THE RELATIVISTIC PLANETARY PERTURBATIONS AND THE ORBITAL MOTION OF THE MOON.

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In this contribution, a review of the calculation of the lunar orbit in the relativistic framework is drawn up. Then, the particular dependence of the lunar orbit motion upon the relativistic perturbations of the motions of the planets themselves is put forth.

INTRODUCTION

Relativistic effects in lunar laser ranging arise when chrono-spatial events occuring in the solar system are considered in a curved space-time. The metric element *ds* of the four-dimensional manifold describing this space-time depends upon the gravitational potentials of all the bodies of the solar system.
One can distinguish two classes of relativistic effects :

1) the measurement process operated with a laser beam between an Earth-station and a Moon retro-reflector which depends upon the relativistic laws governing the path of photons and the behavior of terrestrial clocks;

2) the dynamics of the Earth-Moon system derived from the relativistic laws, governing the separation between the centers of mass of the Earth and Moon, and the rotations of these two bodies.

The modelling of the measurement process can be found in Mulholland (1977) and the relativistic effects in the rotations in Brumberg (1972). We will deal, in this paper, only with a particular aspect of the relativistic dynamical effects in the Earth-Moon separation, or more generally in the lunar orbit motion, that are due to the relativistic perturbations in the motions of the planets themselves.
In section 1, a review of the calculation of the lunar orbit motion in the relativistic framework is drawn up. In section 2, the particular dependence of the lunar orbit motion upon the relativistic perturbations in the motions of the planets themselves is put forth. Then, in section 3, we describe an analytic construction of these planetary perturbations in order to evaluate quantitatively, in section 4, how greatly the lunar orbit motion is affected.
We have adopted for the following calculations the theory of general relativity.

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1. CALCULATION OF THE LUNAR ORBIT MOTION IN THE RELATIVISTIC FRAMEWORK

In the relativistic framework, using the post-Newtonian approximation, the differential equation for the radius-vector $\vec{r}$ between the Earth and Moon can be written symbolically :

$$\ddot{\vec{r}} = \vec{grad}\, U + \frac{1}{c^2} V(\vec{r},\dot{\vec{r}},\ddot{\vec{r}}) \tag{1}$$

U being the Newtonian potential and $\frac{1}{c^2}V(\vec{r},\dot{\vec{r}},\ddot{\vec{r}})$ a function denoting the relativistic acceleration proportional to $1/c^2$ and derived from the metric element ds of the curved space-time (c is the speed of light).

In an analytic theory, one can distinguish two types of perturbations in the lunar orbit motion. The first type is obtained after integration of the newtonian acceleration $\vec{grad}\, U$ in (1). The second type is obtained after integration of the relativistic acceleration proportional to $1/c^2$. Generally speaking, the geodesic of a test particle, in the weak gravitational field of the solar system, can be expanded in series form with the perturbations theory. It is a rough but helpful estimation to consider the main relativistic perturbations of this expansion as proportional to the gravitational radii Gm/c^2 of the masses m present in the physical model (G constant of gravitation) and inversely proportional to the distances l between these masses and the test particle. Table I gives the maximum values for the dimensionless quantity $Gm/c^2 l$ originating in each planet. We give a unique value when the test particle is either the massless Earth or the massless Moon since, in these two cases, the difference is unsignificant. Of course, this does not mean that the differential relativistic effect between the Earth and Moon is null – involving no relativistic perturbations in the lunar orbit motion.

Table I : maximum values for the dimensionless quantity $Gm/c^2 l$ originating in each planet (Me : Mercury ; V : Venus ; M : Mars ; J : Jupiter ; S : Saturn ; U : Uranus ; N : Neptun). The "unit" is 10^{-12}.

Bodies	Sun	Me	V	M	J	S	U	N	Earth upon Moon	Moon upon Earth
$Gm/c^2 l$	10 000	0.00	0.08	0.00	2.50	0.35	0.02	0.02	11.50	0.10

In order to make this table capable of justifying possible approximations in the formulation of the relativistic acceleration proportional to $1/c^2$, we must point out the order of magnitude of the main relativistic perturbations in the lunar orbit motion. They are periodic perturbations

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due to the differential relativistic effect between the Earth and Moon originating in the gravitational radius of the Sun. Their highest amplitude reaches, in General Relativity, a little more than one meter with a half month period (Brumberg, 1972).
From table I and this highest amplitude, we conclude that the planets have, very likely, negligible differential effects in the lunar orbit motion.
Let us note that this table contains, too, the value of Gm/c^2l for the Earth reconsidered with a non-null mass m curving the space-time. But this dimensionless quantity being, in "unit" of 10^{-12}, 11 in comparison with 10 000 for the Sun, the geodesic of the massless Moon will be only slightly changed by this small superimposed curvature. Although, in this case, the relativistic effect is not differential as previously, the periodic perturbations are likely to be very small with respect to the accuracy of the lunar laser ranging data. Schwarzschild's secular advance in the perigee of the Moon (0.″06/cy) is the only effect to be retained.
The table contains also the value for the reciprocal problem, i.e. the Moon with non-null mass curving the space-time in which a massless Earth follows a geodesic. Its effect will be even smaller and thus negligible.
In conclusion, the Moon, as a test particle, follows a geodesic which can be calculated with sufficient approximation in a metric whose metric element ds, as this analysis shows, depends only upon the gravitational potentials of the Sun and Earth. Moreover, the metric is fully defined only under the assumption of a relative motion for the Earth and Sun.
If these conditions are fulfilled, the derivation of equation (1) can be made from the principle of geodesic providing the formulation for the relativistic acceleration proportional to $1/c^2$. With such definitions we can say by analogy with the terminology in the Newtonian framework that the main relativistic perturbations come from a 3-body problem qualified as the relativistic main problem.
In analytical form, this relativistic main problem is treated by Brumberg (1958, 1972) in General Relativity with harmonic coordinates and with circular motions for the point-masses Earth and Sun. Baierlein (1967) deals with the same problem but the motion of the Earth around the Sun is elliptical. Krogh and Baierlein (1968) treat this problem but in a one-parametric formalism including General Relativity and Brans-Dicke theory. Nordtvedt (1973) deals with it also but in the more general PPN (Parametrized Post-Newtonian) formalism. This author considers the Earth and Moon as bodies instead of point-masses, he discovers Nordtvedt's effect which has, however, not been corroborated by lunar laser ranging data (Williams et al, 1975; Shapiro et al, 1975). Brumberg (1981) treats his original problem again but in a four-parametric formalism including two physical parameters (γ, β) and two coordinates parameters (α, ν).

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2. DIRECT AND INDIRECT RELATIVISTIC PERTURBATIONS IN THE LUNAR ORBIT MOTION

Although, we have assumed the relativistic acceleration originating from the planets in $\frac{1}{c^2} V(\vec{r},\dot{\vec{r}},\ddot{\vec{r}})$ to be negligible, this does not mean that we have dismissed entirely their relativistic influence in the lunar orbit motion. The potential U in (1) depends upon their positions and, therefore, upon the relativistic perturbations of their own motions.
The explicit formulation for the potential U which originates from the masses of all the planets, the Sun, the Earth and the Moon is in Chapront-Touzé et Chapront(1980). It can be written as a function depending upon the osculating elements σ of the orbital motions of the planets and upon the Earth-Moon vector $\vec{r}$. A planetary theory provides the osculating elements σ in series form for each planet, the terms of these series being the perturbations $\delta\sigma$ around keplerian ellipses σ_0 :

$$\sigma = \sigma_0 + \delta\sigma$$

One part of $\delta\sigma$ accounts for the relativistic perturbations $\delta\sigma^R$. Since they are very small, the potential U can be linearized :

$$U(\sigma,\vec{r}) = U(\sigma_0,\vec{r}) + \sum_{\sigma} \left(\frac{\partial U}{\partial \sigma} \right)_{\sigma_0} \times \delta\sigma^R \qquad (2)$$

The summation is extended to the six osculating elements of all the planets.
In order to calculate the relativistic perturbations in the lunar orbit motion due to the perturbations $\delta\sigma^R$, we have formally treated the latter in the same way as the Newtonian perturbations of the motions of planets which produce the so-called planetary perturbations in the orbit of the Moon. We have employed Brown's method to find these complementary perturbations to the solution of the Newtonian main problem calculated by Chapront-Touzé (1980). The details of the method can be found in Chapront-Touzé et Chapront(1980). We will recall only that it leads to the following differential system :

$$\frac{dz_i}{dt} = f\left(\frac{\partial\Omega}{\partial w_i}\right)$$
$$\frac{dw_i}{dt} = g\left(\frac{\partial\Omega}{\partial z_i}\right) \qquad i = 1,\ 2,\ 3 \qquad (3)$$

Ω is the disturbing function completing the force function of the Newtonian main problem. The differential system (3) is equivalent to the vectorial equation (1) limited to the Newtonian approximation, i.e. neglecting the relativistic acceleration proportional to $1/c^2$. Therefore, the influence of the relativistic motions of the planets expressed by : $\sum_{\sigma}\left(\frac{\partial U}{\partial \sigma}\right)_{\sigma_0} \times \delta\sigma^R$ enters into the disturbing function Ω in

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the same way, formally, as the Newtonian planetary perturbations into the lunar orbit motion.
The variables z_i and w_i are respectively constants and linear functions of time in the solution of the Newtonian main problem. They are defined in the following way :

$z_1 = \nu$; sidereal mean motion of the Moon

$z_2 = \Gamma$; semi-coefficient in $\sin F$ in latitude which is related to inclination

$z_3 = E$; semi-coefficient in $\sin l$ in longitude which is related to eccentricity

$w_1 = l + \tilde{\omega}$

$w_2 = \tilde{\omega}$

$w_3 = \Omega$; l, $\tilde{\omega}$, Ω being respectively mean mean anomaly, mean longitude of perigee and mean longitude of node of the Moon.

In order to express the right-hand member of (3), one must compute the relativistic perturbations $\delta\sigma^R$ in the osculating elements of the planets. We give the main features of this computation in the next section.
After solving (3) and following the terminology used in the Newtonian framework, we can distinguish two types of relativistic perturbations in the lunar orbit motion due to the planets : the indirect perturbations caused by the relativistic perturbations $\delta\sigma^R_E$ in the Earth motion, and the direct ones caused by the relativistic perturbations $\delta\sigma^R_P$ in the motions of all the other planets.

3. RELATIVISTIC PERTURBATIONS IN THE MOTIONS OF THE PLANETS THEMSELVES

As mentioned above, General Relativity is the theory of gravitation adopted here. We assume that only Schwarzschild's acceleration due to the Sun is of practical importance for the motions of planets. This means that the metric depends only upon the gravitational potential of the Sun and that the massless planets follow geodesics. The equation of the heliocentric motion for the i^{th} planet is :

$$\ddot{\vec{\rho}}_i = -\frac{G(S+m_i)}{\rho_i^3}\vec{\rho}_i + \sum_{j\neq i}^{P} G m_j \left(\frac{\vec{\rho}_j - \vec{\rho}_i}{\Delta_{ij}^3} - \frac{\vec{\rho}_j}{\rho_j^3} \right) + \tag{4}$$

$$+ \frac{GS}{c^2}\left(4\frac{GS}{\rho_i^4}\vec{\rho}_i - \frac{\dot{\rho}_i^2}{\rho_i^3}\vec{\rho}_i + 4\,\frac{\vec{\rho}_i\cdot\dot{\vec{\rho}}_i}{\rho_i^3}\dot{\vec{\rho}}_i \right)$$

with $j = 1, \ldots, P$

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where : $\vec{\rho}_j$ is the heliocentric radius-vector of the j^{th} planet

Δ_{ij} is the mutual distance between the i^{th} and j^{th} planets

S is the mass of the Sun

m_j is the mass of the j^{th} planet.

We have written this equation in isotropic coordinates from Brumberg (1972) chosing $\alpha = 0$ for the coordinates parameter in order to be consistent with the harmonic coordinates generally employed for the main relativistic perturbations in the lunar orbit motion.
The first part of the right-hand member of this equation can be identified formally with Newtonian acceleration. The second part, proportional to the gravitational radius of the Sun GS/c^2, is the relativistic acceleration that we introduced in Gauss's equations. Thus, we have been able to expand the osculating elements in semi-analytical series form with respect to the small parameters GS/c^2 and m_i up to the second order for all the planets. For this purpose, we have used the partial derivatives computed by Bretagnon in his Newtonian theory of planetary motions (Bretagnon, 1980).
We have employed the following planetary osculating elements σ : a, λ, $h = e \sin \tilde{\omega}$, $k = e \cos \tilde{\omega}$, $p = \sin i/2 \sin \Omega$, $q = \sin i/2 \cos \Omega$; a, λ, e, i, $\tilde{\omega}$, Ω being respectively : semi-major axis, mean longitude, eccentricity, inclination, longitude of the perihelion, longitude of the ascending node of the planet.
The results of this computation is given in Lestrade and Bretagnon (1981). The most remarkable non-periodic perturbations are the well-known advances of the perihelia, of course, but also include permanent contractions of the semi-major axes , which are approximately 6 km in isotropic coordinates, depending only slightly upon the eccentricity of the planet. It is worth noting that these contractions are coordinates-dependent and almost disappear in standard coordinates.

4. RESULTS : INDIRECT RELATIVISTIC PERTURBATIONS IN LUNAR ORBIT MOTION

Up to now, the analysis has shown sensitive terms for the indirect relativistic perturbations in the lunar orbit motion. They are due mainly to the following relativistic perturbations in the osculating elements of the Earth (actually, in our computation, of the Earth-Moon barycenter) :

$$10^{10}\,\Delta a\ (\mathrm{UA}) = -394.91 - 22.51 \sin T + 5.17 \cos T + 0.23 \sin 2T + 0.47 \cos 2T + \ldots\ldots$$

$$10^{10}\,\Delta \lambda\ (\mathrm{rd}) = -3.51 \sin T - 15.27 \cos T + \ldots\ldots$$

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$$10^{10}\,\Delta k = -3030.66\,t + 0.08\sin T - 295.82\cos T + - 8.84\sin 2T + 2.03\cos 2T + \ldots\ldots$$

$$10^{10}\,\Delta h = -696.17\,t - 296.14\sin T + 0.08\cos T + + 2.03\sin 2T + 8.84\cos 2T + \ldots\ldots$$

t is the time expressed in centuries and T is the mean mean heliocentric longitude of the Earth (actually, of the Earth-Moon barycenter). These results can be found in Lestrade and Bretagnon (1981).
Table II gives the numbers of periodic terms of the indirect relativistic perturbations in the metrical and angular elements z_i and w_i defined above.

Table II : numbers of periodic terms in the metrical and angular orbital elements z_i and w_i of the Moon due to the relativistic perturbations in the motion of the Earth.

elements:→	ν	E	Γ	$l+\tilde{\omega}$	$\tilde{\omega}$	Ω
accuracy:↓						
$2''10^{-6} \rightarrow 10^{-4}{}''$	2	5	1	10	11	1
$10^{-4}{}'' \rightarrow 2''10^{-3}$	0	0	0	0	4	0

The four largest periodic terms and the two secular terms in $\tilde{\omega}$ and Ω are given below :

$$\delta\tilde{\omega} = +0.6860\,t - 0''.00169\sin(2D - l) - 0''.00101\sin(2D - 2l) - 0''.00061\sin l + 0''.00033\sin(T - 2D + l + 257°)$$

$$\delta\Omega = -0.1979\,t$$

t is the time in centuries ; D and l are Delaunay's arguments.
These two secular terms can be compared to Schwarzschild's advance of the perigee of the Moon due to the gravitational radius of the Earth which is 0''.06 per century and to the geodesic precession due to the solar gravitational radius which amounts to 1''.91 per century both in $\tilde{\omega}$ and Ω.
Another presentation for the results is to compute the perturbations in the true longitude V and in the module r of the radius-vector of the Moon. We find :

$$\delta V\,(10^{-5}{}'') = 14\sin(2D - l) + 6\sin 2D + 5\sin(T + 77°) + 2\sin(T - 2D + 77°) + \ldots\ldots$$

(1" 10^{-5})
(- 55" 10^{-5})

$$\delta r\,(\mathrm{cm}) = -13\cos 2D - 9\cos(2D - l) + 4\cos(4T + 2D + 128°) + 4\cos(T - 2D + 77°) + \ldots\ldots$$

(106 cm)
(-2 cm)

In parentheses we give the values calculated by Brumberg (1972) for the main relativistic perturbations with the same arguments in order to allow comparisons between orders of magnitudes of the two types of perturbations.

CONCLUSION

Complementary terms δr that we have calculated for the module of the radius-vector $\vec{r}$ between the Earth and the Moon are sensitive at the level of accuracy of lunar laser measurement. Moreover, the terms found for the true longitude V of the Moon could be sensitive to VLBI angular measurement.
In fact, in analytical form, both the main relativistic perturbations as given by Brumberg and the indirect relativistic perturbations computed in this paper are to be taken into account to represent accurately the relativistic effects in the lunar orbit motion.
The direct relativistic perturbations have not been considered here, as well as mixed terms (Poisson's terms) in the relativistic perturbations of the motion of the Earth. They are under study.

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