1. Introduction

A graph $G$ is said to possess a perfect matching if there is a subgraph of $G$ consisting of disjoint edges which together cover all the vertices of $G$. Clearly $G$ must then have an even number of vertices. A necessary and sufficient condition for $G$ to possess a perfect matching was obtained by Tutte (3). If $S$ is any set of vertices of $G$, let $p(S)$ denote the number of components of the graph $G - S$ with an odd number of vertices. Then the condition

$$\text{for all } S, p(S) \leq |S|$$

is both necessary and sufficient for the existence of a perfect matching. A simple proof of this result is given in (1).

We consider certain conditions which are sufficient although not necessary. Roughly speaking, $G$ will have a perfect matching if there are enough edges. For example, if $|V(G)| = n$, $n$ even, where $V(G)$ denotes the set of vertices of $G$, and if each vertex is of degree $\geq \frac{n}{2}$, i.e. if each vertex has at least $\frac{n}{2}$ edges incident with it, then it is almost trivial (see § 3) to show that $G$ has a perfect matching. Instead of looking at each vertex separately, we can put a condition on the vertices collectively. If $X$ denotes any subset of $V(G)$, let

$$\Gamma(X) = \{y \in V(G): \ y \text{ is joined by an edge to at least one vertex in } X\}.$$

Following Woodall (4), we define

$$\text{melt } (G) = \max \{c: \forall X \subset V(G), |\Gamma(X)| \geq \min (c | X |, |V(G)|)\}.$$  

Thus melt $(G)$ is the largest number $c$ such that any $k$ vertices of $G$ are collectively adjacent to at least $\min (ck, n)$ vertices. We have already (1) shown that, if $n$ is even,

$$\text{melt } (G) \geq \frac{n}{2} \Rightarrow G \text{ has a perfect matching.} \quad (1)$$

We note that this condition implies that each vertex is of degree $\geq \frac{1}{4}n$. Indeed, we have in general

**Lemma.** If melt $(G) \geq c > 1$, then each vertex of $G$ has degree $\geq \frac{c-1}{c} n$ where $n = |V(G)|$.

**Proof.** Suppose there is a vertex $v$ of degree $\leq \frac{c-1}{c} n$. Then there are...
\[ \geq \frac{n}{c} \] vertices none of which is joined by an edge to \( v \). But these vertices must be joined to at least \( c \cdot \frac{n}{c} = n \) vertices, a contradiction.

In the next section, we combine the two types of condition above to prove

**Theorem 1.** Let \( G \) have \( n \) vertices, \( n \) even. Let \( c \) be any fixed number, \( \frac{1}{4} \leq c \leq \frac{1}{2}, \) and suppose that

(i) each vertex is of degree \( \geq cn \),

(ii) \( \text{melt}(G) \geq \frac{3-4c}{2-2c} \).

Then \( G \) possesses a perfect matching.

**Note 1.** \( c = \frac{1}{2} \) gives the trivial result mentioned above, and \( c = \frac{1}{4} \) gives the result (I).

**Note 2.** The theorem is also true for other values of \( c \), but if \( c > \frac{1}{2} \) condition (i) by itself is sufficient, whereas if \( c < \frac{1}{4} \) then condition (ii) by itself suffices.

**Note 3.** Condition (ii) implies, by the lemma, that each vertex has degree \( \geq 1-2c \geq n \), but this is less than \( cn \) if \( c > \frac{1}{4} \).

**Note 4.** The result is best possible. If \( A, B \) are graphs let \( A + B \) denote the graph obtained by joining every vertex of \( A \) to every vertex of \( B \). Take \( A = aK_3 \cup bK_1 \) and \( B = (a+b-2)K_1 \) where \( K_n \) denotes the complete graph on \( n \) vertices.

Following the suggestion of the referee, who is to be thanked for his careful consideration of the original version of this paper, we shall deduce Theorem 1 from the following stronger theorem which is proved along the same lines but more simply.

**Theorem 2.** Let \( G \) have \( n \) vertices, \( n \) even, and suppose that

\[ |\Gamma(X)| \geq \min \left( 2 \left| X \right| - \frac{n}{2}, n \right) \]

for all sets \( X \) of vertices of \( G \). Then either \( G \) has a perfect matching or there exist subsets \( X, Y \) of \( V(G) \), \( X \not\subseteq Y \), such that

\[ |X| = \frac{1}{2}(3n-6), \quad |Y| = \frac{1}{2}(3n-2), \quad |\Gamma(X)| = 2 \left| X \right| - \frac{n}{2}, \quad |\Gamma(Y)| = 2 \left| Y \right| - \frac{n}{2}. \]

An example of a graph in which the second possibility occurs is \( G = 3K_3 + K_1 \). Theorem 2 is proved in the next section, but we now show that
Theorem 2 implies Theorem 1. We assume Theorem 2 and the hypotheses of Theorem 1. Let $W$ be any set of vertices of $G$. If $|W| > (1-c)n$, then, since the degree of each vertex of $G$ is $\geq cn$, we cannot have a vertex of $G$ joined to no vertex of $W$. Thus $|\Gamma(W)| = V(G)$. So suppose $|W| \leq (1-c)n$. Then

$$|\Gamma(W)| \geq \frac{3-4c}{2-2c} |W| - \frac{n}{2} \tag{2}$$

It follows from Theorem 2 that $G$ possesses a perfect matching unless there exist two sets $X$, $Y$ as in Theorem 2. Then, by (2),

$$\frac{3-4c}{2-2c} |W| = 2 |W| - \frac{n}{2}$$

for $W = X$ and for $W = Y$, giving $|X| = |Y|$, a contradiction.

Theorem 2 is proved in the next section. In the remainder of this paper we shall generalize in one theorem both Theorem 1 and a result of Woodall (4) concerned with the maximum number of disjoint edges in a graph with no perfect matching. Woodall's argument was based on that of (1), and now we in turn extend his result.

2. Proof of Theorem 2

We suppose there is no perfect matching of $G$. Then by Tutte's theorem there is a set $S$ of vertices of $G$ for which $p(S) > |S|$. Using the fact that $p(S) \equiv |S|$ (mod 2), we must then have

$$p(S) \geq |S| + 2.$$  

Case 1. Suppose that $|S| \leq \frac{1}{4}(n-6)$. Let $m$ denote the number of 1-components of $G-S$ (i.e. the number of components with just one vertex). Then

$$n \geq |S| + m + 3(p(S)-m) \tag{3}$$

whence

$$n-m \leq \frac{3}{2}n-2|S|-3. \tag{4}$$

But, if $m>0$,

$$n-m \geq |\Gamma(G-S)| \geq 2 |G-S| - \frac{n}{2} \tag{5}$$

whence

$$n-m \geq \frac{3}{2}n-2|S|.$$
Since (4) and (5) contradict one another, we must have $m = 0$. Thus, from (3),
\[ n \geq 4 \mid S \mid + 6 \]
i.e.
\[ \mid S \mid \leq \frac{1}{4}(n-6), \]
whence
\[ \mid S \mid = \frac{1}{4}(n-6). \]
Equality here implies that each component of $G - S$ must have exactly 3 vertices. If we let $X$ denote the set of vertices in all but one of these components we then have $\mid X \mid = \frac{1}{2}(3n-6)$ and $\mid \Gamma (X) \mid \leq \mid X \mid + \mid S \mid = n-3 = 2 \mid X \mid - \frac{n}{2}$. Similarly, if $Y$ denotes the same set with one more vertex of $G - S$ added, then we also have $\mid Y \mid = \frac{1}{4}(3n-2)$ and $\mid \Gamma (Y) \mid \leq \mid Y \mid + \mid S \mid = n-1 = 2 \mid Y \mid - \frac{n}{2}$.

**Case 2.** Suppose now that $\mid S \mid < \frac{1}{4}(n-6)$. Let $h$ denote the number of vertices in all but the smallest component of $G - S$. Since there are $\geq \mid S \mid + 2$ components of $G - S$, containing between them $n-\mid S \mid$ vertices, we must have
\[ h \geq \frac{\mid S \mid + 1}{\mid S \mid + 2} (n-\mid S \mid). \quad (6) \]
These $h$ vertices can be adjacent to at most $h + \mid S \mid < n$ vertices; on the other hand, they are by hypothesis joined to at least $2h - \frac{n}{2}$ vertices. Thus
\[ h \leq \frac{\mid S \mid + n}{2}. \quad (7) \]
From (6) and (7), eliminating $h$, we obtain
\[ \mid S \mid \geq \frac{1}{4}(n-6), \]
a contradiction.

### 3. Extension to imperfect matchings

A related question is the following. Given a condition on a graph $G$ which does not imply that $G$ possesses a perfect matching, can we estimate how many disjoint edges can be found in $G$? Corresponding to the two types of condition already studied, we have the following results for a graph with $n$ vertices.

1. If each vertex is of degree $\geq cn$, $0 \leq c \leq \frac{1}{4}$, then we can find at least $[cn]$ disjoint edges.

2. If melt $(G) \geq c$, then there are at least
\[ \frac{c}{c+1} \frac{n}{3c-2} \text{ disjoint edges if } 0 < c \leq \frac{1}{4} \]
\[ \left[ \frac{3c-2}{3c} \frac{n}{3c-2} \right] \text{ disjoint edges if } 1 < c \leq \frac{3}{4}. \]
Result 2 is due to Woodall (4), with (1) as the special case \( c = \frac{3}{2} \). Result 1 is almost trivial (although best possible—consider a bipartite graph). For suppose that each vertex is of degree \( \geq k \), and that \( h < k \) disjoint edges have so far been found. If no two remaining vertices are joined by an edge, select any two of them, say \( v_1 \) and \( v_2 \). Then it is easy to see that there must be a pair \( v_3, v_4 \) of vertices, joined by one of the edges already chosen, such that \( v_1 \) is joined to \( v_3 \) and \( v_2 \) to \( v_4 \). With this new pairing we now have \( h+1 \) disjoint edges, and the process can be repeated if \( h+1 < k \). We now state

**Theorem 3.** Let \( G \) be a graph with \( n \) vertices. Suppose that

(i) each vertex is of degree \( \geq dn \),

(ii) \( \text{melt}(G) \geq \frac{3-4d-3f}{2-2d} \),

where \( 4d+3f \geq 1, 2d+3f \leq 1, d \geq 0, f \geq 0 \). Then \( G \) possesses at least \( \left\lfloor \frac{n}{2} (1-f) \right\rfloor \) disjoint edges.

The special case \( f = 0 \) is Theorem 1, and the case \( f = \frac{1}{4}(1-4d) \) is Woodall’s result (9). The referee has suggested that it may be possible to deduce this result from an analogue to Theorem 2 in the same way as Theorem 1 was deduced from Theorem 2. However, we preserve here our original proof. Instead of Tutte’s condition we use Berge’s extension ((2); see also (4) for a simpler proof): for \( G \) to possess at least \( t \) disjoint edges, it is necessary and sufficient that \( p(S) - |S| \leq n - 2t \) for all sets \( S \) of vertices of \( G \). We shall in fact prove that, for all \( S \),

\[
p(S) \leq |S| + nf + \frac{3}{2}
\]

since this will imply that there are at least \( \frac{n}{2} (1-f) - \frac{3}{4} \) and hence at least \( \left\lfloor \frac{n}{2} (1-f) \right\rfloor \) disjoint edges.

**4. Proof of Theorem 3**

In view of the above remarks, we may suppose that there exists a set \( S \) of vertices of \( G \) such that

\[
p(S) > |S| + nf + \frac{3}{2}
\]

and show that this leads to a contradiction.

**Case 1.** \( |S| \geq dn \). Let \( m \) denote the number of 1-components in \( G - S \). If \( m = 0 \),

\[
n \geq |S| + 3p(S) > 4 |S| + 3fn \geq (4d+3f)n \geq n,
\]

so we must have \( m > 0 \). Thus

\[
n - m \geq |\Gamma(G - S)| \geq \frac{3-4d-3f}{2-2d} (n - |S|),
\]
whence

\[ m \leq \frac{3 - 4d - 3f}{2 - 2d} |S| - \frac{1 - 2d - 3f}{2 - 2d} n. \quad (11) \]

But we also have, from (3), ignoring the term \( \frac{f}{2} \) in (10),

\[ n > 4 |S| - 2m + 3nf, \]
\[ m > 2 |S| - \frac{1}{2}(1 - 3f)n. \]

Eliminating \( m \) from (11) and (12), we obtain \( |S| < dn \), a contradiction.

**Case 2.** \( |S| < dn \). Here there can be no 1-components, so that each odd component contains at least

\[ \max (3, \frac{dn - |S|}{n} + 1) \quad (13) \]

vertices. From now on we can assume that \( 4d + 3f > 1 \).

**Case 2(a).** Suppose there is at least one 3-component. Then (13) yields

\[ dn - |S| = \beta, \quad 0 < \beta \leq 2. \quad (14) \]

Then (3) and (10) give

\[ n > 4 |S| + 3nf + 5 = (4d + 3f)n - 4\beta + 5 \]

so that

\[ n(4d + 3f - 1) < 4\beta - 5. \quad (15) \]

Considering on the other hand all but one of the odd components we have, from the definition of melt \( (G) \),

\[ n - 3 \geq \frac{3 - 4d - 3f}{2 - 2d} (n - |S| - 3). \]

Substituting for \( |S| \) from (14), this gives

\[ n(1 - d)(4d + 3f - 1) \geq (3 - \beta)(4d + 3f - 1) - 6d + 2\beta > 2\beta - 6d. \]

Thus, by (15), we must have

\[ 2\beta - 6d < (1 - d)(4\beta - 5) \]

whence

\[ d > \frac{1}{2}. \]

It follows that

\[ 4d + 3f - 1 > \frac{1}{2}. \quad (16) \]

(15) and (16), with \( \beta \leq 2 \), now yield \( n < 9 \), and a contradiction easily follows.

**Case 2 (b).** Suppose now there is no 3-component. Here we shall show that \( |S| \) is bounded. First of all, if \( |S| < \frac{1}{4}dn \), then

\[ n > |S| + (|S| + nf + \frac{f}{2})(\frac{1}{4}dn + 1) \]

so that

\[ \frac{1}{4}dn |S| < |S|(2 + \frac{1}{4}dn) < n(1 - f - \frac{3}{8}d - \frac{1}{4}fdn). \quad (17) \]
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If \( dn<4 \) then \(|S|<1 \) whereas, if \( dn \geq 4 \), then \( \frac{1}{2}fdn \geq 2f \) and (17) yields \(|S|<6 \). Secondly, if \( \frac{1}{2}dn \leq |S|<dn \), then

\[
n > |S| + (|S| + nf)(dn - |S| + 1)\]

whence

\[
dn - |S| < \frac{n + nf}{|S| + nf} - 2 < \frac{1 + f}{\frac{1}{2}d + f} - 2 < 8.
\]

Thus

\[
8 + \frac{1}{2}n(1 - 5f) > 8 + |S| > dn,
\]

\[
40 > (6d + 5f - 1)n > \frac{1}{2}n,
\]

so that \( n < 80 \). But

\[
n \geq |S| + 5(|S| + 2)
\]

so we must have \(|S| \leq 11\).

Thus in any case, \(|S| \leq 11\). It remains finally to consider each possible value of \(|S|\) in turn. In each case we argue as follows.

Let \( h \) denote the number of vertices in all but one of the odd components of \( G - S \). Then

\[
|S| + h \geq \frac{3 - 4d - 3f}{2 - 2d} h
\]

whence

\[
|S| \geq \frac{1 - 2d - 3f}{2 - 2d} h.
\]

Thus

\[
|S| \geq \frac{1 - 2d - 3f}{2 - 2d} \cdot 5 \cdot (|S| + 1). \tag{18}
\]

For any specific value of \(|S|\), (18) gives a lower bound for \( d \), and so for \( dn \). For example, if \(|S| = 5\), (18) yields

\[
d \geq \frac{3}{2} - \frac{3}{2}f
\]

and

\[
dn \geq \frac{3}{2}n - \frac{3}{2} \theta
\]

where \( \theta = fn \). Having obtained this bound for \( dn \), we obtain a contradiction by estimating \( n \) in two different ways. For we have

\[
n > |S| + 5(|S| + \theta + 1)
\]

and also

\[
n > (|S| + \theta + 1)(dn - |S| + 1).
\]

With \(|S| = 5\), these give

\[
n > 35 + 5 \theta \tag{19}
\]

and

\[
n > (6 + \theta)(\frac{3}{2}n - \frac{3}{2} \theta - 4)
\]

i.e.

\[
n < \frac{9 \theta^2 + 74 \theta + 120}{2 \theta + 7}. \tag{20}
\]

(19) and (20) contradict one another. The theorem is now proved.
REFERENCES


(4) D. R. Woodall, The melting point of a graph, and its Anderson number (to appear).

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