 and 9.2]. □

It is well known that a module C is FP-injective if and only if C is pure in every module that contains it [15, 18]; a complex C is FP-injective if and only if C is pure in every complex that contains it [24]. Here we get the following result.

LEMMA 4.6. *Let R be a ring. Then the following conditions are equivalent for a complex C of left R-modules:*

(1) *C is* C-E *FP-injective;*

(2) *for any* C-E *exact sequence of complexes $0 \to C \to B \to A \to 0$, C is* C-E *pure in B as a subcomplex;*

(3) *for any* C-E *exact sequence of complexes $0 \to C \to I \to L \to 0$ with I* C-E *injective, C is* C-E *pure in I as a subcomplex;*

(4) *C is* C-E *pure in $I(C)$, where $I(C)$ is the* C-E *injective envelope of C;*

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(5) *for every* C-E *exact sequence of complexes* $0 \to X \to Y \to Z \to 0$ *with* Z C-E *finitely presented, the functor* $\mathrm{Hom}(-, C)$ *preserve the exactness;*

(6) *every* C-E *exact sequence of complexes* $0 \to C \to I \to L \to 0$ *with* L C-E *finitely presented splits.*

Proof. (1) ⇒ (2). Let $0 \to C \to B \to A \to 0$ be a C-E exact sequence and P a C-E finitely presented complex. Then

$$0 \to \underline{\mathrm{Hom}}(P, C) \to \underline{\mathrm{Hom}}(P, B) \to \underline{\mathrm{Hom}}(P, A) \to \overline{\underline{\mathrm{Ext}}}^1(P, C) = 0$$

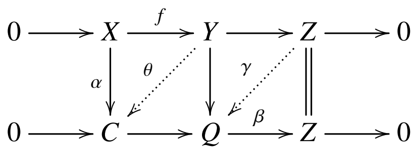
is exact. So, C is C-E pure in B by Lemma 3.8.

(2) ⇒ (3) ⇒ (4), (1) ⇒ (5) and (5) ⇒ (6) are obvious.

(4) ⇒ (1). Let P be any C-E finitely presented complex. Then $\underline{\mathrm{Hom}}(P, I(C)) \to \underline{\mathrm{Hom}}(P, I(C)/C) \to 0$ is exact and so $\overline{\underline{\mathrm{Ext}}}^1(P, C) = 0$, which means that C is C-E FP-injective.

(5) ⇒ (1). Let P be any C-E finitely presented complex. Then there exists a C-E exact sequence of complexes $0 \to X \to F \to P \to 0$ with F C-E projective. So, $\overline{\underline{\mathrm{Ext}}}^1(P, C) = 0$ and C is C-E FP-injective.

(6) ⇒ (5). Let $0 \to X \to Y \to Z \to 0$ be a C-E exact sequence with Z C-E finitely presented. For a morphism $\alpha : X \to C$, we form the following pushout diagram:



By (6), the sequence

$$0 \to C \to Q \to Z \to 0$$

splits and so there exists γ such that $\beta\gamma = 1$. Thus, there exists θ such that $\theta f = \alpha$ by the homotopy lemma. So, (5) follows. □

Using Lemmas 2.4 and 3.6 and the standard homological method, the following results can be obtained.

Lemma 4.7. *The following conditions are equivalent for a complex* C*:*

(1) C *is* C-E *flat;*

(2) $\overline{\underline{\mathrm{Tor}}}_1(P, C)=0$ *for any* C-E *finitely presented complex* P*;*

(3) $C, C/\mathrm{B}(C)$ *are complexes consisting of flat modules.*

Lemma 4.8. *Let* Y, X *be two complexes over an arbitrary ring* R*. Then* $\overline{\underline{\mathrm{Ext}}}^1(Y, X^+) \cong \overline{\underline{\mathrm{Tor}}}_1(Y, X)^+$.

Lemma 4.9.

(1) *For any ring* R*, a complex* C *is* C-E *flat if and only if* C^+ *is* C-E *FP-injective.*

(2) *If* R *is right coherent, then a complex* C *is* C-E *FP-injective if and only* C^+ *is* C-E *flat.*

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PROOF. (1) It follows by Lemmas 4.7 and 4.8.

(2) "$\Rightarrow$" It follows by Lemmas 4.5 and 4.7.

"$\Leftarrow$" Let C^+ be C-E flat, $0 \to C \to B \to A \to 0$ be C-E exact and P C-E be finitely presented. Then $0 \to A^+ \to B^+ \to C^+ \to 0$ is C-E pure exact. Thus, $0 \to \underline{\mathrm{Hom}}(P, A^+) \to \underline{\mathrm{Hom}}(P, B^+) \to \underline{\mathrm{Hom}}(P, C^+) \to 0$ is exact, which implies that $0 \to (P \overline{\otimes} A)^+ \to (P \overline{\otimes} B)^+ \to (P \overline{\otimes} C)^+ \to 0$ is exact. So, $0 \to P \overline{\otimes} C \to P \overline{\otimes} B \to P \overline{\otimes} A \to 0$ is exact and hence C is C-E FP-injective by Lemma 4.6. □

PROOF OF THEOREM 4.4. "$\Rightarrow$" It follows by Lemma 4.5.

"$\Leftarrow$" Let C_n and $\mathrm{Z}_n(C)$ be FP-injective modules. Note that

$$(C^+)_n = C^+_{-n}, \quad (C^+)_n/\mathrm{B}_n(C^+) = (\mathrm{Z}_{-n}(C))^+$$

for all $n \in \mathbb{Z}$. Then C^+ is a C-E flat complex by Lemma 4.7. Therefore, C is C-E FP-injective using Lemma 4.9. □

Here we define the following terms for any class, $\mathcal{X}$, of complexes.

(1) A class, $\mathcal{X}$, of complexes is said to be closed under C-E extensions if for every short C-E exact sequence $0 \to X' \to X \to X'' \to 0$ with $X'' \in \mathcal{X}$ and $X' \in \mathcal{X}$, $X \in \mathcal{X}$.

(2) We call $\mathcal{X}$ C-E projectively resolving if $\mathrm{CE}(\mathcal{P}) \subseteq \mathcal{X}$ and, for every short C-E exact sequence $0 \to X' \to X \to X'' \to 0$ with $X'' \in \mathcal{X}$, the conditions $X' \in \mathcal{X}$ and $X \in \mathcal{X}$ are equivalent.

(3) We call $\mathcal{X}$ C-E injectively resolving if $\mathrm{CE}(\mathcal{I}) \subseteq \mathcal{X}$ and, for every short C-E exact sequence $0 \to X' \to X \to X'' \to 0$ with $X' \in \mathcal{X}$, the conditions $X'' \in \mathcal{X}$ and $X \in \mathcal{X}$ are equivalent.

PROPOSITION 4.10.

(1) *The class of all* C-E *FP-injective complexes is closed under* C-E *extensions and* C-E *pure subcomplexes.*

(2) *The class of all* C-E *flat complexes is closed under* C-E *extensions and direct sums,* C-E *pure subcomplexes and* C-E *pure quotient complexes.*

PROOF. (1) It is easy to show that the class of all C-E FP-injective complexes is closed under C-E extensions.

Let B be a C-E pure subcomplex of a C-E FP-injective complex A and D a C-E finitely presented complex. Then $0 \to B \to A \to A/B \to 0$ is C-E exact. Thus, $0 \to \underline{\mathrm{Hom}}(D, B) \to \underline{\mathrm{Hom}}(D, A) \to \underline{\mathrm{Hom}}(D, A/B) \to 0$ is exact. Note that $\overline{\underline{\mathrm{Ext}}}^1(D, A) = 0$ and so $\overline{\underline{\mathrm{Ext}}}^1(D, B) = 0$. Therefore, B is C-E FP-injective.

(2) It is easy to show that the class of all C-E flat complexes is closed under C-E extensions and direct sums.

Let B be a C-E pure subcomplex of a C-E flat complex A. Then $0 \to B \to A \to A/B \to 0$ is C-E pure exact. Thus, $0 \to (A/B)^+ \to A^+ \to B^+ \to 0$ is split. We have that B^+ is C-E FP-injective, since A^+ is C-E FP-injective by Lemma 4.8, and so B is C-E flat.

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Let S be a C-E pure subcomplex of a C-E-flat complex C. Then the C-E pure exact sequence $0 \to S \to C \to C/S \to 0$ induces the split exact sequence $0 \to (C/S)^+ \to C^+ \to S^+ \to 0$. Thus, $(C/S)^+$ is C-E FP-injective, since C^+ is C-E FP-injective by Lemma 4.9. So, C/S is C-E flat by Lemma 4.9 again. □

5. A note on coherent rings

Coherent rings have been characterized in various ways. The deepest result is the one due to Chase [4], which claims that the ring R is left coherent if and only if products of flat right R-modules are again flat if and only if products of copies of R are flat right R-modules. For other characterizations of coherency, see Chen, Ding, Glaz, Matlis, Stenström [5–7, 11, 16, 17, 21] and so on.

In this section, we give some characterizations of coherent rings, using C-E FP-injective and C-E flat complexes.

THEOREM 5.1. *The following statements are equivalent for any ring R:*

(1) *R is left coherent;*
(2) *every direct product of* C-E *flat complexes of right R-modules is* C-E *flat;*
(3) *every direct limit of* C-E *FP-injective complexes of left R-modules is* C-E *FP-injective;*
(4) *a complex C of left R-modules is* C-E *FP-injective if and only if C^+ is* C-E *flat;*
(5) *a complex C of left R-modules is* C-E *FP-injective if and only if C^{++} is* C-E *FP-injective;*
(6) *a complex C of right R-modules is* C-E *flat if and only if C^{++} is* C-E *flat;*
(7) *the class of* C-E *FP-injective complexes is* C-E *injectively resolving;*
(8) *let $0 \to A \to B \to C \to 0$ be a* C-E *short exact sequence in $\mathscr{C}(R)$. If A, B are* C-E *FP-injective complexes, then C is* C-E *FP-injective;*
(9) *if C is a* C-E *FP-injective complex and S is a* C-E *pure subcomplex of C, then C/S is* C-E *FP-injective.*

To prove Theorem 5.1, we need the following lemmas.

LEMMA 5.2. *Let $\{C^i\}_{i\in I}$ be a direct system of complexes and D a* C-E *finitely presented complex. Then* $\mathrm{Hom}(D, \lim_{\to} C^i) \cong \lim_{\to} \mathrm{Hom}(D, C^i)$.

PROOF. It follows from Stenström [22, Ch. V, Proposition 3.4]. □

LEMMA 5.3. *Let $\{C^i\}_{i\in I}$ be a family of complexes and D a* C-E *finitely generated complex. Then* $\underline{\mathrm{Hom}}(D, \bigoplus_{i\in I} C^i) \cong \bigoplus_{i\in I} \underline{\mathrm{Hom}}(D, C^i)$ *as complexes.*

PROOF. It is easy by [8, Proposition 2.5.16]. □

By the standard homological method, Lemmas 5.2 and 5.3, we have the following results.

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Lemma 5.4.

(1) *Let R be a coherent ring, D a* C-E *finitely presented complex and $(C^i)_{i\in I}$ a direct system of complexes. Then* $\overline{\mathrm{Ext}}^1(D, \lim_{\rightarrow} C^i) \cong \lim_{\rightarrow} \overline{\mathrm{Ext}}^1(D, C^i)$.

(2) *Let R be any ring, D a* C-E *finitely presented complex and $(C^i)_{i\in I}$ a family of complexes. Then* $\underline{\mathrm{Ext}}^1(D, \bigoplus_{i\in I} C^i) \cong \bigoplus_{i\in I} \underline{\mathrm{Ext}}^1(D, C^i)$.

Remark 5.5.

(1) The class of all C-E FP-injective complexes is closed under direct sums.

(2) If R is a coherent ring and C is a C-E FP-injective complex, then $\overline{\mathrm{Ext}}^n(P, C) = 0$ for any C-E finitely presented complex P and all $n \geq 1$.

Proof. (1) It follows by Lemma 5.4(2).

(2) Let C be a C-E FP-injective complex and P a C-E finitely presented complex. Then there is a C-E exact sequence $0 \to K \to P \to D \to 0$ with P C-E finitely generated and C-E projective and K C-E finitely generated. Thus, it is clear that $\overline{\mathrm{Ext}}^n(P, C) = 0$, since R is a coherent ring. □

Lemma 5.6. *Let $\{C^i\}_{i\in I}$ be a family of complexes and D a* C-E *finitely presented complex. Then $D \overline{\otimes} \prod_{i\in I} C^i \cong \prod_{i\in I}(D \overline{\otimes} C^i)$ as complexes.*

Proof. Firstly,

$$\alpha : D \otimes_R \prod_{i\in I} C^i \longrightarrow \prod_{i\in I}(D \otimes_R C^i)$$

defined by $x \longmapsto ((D \otimes_R \pi^i)(x))_{i\in I}$ is an isomorphism, where $x = d \otimes c \in (D \otimes_R \prod_{i\in I} C^i)_l$ and $\pi^j : \prod_{i\in I} C^i \longrightarrow C^j$ is the natural projection (see [8, Proposition 2.5.17]).

Secondly, we will show that $D \overline{\otimes} \prod_{i\in I} C^i \cong \prod_{i\in I}(D \overline{\otimes} C^i)$. We have the following commutative diagram:

$$\begin{array}{ccccc}
\Big(D \otimes_R \prod_{i\in I} C^i\Big)_l & \longrightarrow & \dfrac{\big(D\otimes_R \prod_{i\in I} C^i\big)_l}{\mathrm{B}_l\big(D\otimes_R \prod_{i\in I} C^i\big)} & \longrightarrow & 0 \\
\alpha_l \big\downarrow & & \beta_l \big\downarrow & & \\
\Big(\prod_{i\in I} D \otimes_R C^i\Big)_l & \longrightarrow & \dfrac{\big(D\otimes_R \prod_{i\in I} C^i\big)_l}{\mathrm{B}_l\big(\prod_{i\in I} D\otimes_R C^i\big)} & \longrightarrow & 0
\end{array}$$

where $\beta : ((D \otimes_R \prod_{i\in I} C^i)_l / \mathrm{B}_l(D \otimes_R \prod_{i\in I} C^i)) \longrightarrow ((D \otimes_R \prod_{i\in I} C^i)_l / \mathrm{B}_l(\prod_{i\in I} D \otimes_R C^i))$ is given by the assignment

$$d \otimes c + \mathrm{B}\Big(D \otimes_R \prod_{i\in I} C^i\Big) \longrightarrow \alpha(d \otimes c) + \mathrm{B}\Big(\prod_{i\in I} D \otimes_R C^i\Big)$$

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for any $d \otimes c \in (D \otimes_R \prod_{i\in I} C^i)_l$. Thus, β is a graded isomorphism of graded modules with degree 0. Moreover,

$$\beta\delta^{D\overline{\otimes}\prod_{i\in I}C^i}\Big(d\otimes c + \mathrm{B}\Big(D\otimes_R \prod_{i\in I} C^i\Big)\Big) = \beta(\delta^D(d)\otimes c) = \alpha(\delta^D(d)\otimes c) = (\delta^D(d)\otimes \pi^i(c))_{i\in I}$$

and

$$\begin{aligned}
&\delta^{\prod_{i\in I}(D\overline{\otimes}C^i)}\beta\Big(d\otimes c + \mathrm{B}\Big(D\otimes_R \prod_{i\in I} C^i\Big)\Big)\\
&\quad = \delta^{\prod_{i\in I}(D\overline{\otimes}C^i)}\Big(\alpha(d\otimes c) + \mathrm{B}\Big(D\otimes_R \prod_{i\in I} C^i\Big)\Big)\\
&\quad = \delta^{\prod_{i\in I}(D\overline{\otimes}C^i)}\alpha(d\otimes c) = (\delta^{D\overline{\otimes}C^i}\alpha(d\otimes c))_{i\in I} = (\delta^D(d)\otimes\pi^i(c))_{i\in I}.
\end{aligned}$$

Therefore, β is an isomorphism of complexes. □

Lemma 5.7. *Let $\{C^i\}_{i\in I}$ be a family of complexes. Then:*

(1) $\bigoplus_{i\in I} C^i$ *is a* C-E *pure subcomplex of* $\prod_{i\in I} C^i$*;*
(2) $\prod_{i\in I} C^i$ *is a* C-E *pure subcomplex of* $\prod_{i\in I}(C^i)^{++}$.

Proof. (1) For any C-E finitely presented complex P, we have the following commutative diagram by Lemma 5.6:

$$\begin{array}{ccccc}
 & & \Big(\bigoplus_{i\in I} C^i\Big)\overline{\otimes}P & \longrightarrow & \Big(\prod_{i\in I} C^i\Big)\overline{\otimes}P\\
 & & \cong\downarrow & & \downarrow\cong\\
0 & \longrightarrow & \bigoplus_{i\in I}(C^i\overline{\otimes}P) & \longrightarrow & \prod_{i\in I}(C^i\overline{\otimes}P)
\end{array}$$

Hence, $\bigoplus_{i\in I} C^i$ is a C-E pure subcomplex of $\prod_{i\in I} C^i$.

(2) It is similar to the proof of (1), since C^i is a C-E pure subcomplex of $(C^i)^{++}$ for each $i\in I$. □

Proof of Theorem 5.1. (1) $\Rightarrow$ (2). Let $\{C^i\}_{i\in I}$ be a family of C-E flat complexes of right R-modules. Then $\prod_{i\in I} C^i_n, \mathrm{B}_n(\prod_{i\in I} C^i), \mathrm{Z}_n(\prod_{i\in I} C^i), \mathrm{H}_n(\prod_{i\in I} C^i)$ are flat in R-Mod for all $n\in\mathbb{Z}$. So, (2) follows.

(2) $\Rightarrow$ (1). Let $\{M_i\}_{i\in I}$ be a family of flat right R-modules. Then $\underline{M_i}$ is C-E flat in $\mathscr{C}(R)$ for $i\in I$. So, $\prod_{i\in I}\underline{M_i}$ is C-E flat by (2), which implies that $\prod_{i\in I}\overline{M_i}$ is flat. Hence, R is left coherent.

(1) $\Rightarrow$ (3). It follows by Lemma 5.4.

(3) $\Rightarrow$ (1). It follows by a similar argument of (2) $\Rightarrow$ (1).

(1) $\Rightarrow$ (4) is easy.

(4) $\Rightarrow$ (5). Let C be a complex of left R-modules. If C is C-E FP-injective, then C^+ is C-E flat by (4) and so C^{++} is C-E FP-injective by Lemma 4.9. Conversely, if C^{++}

is C-E FP-injective, then C is a C-E pure subcomplex of C^{++} by Proposition 3.10. So, C is C-E FP-injective by Proposition 4.10.

(5) $\Rightarrow$ (6). If C is a C-E flat complex of right R-modules, then C^+ is a C-E FP-injective complex of left R-modules by Lemma 4.9. Hence, C^{+++} is C-E FP-injective by (5). Thus, C^{++} is C-E flat by Lemma 4.9. Conversely, if C^{++} is C-E flat, then C is C-E flat by Proposition 4.10.

(6) $\Rightarrow$ (2). Let $\{C^i\}_{i\in I}$ be a family of C-E flat complexes of right R-modules. Then $\bigoplus_{i\in I} C^i$ is C-E flat, so $(\bigoplus_{i\in I} C^i)^{++} \cong (\prod_{i\in I} {C^i}^+)^+$ is C-E flat by (6). But $\bigoplus_{i\in I} (C^i)^+$ is a C-E pure subcomplex of $\prod_{i\in I} (C^i)^+$ by Lemma 5.6 and so $(\prod_{i\in I} (C^i)^+)^+ \to (\bigoplus_{i\in I} (C^i)^+)^+ \to 0$ splits. Thus, $\prod_{i\in I} (C^i)^{++} \cong (\bigoplus_{i\in I} (C^i)^+)^+$ is C-E flat. Since $\prod_{i\in I} C^i$ is a C-E pure subcomplex of $\prod_{i\in I}(C^i)^{++}$ by Lemma 5.7, $\prod_{i\in I} C^i$ is C-E flat by Proposition 4.10.

(1) $\Rightarrow$ (7). It is obvious that the class of all C-E FP-injective complexes is closed under C-E extensions.

Let $0 \to A' \to A \to A'' \to 0$ be a C-E exact sequence with A' and A C-E FP-injective. By the above remark, it is not hard to see that A'' is C-E FP-injective.

(7) $\Rightarrow$ (8) is obvious.

(8) $\Rightarrow$ (1). We note that R is a left coherent ring if and only if every factor module of an FP-injective module by a pure submodule is FP-injective (see [25]). Let N be a pure submodule of a left R-module M with M FP-injective. Then

$$0 \to \underline{N} \to \underline{M} \to \underline{M/N} \to 0$$

is C-E pure exact and $\underline{M}$ is C-E FP-injective. So, $\underline{M/N}$ is a C-E FP-injective complex by (8) and hence M/N is an FP-injective module.

(8) $\Rightarrow$ (9). It is clear by Proposition 4.10 and (8).

(9) $\Rightarrow$ (8). Let $0 \to A \to B \to C \to 0$ be a C-E exact sequence in $\mathscr{C}(R)$ with A and B C-E FP-injective. Then the sequence

$$0 \to A \to B \to C \to 0$$

is C-E pure exact, since A is C-E FP-injective. Therefore, C is C-E FP-injective.

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