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COHOMOLOGY OF QUANTUM GROUPS: THE QUANTUM DIMENSION

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ABSTRACT This paper uses the notion of the quantum dimension to obtain new results on the cohomology and representation theory of quantum groups at a root of unity In particular, we consider the elementary theory of support varieties for quantum groups

Let G be a finite group and k a field of characteristic $p \geq 0$. A trivial fact in representation theory states that if V is a finite dimensional kG-module whose dimension is not divisible by p, then the trivial module k occurs as a direct summand of $V \otimes V^*$. One can ask for a similar result for representations of quantum groups G_q (or quantum enveloping algebras $\mathfrak{U}_q(\mathfrak{g})$). Here, the field k generally has characteristic zero, and the "quantum world" is complicated by several non-commutative phenomena: for example, the canonical isomorphisms $U \otimes V \cong V \otimes U$ and $V^{**} \cong V$ of vector spaces are usually *not* module morphisms. Nevertheless, one may formulate a satisfactory analogue of the above finite group-theoretic result by making use of the *quantum dimension* of a module. In this paper, we study this concept in some detail, and then apply it to obtain some new results centering on the cohomology of quantum groups.

We begin in Section 1 with the preliminary notion of the generic dimension of certain kinds of graded modules. The quantum dimension and several of its properties are developed in Section 2. For example, suppose q is a root of unity and $G_q \longrightarrow G$ is the natural Frobenius (or "covering") map onto the associated semisimple algebraic group G. Then for a G_q-module V, the quantum dimension $\dim_q V$ agrees with the usual dimension $\dim V$ if and only if the action of G_q on V factors through G.

In Section 3, we relate the quantum dimension to the original question of splitting the trivial module k off from $V \otimes V^*$ in Theorem 3.3. Several interesting applications are also presented here; see, in particular, Corollary 3.6 concerning Weyl modules.

We adapt the notion of quantum dimension to module categories for Levi subgroups in Section 4. Section 5 then turns to cohomological questions involving quantum groups. Using the notion of quantum dimension, together with recent work of Ginzburg-Kumar [10] on the calculation of the cohomology ring for the finite quantum groups, we indicate how to "quantize" some of the theory of support varieties. (See [7], [8], [14], [15] and [16] for this theory in the context of algebraic groups.) We have only indicated a few

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results in this direction, namely, those which seem most closely to illustrate the use of the quantum dimension. We expect to return to this topic in more detail in a sequel.

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All X-graded vector spaces considered in §1 and all modules for quantum groups or quantum enveloping algebras considered in §§2–5 are assumed to be finite dimensional over the ground field k.

List of Notation

k	A field, whose characteristic is arbitrary in §1, and 0 in other sections.
Φ	Irreducible root system (finite, crystallographic) in a Euclidean space $\mathbb{E}$ having inner product $(\ ,\ )$. Usually, $\mathbb{E}$ is the span of Φ. However, if Φ has type A_{n-1}, the following exceptional case is also allowed: $\mathbb{E}$ has a basis $e_1, e_2, \ldots, e_n$ with $\Phi = \{e_i - e_j \mid i \neq j\}$.
Φ^+, Π	Fixed set of positive roots and simple roots, respectively.
h, ρ	The Coxeter number and the Weyl weight $\frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha$ of Φ, respectively.
$\alpha^\vee$	$2\alpha/(\alpha,\alpha)$, the dual root of $\alpha \in \Phi$.
W, Q	Weyl group of Φ and the $\mathbb{Z}$-span of Φ, respectively.
X	Integral weight lattice of Φ (in the exceptional case above, X is the $\mathbb{Z}$-span of e_i's).
X_+	$\{\lambda \in X \mid (\lambda, \alpha^\vee) \geq 0, \forall \alpha \in \Pi\}$, the set of dominant integral weights.
l	An odd integer > 1. (If Φ has component of type G_2, l is assumed to be prime to 3.)
C_l	$\{\lambda \in \mathbb{E} \mid 0 < (\lambda+\rho, \alpha^\vee) < l, \forall \alpha \in \Phi^+\}$, the bottom l-alcove.
$\mathcal{T}_l$	Group of translations $t_{l\lambda}\colon \mathbb{E} \longrightarrow \mathbb{E}$, $x \longmapsto x + l\lambda$ for $\lambda \in Q$.
W_l	$W \ltimes \mathcal{T}_l$, the affine Weyl group of Φ with parameter l. For $w \in W_l$ and $x \in \mathbb{E}$, write $w \cdot x = w(x+\rho) - \rho$.
ℓ	The Coxeter length function on W_l relative to Φ^+.
$\chi(\lambda)$	$\frac{\sum_{w\in W}(-1)^{\ell(w)}e^{w(\lambda+\rho)}}{\sum_{w\in W}(-1)^{\ell(w)}e^{w(\rho)}} \in \mathbb{Z}X$, the Weyl character for $\lambda \in X$.

1. **Generic dimensions.** In this section, we assume, for convenience, that the root system Φ is indecomposable. However, our definitions and results can be extended to the general case without any difficulties; everything works out componentwise.

Denote by α_0 the highest short root of Φ, and define, for any root $\alpha \in \Phi$, an integer $d_\alpha = (\alpha,\alpha)/(\alpha_0,\alpha_0)$. Note that $d_\alpha \in \{1,2,3\}$ and the matrix $\left(d_\beta(\alpha,\beta^\vee)\right)_{\alpha,\beta\in\Pi}$ is symmetric. Therefore, the d_α's (for $\alpha \in \Pi$) are exactly the d_i's in the definition of the quantum enveloping algebra[1] associated with Φ, see [3], [20].

[1] One should follow [20] here rather than [3]; otherwise the classical limit of the quantum enveloping algebra would not be the enveloping algebra of the Lie algebra with root system Φ.

Suppose for a moment that rank $X = \operatorname{rank}\Phi$. Any $\lambda \in X$ can be expressed as a rational linear combination of simple roots: $\lambda = \sum_{\alpha\in\Pi} r_\alpha \alpha$. The *weighted height* $\mathrm{wht}(\lambda)$ of λ is defined to be $\sum_{\alpha\in\Pi} r_\alpha d_\alpha$. If rank $X > \operatorname{rank}\Phi$, the weighted height of $\lambda \in X$ is defined to be the weighted height of the orthogonal projection of λ in $\mathbb{R}\Phi$.

LEMMA 1.1. *For any $\lambda \in X$,* $\mathrm{wht}(\lambda) = 2(\lambda,\rho)/(\alpha_0,\alpha_0) = \frac{1}{2}\sum_{\alpha\in\Phi^+} d_\alpha(\lambda,\alpha^\vee)$. *In particular,* $\mathrm{wht}(\lambda)$ *is a half integer.*

PROOF. For $\alpha \in \Pi$, we have $2(\alpha,\rho)/(\alpha_0,\alpha_0) = (\rho,\alpha^\vee)(\alpha,\alpha)/(\alpha_0,\alpha_0) = d_\alpha$, so $\mathrm{wht}(\lambda) = 2(\lambda,\rho)/(\alpha_0,\alpha_0)$ for any $\lambda \in X$. Also, $2(\lambda,\rho)/(\alpha_0,\alpha_0) = \sum_{\alpha\in\Phi^+}(\lambda,\alpha)/(\alpha_0,\alpha_0) = \frac{1}{2}\sum_{\alpha\in\Phi^+} d_\alpha(\lambda,\alpha^\vee)$. ∎

Let V be an X-graded vector space; that is, there is a family of subspaces $\{V_\lambda\}$ indexed by X such that $V = \bigoplus_{\lambda\in X} V_\lambda$. In this general setting, the *formal character* of V is the element $\mathrm{ch}(V)$ in the group ring $\mathbb{Z}X$ defined by $\mathrm{ch}(V) = \sum_{\lambda\in X}(\dim V_\lambda)e^\lambda$.

DEFINITION 1.2. The *generic dimension* of an X-graded vector space $V = \bigoplus_{\lambda\in X} V_\lambda$ is the element

$$\dim_{\mathrm{gen}} V = \sum_{\lambda\in X}(\dim V_\lambda)t^{-2\,\mathrm{wht}(\lambda)} \in \mathbb{Z}[t,t^{-1}],$$

t being an indeterminate.

Clearly, $\dim_{\mathrm{gen}}$ is a ring homomorphism from the Grothendieck ring of the category of X-graded vectors spaces to $\mathbb{Z}[t,t^{-1}]$. We also have the following result.

THEOREM 1.3. *Let V be an X-graded vector space with* $\mathrm{ch}(V) = \chi(\lambda)$ *for some* $\lambda \in X_+$. *Then*

$$\dim_{\mathrm{gen}} V = \prod_{\alpha\in\Phi^+} \frac{t^{-d_\alpha(\lambda+\rho,\alpha^\vee)} - t^{d_\alpha(\lambda+\rho,\alpha^\vee)}}{t^{-d_\alpha(\rho,\alpha^\vee)} - t^{d_\alpha(\rho,\alpha^\vee)}}. \tag{1.3.1}$$

PROOF. For any $\mu \in \mathbb{E}$, consider the subring R_μ of $\mathbb{Z}X$ generated by all elements of the form e^ξ with $4(\xi,\mu)/(\alpha_0,\alpha_0) \in \mathbb{Z}$. Clearly, the map $e^\xi \longmapsto t^{-4(\xi,\mu)/(\alpha_0,\alpha_0)}$ extends to a ring homomorphism $\psi_\mu\colon R_\mu \longrightarrow \mathbb{Z}[t,t^{-1}]$. Thanks to Lemma 1.1, $\dim_{\mathrm{gen}} V = \psi_\rho\,\mathrm{ch}(V)$.

Let $D(\mu) = \sum_{w\in W}(-1)^{\ell(w)}e^{w(\mu)}$. It is well-known that $D(\rho) = e^\rho\prod_{\alpha\in\Phi^+}(1-e^{-\alpha})$. Thus,

$$\begin{aligned}\psi_\rho\big(D(\mu)\big) &= \sum_{w\in W}(-1)^{\ell(w)}t^{-4(w(\mu),\rho)/(\alpha_0,\alpha_0)} = \sum_{w\in W}(-1)^{\ell(w)}t^{-4(w(\rho),\mu)/(\alpha_0,\alpha_0)}\\ &= \psi_\mu\big(D(\rho)\big) = t^{-4(\rho,\mu)/(\alpha_0,\alpha_0)}\prod_{\alpha\in\Phi^+}(1-t^{4(\alpha,\mu)/(\alpha_0,\alpha_0)})\\ &= \prod_{\alpha\in\Phi^+}(t^{-2(\alpha,\mu)/(\alpha_0,\alpha_0)} - t^{2(\alpha,\mu)/(\alpha_0,\alpha_0)}) = \prod_{\alpha\in\Phi^+}(t^{-d_\alpha(\mu,\alpha^\vee)} - t^{d_\alpha(\mu,\alpha^\vee)}).\end{aligned}$$

Since $\mathrm{ch}(V) = D(\lambda+\rho)/D(\rho)$, the result follows from the above calculation. ∎

Define the *generic degree* of an element $\lambda \in X$ by $\deg_{\mathrm{gen}}(\lambda) = \psi_\rho\big(\chi(\lambda)\big)$. The argument in the above proof is valid also for non-dominant λ. Thus, $\deg_{\mathrm{gen}}(\lambda)$ can be calculated by using the right-hand side of the formula (1.3.1). Since $\chi(w\cdot\lambda) = (-1)^{\ell(w)}\chi(\lambda)$ for $w \in W$ and $\lambda \in X$, the following result is immediate:

LEMMA 1.4. *For any $w \in W$ and $\lambda \in X$,* $\deg_{\mathrm{gen}}(w \cdot \lambda) = (-1)^{\ell(w)} \deg_{\mathrm{gen}}(\lambda)$.

Given a ring R and a ring homomorphism $\varphi : \mathbb{Z}[t, t^{-1}] \longrightarrow R$, we obtain a *specialization* $\dim_\varphi V \equiv \varphi(\dim_{\mathrm{gen}} V)$ of $\dim_{\mathrm{gen}} V$ for an X-graded vector space V. Also, for $\lambda \in X$, we have the specialization $\deg_\varphi(\lambda) = \varphi\big(\deg_{\mathrm{gen}}(\lambda)\big)$ of the generic degree $\deg_{\mathrm{gen}}(\lambda)$. (Note that $\deg_\varphi(\lambda)$ always makes sense, even if the denominator of the right-hand side of (1.3.1) is zero; in fact, $\deg_{\mathrm{gen}}(\lambda)$ is a polynomial in t and t^{-1}.)

EXAMPLE 1.5. The ordinary dimension $\dim V$ of an X-graded vector space V is the specialization of $\dim_{\mathrm{gen}} V$ under the ring homomorphism $\mathbb{Z}[t, t^{-1}] \longrightarrow \mathbb{Z}$ given by $t \longmapsto 1$. In particular, the (ordinary) Weyl dimension formula follows from Theorem 1.3 by letting $t \longrightarrow 1$. (Recall that $\lim_{t \to 1} \frac{t^{-n} - t^n}{t^{-1} - t} = n$.)[2]

EXAMPLE 1.6. A very elementary example of specialization of the generic dimension occurs when $\operatorname{char} k = p > 0$. Then the ring homomorphism $\mathbb{Z}[t, t^{-1}] \longrightarrow k$ given by $1_{\mathbb{Z}} \longmapsto 1_k$ and $t \longmapsto 1_k$ specializes the generic dimension of an X-graded vector space V to its *modular dimension* $\dim^p V$, the image of $\dim V$ under the ring homomorphism $\mathbb{Z} \longrightarrow k$ given by $1_{\mathbb{Z}} \longmapsto 1_k$. It can be defined for any vector space over k, without referring to X.

Also, the generic degree $\deg_{\mathrm{gen}}(\lambda)$ for $\lambda \in X$ specializes to the *modular degree* $\deg^p(\lambda)$ of λ. By (1.3.1), if $p \geq h$, then $\deg^p(\lambda + p\mu) = \deg^p(\lambda)$ for any $\lambda, \mu \in X$.

Assume that k is algebraically closed, and that Φ is a root system with $p \geq h$. Let G be a simply connected, semisimple algebraic group over k with root system Φ. For $\lambda \in X_+$, let $V(\lambda)$ denote the Weyl module with highest weight λ. Then

- $\dim^p V(\lambda) \neq 0$ iff λ is p-regular;
- If $w \in W_p$ with $w \cdot \lambda \in X_+$, then $\dim^p V(w \cdot \lambda) = (-1)^{\ell(w)} \dim^p V(\lambda)$.

The first assertion (which is well-known, of course) follows from the Weyl dimension formula. If $w = t_{p\mu} w'$, where $\mu \in Q$ and $w' \in W$, then $\ell(w) \cong \ell(w') \pmod 2$, and the second assertion then follows directly from Lemma 1.4; see also [15, (3.9)].

2. **Quantum dimensions.** In the section, we assume that $\operatorname{char} k = 0$ and let $q \in k$ be a primitive l-th root of unity with l odd. And, if Φ has a component of type G_2, it is also required that l is not divisible by 3. Moreover, we assume that $l \geq h$, so that l-regular weights exist. The ring homomorphism $\mathbb{Z}[t, t^{-1}] \longrightarrow k$ sending $1_{\mathbb{Z}} \longmapsto 1_k$ and $t \longmapsto q$ defines a specialization $\dim_q V$, called the *quantum q-dimension* of V, of the generic dimension $\dim_{\mathrm{gen}} V$ of an X-graded vector space V over k[3]. Also, the *quantum q-degree* $\deg_q(\lambda)$ for $\lambda \in X$ is defined to be the specialization of $\deg_{\mathrm{gen}}(\lambda)$.

We will work in the following categories (in [22] we use $\mathcal{C}$ instead of $\mathcal{G}$).

(1) $\mathcal{G}$ is the category of rational G-modules, G being the (split) simply connected semisimple algebraic group with root system Φ defined on k, or, equivalently, the category of $\mathfrak{g}$-modules, $\mathfrak{g}$ being the Lie algebra of G. One exception: if Φ is of type A_{n-1},

[2] An *unweighted* version of the generic dimension is discussed in [18, §10.10]. It is used there to give a proof of the Weyl dimension formula.

[3] As indicated in the introduction, this notion has arisen previously in the literature. See, for example, [1], [9].

$G = \mathrm{GL}(n)$ is also allowed. The irreducible objects in $\mathcal{G}$ are indexed via highest weights by X_+. Denote the irreducible object with highest weight $\lambda \in X_+$ by $L(\lambda)$.

(2) $\mathcal{G}_q$ is the category of the integral modules of type 1 over the standard arithmetic quantization $\mathfrak{U}_q = \mathfrak{U}_q(\mathfrak{g})$ (see, for example, [20], [3]) of the enveloping algebra $\mathfrak{U}(\mathfrak{g})$ of $\mathfrak{g}$ with parameter q, or, if Φ is of type A_{n-1}, the category of rational modules of the quantum linear group $G_q = \mathrm{SL}_q(n)$ or $\mathrm{GL}_q(n)$, see [21]. It is a highest weight category in the sense of [5]. It has weight poset X_+, and for $\lambda \in X_+$, the corresponding irreducible, Weyl and "induced" objects are denoted by $L^q(\lambda)$, $V^q(\lambda)$ and $A^q(\lambda)$, respectively. Recall that $\mathrm{ch}\, V^q(\lambda) = \mathrm{ch}\, A^q(\lambda) = \chi(\lambda)$ [3, (5.12)], [21, (10.4.4)].

Recall that $\mathfrak{U}_q$ is generated by $E_i, E_i^{(l)}, F_i, F_i^{(l)}$ and $K_i^{\pm 1}$ for $i = 1, 2, \ldots, \mathrm{rank}\,\Phi$, and it has a decomposition $\mathfrak{U}_q = \mathfrak{U}_q^+\mathfrak{U}_q^0\mathfrak{U}_q^-$, where $\mathfrak{U}_q^+$ (resp., $\mathfrak{U}_q^-$) is the subalgebra generated by E_i and $E_i^{(l)}$ (resp., F_i and $F_i^{(l)}$) for all i, and $\mathfrak{U}_q^0$ is a commutative and cocommutative sub-Hopf algebra generated by $K_i^{\pm 1}$ and some other elements.

(3) $\hat{\mathcal{G}}_q^1$ is the category of integral modules of type 1 for the subalgebra $\hat{\mathbf{u}}_q$ of $\mathfrak{U}_q$ generated by $\mathfrak{U}^0, E_i$ and F_i for $i = 1, 2, \ldots, \mathrm{rank}\,\Phi$, or, if Φ is of type A_{n-1}, the category of rational modules for the quantum group $(G_q)_1T$ as defined in [21]. It is a highest weight category with weight poset X. Denote the irreducible, Weyl, and "induced" objects corresponding to $\lambda \in X$ by $\hat{L}_1^q(\lambda)$, $\hat{V}_1^q(\lambda)$ and $\hat{A}_1^q(\lambda)$, respectively.

(4) $\mathcal{G}_q^1$ is the category of integral modules of type 1 for the subalgebra $\mathbf{u}_q$ of $\mathfrak{U}_q$ generated by E_i, F_i and $K_i^{\pm 1}$ for $i = 1, 2, \ldots, \mathrm{rank}\,\Phi$, or, if Φ is of type A_{n-1}, the category of rational modules for the Frobenius kernel $(G_q)_1$ as defined in [21]. This is not a highest weight category. However, we still can use X (more precisely, X/lX) to index the irreducibles: $L_1^q(\lambda)$ is the restriction of $\hat{L}_1^q(\lambda)$ for $\lambda \in X$. We also have the Weyl objects $V_1^q(\lambda) = \hat{V}_1^q(\lambda)|_{\mathcal{G}_q^1}$ and the "induced" objects $A_1^q(\lambda) = \hat{A}_1^q(\lambda)|_{\mathcal{G}_q^1}$.

Although an object in $\mathcal{G}_q^1$ is not, generally speaking, an X-graded object, it is X/lX-graded. And, since q is an l-th root of unity and $\mathrm{wht}(\mu) \in \frac{1}{2}\mathbb{Z}$, the quantum q-dimension for any X/lX-graded vector space, hence for any object in $\mathcal{G}_q^1$, is defined.

In the following proposition, $\mathcal{H}$ denotes the category of rational modules for the "diagonal" maximal torus T of G, or the category of integral $\mathfrak{U}_q^0$-modules of type 1. There are "Frobenius twist functors" $F\colon \mathcal{G} \longrightarrow \mathcal{G}_q$ and $F\colon \mathcal{H} \longrightarrow \hat{\mathcal{G}}_q^1$. If $V \in \mathrm{Ob}(\mathcal{G})$ or $\mathrm{Ob}(\mathcal{H})$, $F(V)$ will be denoted by $V^{(l)}$.

PROPOSITION 2.1. *(1) For $U \in \mathrm{Ob}(\mathcal{G})$ and $V \in \mathrm{Ob}(\mathcal{G}_q)$ (resp., $U \in \mathrm{Ob}(\mathcal{H})$ and $V \in \mathrm{Ob}(\hat{\mathcal{G}}_q^1)$), we have* $\dim_q(U^{(l)} \otimes V) = \dim U \cdot \dim_q V$.

(2) Suppose that $\mathrm{rank}\, X = \mathrm{rank}\,\Phi$. *If $V \in \mathrm{Ob}(\mathcal{G}_q)$ (resp., $V \in \mathrm{Ob}(\hat{\mathcal{G}}_q^1)$, $V \in \mathrm{Ob}(\mathcal{G}_q^1)$) is irreducible, then* $\dim_q V = \dim V$ *iff $V = U^{(l)}$ for some $U \in \mathrm{Ob}(\mathcal{G})$ (resp., $V = U^{(l)}$ for some $U \in \mathrm{Ob}(\mathcal{H})$, V is trivial as an object in $\mathcal{G}_q^1$). If $l > h$, the same is true for any object V in one of the above categories.*

PROOF. Clearly, $\dim_q U^{(l)} = \dim U$. Thus, (1) and one direction of (2) follow.

For the other direction of (2), we first prove that if $\dim_q V = \dim V$ for V in $\mathcal{G}_q$, $\hat{\mathcal{G}}_q^1$ or $\mathcal{G}_q^1$, then $V_\lambda \neq 0$ implies $\lambda \in lX$. We may assume that $q \in \mathbb{C}$. Then

$$|\dim_q V| \leq \sum_{\lambda \in X} \dim V_\lambda |q^{-2\,\mathrm{wht}(\lambda)}| = \sum_{\lambda \in X} \dim V_\lambda = \dim V.$$

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Clearly, the equality holds iff all $q^{-2\,\mathrm{wht}(\lambda)}$ with $V_\lambda \neq 0$ are the same. Therefore,

$$\dim_q V = \dim V \iff q^{-2\,\mathrm{wht}(\lambda)} = 1 \text{ for all weights } \lambda \text{ of } V \iff 2\,\mathrm{wht}(\lambda) \equiv 0 \pmod{l} \text{ for all weights } \lambda \text{ of } V.$$

It is easy to see that if λ is a weight of V, then for any $\alpha \in \Pi$, there is an element $\nu \in X$ such that $s_\alpha(\lambda) + l\nu$ is also a weight of V. (In particular, if $V \in \mathrm{Ob}(\mathcal{G}_q)$, we always can choose $\nu = 0$). Thus, under the assumption that $\dim_q V = \dim V$,

$$2(\lambda, \alpha^\vee)d_\alpha = 2\,\mathrm{wht}\big((\lambda, \alpha^\vee)\alpha\big) \equiv 2\,\mathrm{wht}\big(\lambda - s_\alpha(\lambda)\big) \equiv 0 \pmod{l}.$$

It follows that $(\lambda, \alpha^\vee) \equiv 0 \pmod{l}$ for any $\alpha \in \Pi$, *i.e.*, $\lambda \in lX$, as required. In particular, if V is irreducible with $\dim_q V = \dim V$, then its highest weight is in lX_+, forcing the result for V.

If V is not irreducible and does not have the form of $U^{(l)}$, then V is not completely reducible. Thus, there exist λ, μ with $\mathrm{Ext}^1_{\mathcal{G}_q}\big(L^q(l\lambda), L^q(l\mu)\big) \neq 0$ ($\mathcal{G}_q$ case), or $\mathrm{Ext}^1_{\hat{\mathcal{G}}_q^1}\big(\hat{L}_1^q(l\lambda), \hat{L}_1^q(l\mu)\big) \neq 0$ ($\hat{\mathcal{G}}_q^1$ case), or $\mathrm{Ext}^1_{\mathcal{G}_q^1}(k,k) \neq 0$. Since the restriction from $\mathcal{G}_q$ or $\hat{\mathcal{G}}_q^1$ to $\mathcal{G}_q^1$ preserves the socle of an object (see, for example, [22, (3.2)]) the above extension groups for $\mathcal{G}_q$ and $\hat{\mathcal{G}}_q^1$ remain nonzero when restricted to $\mathcal{G}_q^1$, giving also $\mathrm{Ext}^1_{\mathcal{G}_q}(k,k) \neq 0$. This contradicts to $H^1(\mathcal{G}_q^1, k) = 0$ (see Theorem 5.1), proving (2). ∎

One cannot expect that Proposition 2.1(2) holds if $\dim \mathbb{E} = \operatorname{rank} X > \operatorname{rank} \Phi$. This is because the extra dimension of $\mathbb{E}$ is ignored in defining the quantum dimension.

The following proposition is the quantum analogue of Example 1.6. Statements (1) and (2) are contained in [1, (3.2) and (3.3)].

PROPOSITION 2.2. *Let $\lambda \in X_+$. Then*

(1) We have

$$\dim_q V^q(\lambda) = \prod_{\alpha \in \Phi^+} \frac{q^{-d_\alpha(\lambda+\rho,\alpha^\vee)} - q^{d_\alpha(\lambda+\rho,\alpha^\vee)}}{q^{-d_\alpha(\rho,\alpha^\vee)} - q^{d_\alpha(\rho,\alpha^\vee)}};$$

(2) $\dim_q V^q(\lambda) \neq 0$ *iff λ is l-regular;*

(3) If $w \in W_l$ satisfies $w \cdot \lambda \in X_+$, then

$$\dim_q V^q(w \cdot \lambda) = (-1)^{\ell(w)} \dim_q V^q(\lambda).$$

PROOF. Since $l \geq h$, 0 is l-regular. By (1.3.1), (1) and (2) follow from the fact that the image of $\psi_\rho\big(D(\mu+\rho)\big)$ (see the proof of Theorem 1.3) under the specialization $t \longmapsto q$ vanishes iff μ is not l-regular. This is an easy exercise.

(1), in fact, is a formula for $\deg_q(\lambda)$ ($\lambda \in X$), and implies that $\deg_q(\lambda + l\mu) = \deg_q(\lambda)$, for all $\lambda, \mu \in X$. Thus, by Lemma 1.4, for $w = t_{l\mu}w' \in W_l$ with $\mu \in Q$ and $w' \in W$, we have $\deg_q(w \cdot \lambda) = \deg_q(w' \cdot \lambda + l\mu) = \deg_q(w' \cdot \lambda) = (-1)^{\ell(w')}\deg_q(\lambda) = (-1)^{\ell(w)}\deg_q(\lambda)$. Therefore, $\dim_q V^q(w \cdot \lambda) = (-1)^{\ell(w)} \dim_q V^q(\lambda)$. ∎

REMARK 2.3. The same argument shows $\dim_q V^q(w \cdot \lambda)$ equals $\dim_q V^q(\lambda)$ up to sign for $w \in \tilde{W}_l$, where $\tilde{W}_l$ is the extended affine Weyl group of Φ with parameter l, which is, by definition, generated by W and all translations of the form $x \longmapsto x + l\nu$ with $\nu \in X$.

COROLLARY 2.4. *For any* $\lambda \in X$, *we have* $\dim_q \hat{V}_1^q(\lambda) = \dim_q V_1^q(\lambda) = 0$.

PROOF. Since $\operatorname{ch} \hat{V}_1^q(\lambda) = e^{\lambda-(l-1)\rho} \operatorname{ch} V^q\big((l-1)\rho\big)$, we can apply Proposition 2.2(2). ∎

COROLLARY 2.5. *Let* V *be an indecomposable object in* $\mathcal{G}_q$, $\hat{\mathcal{G}}_q^1$ *or* $\mathcal{G}_q^1$. *Then* $\dim_q V = n_V \dim_q V^q(\lambda)$ *for some* $n_V \in \mathbb{Z}$, *where* $\lambda \in \bar{C}_l$ *is* W_l*-conjugate (under the dot action) to the highest weight of a composition factor of* V.

PROOF. By the linkage principle, if $V \in \mathrm{Ob}(\mathcal{G}_q)$, $\operatorname{ch}(V)$ is a $\mathbb{Z}$-linear combination of $\operatorname{ch} V^q(w \cdot \lambda)$ for $w \in W_l$ dominant. The result follows from Proposition 2.2(3).

We also have linkage principles for $\hat{\mathcal{G}}_q^1$ and $\mathcal{G}_q^1$ (see [4, §2.9] and [22, (2.8)]), where the irreducible objects in these categories are, up to a tensor factor of the form $l\mu$, the restrictions of irreducible objects in $\mathcal{G}_q$. The result follows formally. ∎

PROPOSITION 2.6. *Let* V *be an object in* $\mathcal{G}_q$, $\hat{\mathcal{G}}_q^1$ *or* $\mathcal{G}_q^1$. *Then:*

(1) $\dim_q V \in \mathbb{Q}[q+q^{-1}] \subset \mathbb{Q}[(q-q^{-1})\sqrt{-1}]$; *in particular,* $\dim_q V$ *is totally real (i.e., its image is a real number under any field embedding* $\mathbb{Q}[q] \hookrightarrow \mathbb{C}$*);*

(2) $\dim_q V = \dim_{q^{-1}} V$;

(3) $\dim_q V = \dim_q V^*$.

PROOF. (1) Suppose $q \in \mathbb{C}$. Then $q = \cos\theta + \sqrt{-1}\sin\theta$ for $\theta = 2r\pi/l$ with $(r,l) = 1$. Thus, for $s \in \mathbb{Z}$, $q^{-s} - q^s = -2\sqrt{-1}\sin s\theta$. By Theorem (1.3), we obtain, for $\lambda \in X_+$, that

$$\dim_q V^q(\lambda) = \deg_q(\lambda) = \prod_{\alpha\in\Phi^+} \frac{\sin d_\alpha(\lambda+\rho,\alpha^\vee)\theta}{\sin d_\alpha(\rho,\alpha^\vee)\theta}. \tag{2.6.1}$$

This, together with Corollary 2.5, shows that $\dim_q V$ is totally real. We need to prove that $\dim_q V \in \mathbb{Q}[\cos\theta] \subset \mathbb{Q}[\sin\theta]$. By (2.6.1), $\dim_q V^q(\lambda)$ is a product of factors of the form $(\sin s\theta \sin\theta)/(\sin s'\theta\sin\theta)$ for $s, s' \in \mathbb{Z}$. Or, being a real number, $\dim_q V$ can be calculated by the formula that

(2.6.2)
$$\dim_q V = \sum_{\mu\in X} (\dim V_\mu)\cos\big(2\,\mathrm{wht}(\mu)\theta\big), \quad \text{for } V \in \mathrm{Ob}(\mathcal{G}_q), \mathrm{Ob}(\hat{\mathcal{G}}_q^1) \text{ or } \mathrm{Ob}(\mathcal{G}_q^1).$$

Therefore, to prove $\dim_q V \in \mathbb{Q}[\cos\theta]$, it is enough to show that, for $s \in \mathbb{Z}$ (or equivalently, for $s \in \mathbb{Z}^+$) and a variable x, $\cos sx$ and $\sin sx \sin x$ are polynomials in $\cos x$ with coefficients in $\mathbb{Z}$. This can be down simultaneously by induction on s. Finally, l being odd, $\cos\theta = \cos(l+1)\theta = \cos(\frac{l+1}{2}\cdot 2\theta)$ is a polynomial in $\cos 2\theta = 1 - 2\sin^2\theta$ with coefficients in $\mathbb{Z}$. This shows that $\mathbb{Q}[\cos\theta] \subset \mathbb{Q}[\sin\theta]$.

(2) We have $\dim_q V^q(\lambda) = \dim_{q^{-1}} V^q(\lambda)$ by the formula in Theorem 1.3. Now use Corollary 2.5.

(3) From the definition we have $\dim_q V^* = \dim_{q^{-1}} V$, so (3) follows from (2). ∎

Observe $\mathbb{Q}[(q-q^{-1})\sqrt{-1}]$ may not be contained in $\mathbb{Q}[q+q^{-1}]$, *e.g.*, let $q = e^{2\pi i/3}$.

THEOREM 2.7. *Assume that the quantum Lusztig conjecture is true for $\mathcal{G}_q$.*[4] *Then*
(1) For $\lambda \in X_+$, $\dim_q L^q(\lambda) \neq 0$ iff λ is l-regular;
(2) For $\lambda \in X$, $\dim_q \hat{L}_1^q(\lambda) \neq 0$ iff λ is l-regular;
(3) For $\lambda \in X$, $\dim_q L_1^q(\lambda) \neq 0$ iff λ is l-regular.

PROOF. (1) If λ is not l-regular, Proposition 2.2, together with Corollary 2.5, shows that $\dim_q L^q(\lambda) = 0$. Now assume that $\lambda \in C_l \cap X_+$ and $w \in W_l$ is dominant. We have

$$\begin{aligned}\dim_q L^q(w\cdot\lambda) &= \sum_{y\in W_l \text{ dominant}} (-1)^{\ell(y)-\ell(w)} \mathcal{P}_{yw_0,ww_0}(-1)\dim_q V^q(y\cdot\lambda)\\ &= \Big((-1)^{\ell(w)} \sum_{y\in W_l \text{ dominant}} \mathcal{P}_{yw_0,ww_0}(-1)\Big)\dim_q V^q(\lambda),\end{aligned}$$

where the Kazhdan-Lusztig polynomials $\mathcal{P}_{yw_0,ww_0}$'s are regarded as polynomials in t^2. It is known (see, for example, [12, §7.12]) that $\mathcal{P}_{yw_0,ww_0}$ has nonnegative coefficients (and there do exist some nonzero polynomials among these $\mathcal{P}_{yw_0,ww_0}$'s), so

$$n_{L^q(w\cdot\lambda)} = (-1)^{\ell(w)} \sum_{y\in W_l \text{ dominant}} \mathcal{P}_{yw_0,ww_0}(-1) \neq 0,$$

since $\operatorname{char} k = 0$. Also, $\dim_q V^q(\lambda) \neq 0$, by Proposition 2.2. Therefore, $\dim_q L^q(w\cdot\lambda) \neq 0$.

(2) and (3) now follow from (1), since $\dim_q \hat{L}_1^q(\lambda) = \dim_q L_1^q(\lambda) = \dim_q L^q(\lambda')$, where λ' is the unique l-restricted dominant weight in the coset $\lambda + lX$. ∎

REMARK 2.8. The analogues result for modular characters fails: $p(p-1)\rho$ is p-regular, but $\dim^p L\big(p(p-1)\rho\big) = 0$. Even though, we have a weaker result as follows.

For two primes p and q, the affine Weyl groups W_p and W_q (with respect to the same root system Φ) are canonically isomorphic. The isomorphism is the identity on W, and maps $t_{p\alpha} \in W_p$ to $t_{q\alpha} \in W_q$ for $\alpha \in \Phi$. Clearly, the isomorphism sends dominant elements to dominant elements, where $w \in W_p$ is *dominant* provided $w\cdot C_p$ is a dominant alcove. Fix a dominant affine Weyl element w. As in the above proof, we have (see [14, (4.17)]):

Assume the Lusztig conjecture holds for groups G with root system Φ and $p \geq h$. There exists a positive integer $M_w \geq h$ depending only on w such that for any $p \geq M_w$ and any $\lambda \in C_p \cap X_+$, we have $\dim^p L(w\cdot\lambda) \neq 0$. Thus, there is a positive integer M_Φ such that for any restricted dominant weight μ, $\dim^p L(\mu) \neq 0$ provided $p \geq M_\Phi$.

3. **An isomorphism between a module and its double dual.** As in the Appendix, denote by γ the antipode of a Hopf algebra.

LEMMA 3.1. *Let $G_q = \mathrm{GL}_q(n)$ or $\mathrm{SL}_q(n)$. Let T_q be the diagonal maximal torus in G_q, and H be a closed subgroup of G_q. If $f \in K[H]$ is a weight vector of weight μ with respect to the coadjoint action of T_q, then $\gamma^2(f) = q^{-2\,\mathrm{wht}(\mu)} f$.*

PROOF. We only need to verify the result for the (image in $K[H]$) of the coordinate function X_{ij}. The weight of X_{ij} under the coadjoint action is $\mu = X_{ii}^{-1}X_{jj}$, whose weighted

[4] See [19]

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height is $i-j$ (see [21, §6.4]). On the other hand, by [21, (5.4.2)], $\gamma^2(X_{ij}) = q^{-2(i-j)}X_{ij} = q^{-2\,\mathrm{wht}(\mu)}X_{ij}$, as required. ∎

Now we can prove the following isomorphism theorem, which, in the context of quantum enveloping algebras, is contained in [1, (3.6)].

THEOREM 3.2. *Let V be an object in $\mathcal{G}_q$, $\hat{\mathcal{G}}_q^1$ or $\mathcal{G}_q^1$. For $v \in V$, the canonical image of v in V^{**} is denoted by $\bar{v}$. Then the linear map $\theta\colon V \longrightarrow V^{**}$ defined by $\theta(v_\lambda) = q^{2\delta\,\mathrm{wht}(\lambda)}\bar{v}_\lambda$ for any weight vector $v_\lambda \in V_\lambda$ is an isomorphism in the category to which V belongs. Here $\delta = 1$ if we consider the case of quantum linear groups, and $\delta = -1$ if quantum enveloping algebras are in consideration.*

PROOF. We only give a proof for quantum groups. Let the structure maps of V and V^{**} be τ and τ^{**}, respectively. We may assume that $\tau(v_\lambda) = \sum_{\mu\in X} v'_\mu \otimes f_{\lambda-\mu}$, where $v'_\mu \in V_\mu$, and f_ν has weight ν under the coadjoint action of T_q (see [21, (6.4.3)]) (or $(T_q)_1$ if $V \in \mathrm{Ob}(\mathcal{G}_q^1)$, the proof of [21, (6.4.3)] works for this situation also). Then, by Lemma 3.1,

$$\begin{aligned}\tau^{**}\circ\theta(v_\lambda) &= \tau^{**}(q^{2\,\mathrm{wht}(\lambda)}\bar{v}_\lambda) = q^{2\,\mathrm{wht}(\lambda)}\sum_\mu \bar{v'}_\mu \otimes \gamma^2(f_{\lambda-\mu})\\ &= \sum_\mu q^{2\,\mathrm{wht}(\mu)}\bar{v'}_\mu \otimes f_{\lambda-\mu} = (\theta\otimes\mathrm{id})\circ\tau(v_\lambda).\end{aligned}$$ ∎

THEOREM 3.3. *Let V be an object in $\mathcal{D} = \mathcal{G}_q$, $\hat{\mathcal{G}}_q^1$ or $\mathcal{G}_q^1$. Then there is a morphism $\pi_q\colon V\otimes V^* \longrightarrow k$ whose composition with the canonical morphism $\kappa\colon k \longrightarrow V\otimes V^*$ is the morphism $k \longrightarrow k$ sending $a \in k$ to $(\dim_q V)a$. Therefore:*

(1) If $\dim_q V \neq 0$, then k is a direct summand of $V\otimes V^*$;

(2) If $\mathrm{End}_{\mathcal{D}}(V) = k\cdot\mathrm{id}_V$, then k is a direct summand of $V\otimes V^*$ iff $\dim_q V \neq 0$.

PROOF. By Lemma A.5, the canonical map $\pi\colon V^{**}\otimes V^* \longrightarrow k$, sending $\bar{v}\otimes f$ to $f(v)$ for $v \in V$ and $f \in V^*$, is a morphism. Consider the morphism $\pi_q = \pi\circ(\theta\otimes\mathrm{id}_{V^*})$, θ being as Theorem 3.2. Using a basis of V consisting of weight vectors, we see immediately that $\pi_q\circ\kappa(1_k) = \sum_\lambda \dim(V_\lambda)q^{2\delta\,\mathrm{wht}(\lambda)}$, which is, by definition, $\dim_q V$ or $\dim_{q^{-1}} V$. If $\dim_{q^{-1}} V$ is obtained, we can use Proposition 2.6.

The composition of $\kappa\colon k \longrightarrow V\otimes V^*$ and $\pi_q\colon V\otimes V^* \longrightarrow k$ gives a morphism $\pi_q\circ\kappa\colon k \longrightarrow k$ sending $a \in k$ to $(\dim_q V)a$. The assertion (1) and part of (2) follow immediately. To prove (2), we can assume k is a direct summand of $V\otimes V^*$. By hypothesis, there is a unique (up to scalar) homomorphism $k \longrightarrow V\otimes V^* = \mathrm{End}_k V$; taking duals, there is also a unique (up to scalar) homomorphism $V\otimes V^* \longrightarrow k$. Thus, κ equals (up to scalar) the canonical injection determined by the direct sum decomposition of $V\otimes V^*$, and π_q is (up to scalar) the projection. Thus, $\pi_q\circ\kappa \neq 0$ and hence $\dim_q V \neq 0$. ∎

Note that the objects $L^q(\lambda)$, $V^q(\lambda)$ and $A^q(\lambda)$ in $\mathcal{G}_q$ all have the property mentioned in Theorem 3.3(2). Thus, for these modules, the conclusion in Theorem 3.3(2) holds. The same is true for the objects $\hat{L}_1^q(\lambda)$, $\hat{V}_1^q(\lambda)$ and $\hat{A}_1^q(\lambda)$ in the category $\hat{\mathcal{G}}_q^1$.

REMARK 3.4. A result similar to Theorem 3.3 holds for modular dimension:
Let char $k = p > 0$ and V be a finite dimensional (in the usual sense) (co)module over a Hopf k-algebra $\mathfrak{H}$ whose antipode is an involution. Then:

(1) If $\dim^p V \neq 0$, then the trivial (co)module k is a direct summand of $V \otimes V^*$.

(2) If $\mathrm{End}_{\mathfrak{H}}(V) = k \cdot \mathrm{id}_V$ (*e.g.*, V is absolutely irreducible), then the trivial (co)module k is a direct summand of $V \otimes V^*$ iff $\dim^p V \neq 0$.

COROLLARY 3.5. *If I is an injective (= projective) object (in particular, if I is an indecomposable, injective object) in $\mathcal{G}_q$, $\hat{\mathcal{G}}_q^1$ or $\mathcal{G}_q^1$, then* $\dim_q I = 0$.

PROOF. If not, Theorem 3.3 forces k to be an injective object. As is well-known, this is not the case. ∎

The following is a very easy application of Corollary 3.5.

COROLLARY 3.6. *Suppose that $\lambda \in X_+$ is l-regular. Then*

$$\sum_{\substack{w \in W_l \\ w \cdot \lambda \in X_+}} (-1)^{\ell(w)} [V^q(w \cdot \lambda) : L^q(\lambda)] = 0.$$

PROOF. The $\mathcal{G}_q$-injective envelope $I^q(\lambda)$ of $L^q(\lambda)$ has a good filtration with $A^q(w \cdot \lambda)$ occurring $[V^q(w \cdot \lambda) : L^q(\lambda)]$ times as a section of the filtration (*cf.* [5, (3.11)]). Since $A^q(w \cdot \lambda)$ and $V^q(w \cdot \lambda)$ have the same formal character (thus the same quantum q-dimension), we see that

$$\sum_{\substack{w \in W_l \\ w \cdot \lambda \in X_+}} [V^q(w \cdot \lambda) : L^q(\lambda)] \dim_q V^q(w \cdot \lambda) = \dim_q I^q(\lambda) = 0.$$

Now use Proposition 2.2(3), noting that $\dim_q V(\lambda) \neq 0$ by Proposition 2.2(2). The result follows. ∎

4. **Generic and quantum dimensions for objects in a Levi category.** In this section, we will consider a new class of categories related to the quantum enveloping algebra $\mathfrak{U}_q$ or the quantum group G_q—the *Levi categories* $\mathcal{G}_{q,I}$.

For a subset I of $\{1, 2, \ldots, \mathrm{rank}(\Phi)\}$, the corresponding subset of Π is also denoted by I. (Note that, once the quantum group or quantum enveloping algebra is defined, the set Π is numbered.) Let Φ_I be the subroot system of Φ generated by I.

In the quantum group case, $\mathcal{G}_{q,I}$ is the category of modules for the *Levi subgroup* $G_{q,I}$ as defined in [21, §6.1]. Recall that the coordinate algebra $k[G_q]$ is generated by the coordinate functions X_{ij} (and D_q^{-1} if $G_q = \mathrm{GL}_q(n)$, D_q being the quantum determinant). Then $k[G_{q,I}]$ is the quotient algebra of $k[G_q]$ by the ideal generated by all X_{ij} with $i \neq j$ and $(i,j) \notin I^+ \times I^+$, where $I^+ = I \cup \{i+1 \mid i \in I\}$. If I is connected in the obvious sense, $G_{q,I}$ is essentially the "central product" of the quantum group $\mathrm{SL}_q(r)$, $r = \mathrm{card}\, I^+$, and a central torus. If I is disconnected, $G_{q,I}$ is the "central product" of $\mathrm{SL}_q(r)$'s for various

r together with a central torus. Thus, the representation theory of $\mathrm{SL}_q(n)$ is applicable to the category $\mathcal{G}_{q,I}$.

In the quantum enveloping case, $\mathcal{G}_{q,I}$ is the category of integral modules of type 1 for the *Levi subalgebra* $\mathfrak{U}_{q,I}$ of $\mathfrak{U}_q$ generated by $\mathfrak{U}_q^0$ and E_i, $E_i^{(l)}$, F_i and $F_i^{(l)}$ with $i \in I$. Clearly, $\mathfrak{U}_{q,I}$ is the quantum enveloping algebra corresponding to the root system Φ_I with parameter q' (where $q' = q$ if Φ_I contains a short root of Φ, and $q' = q^{d_\alpha}$ for any $\alpha \in \Phi_I$ otherwise), regardless of the size of $\mathfrak{U}^0$. We can use the representation theory of quantum enveloping algebras to the category $\mathcal{G}_{q,I}$.

We also consider categories $\mathcal{G}_I$ (the analogue of $\mathcal{G}$), $\hat{\mathcal{G}}^1_{q,I}$ (the analogue of $\hat{\mathcal{G}}^1_q$) and $\mathcal{G}^1_{q,I}$ (the analogue of $\mathcal{G}^1_q$). For example, $\mathcal{G}_I$ is simply the category of rational modules for the Levi subgroup G_I of G corresponding to I. The details are left to the reader.

The objects in $\mathcal{G}_{q,I}$ are X-graded, so their generic and quantum dimensions are defined. In this direction, we have the following result, generalizing Theorem 1.3.

PROPOSITION 4.1. *Let $V^{q,I}(\lambda)$ be the Weyl module in $\mathcal{G}_{q,I}$ with highest weight λ, where $\lambda \in X$ is dominant with respect to I (i.e., $(\lambda, \alpha^\vee) \geq 0$ for all $\alpha \in I$). Then*

$$\dim_{\mathrm{gen}} V^{q,I}(\lambda) = c(\lambda) \prod_{\alpha \in \Phi_I^+} \frac{t^{-d_\alpha(\lambda+\rho_I,\alpha^\vee)} - t^{d_\alpha(\lambda+\rho_I,\alpha^\vee)}}{t^{-d_\alpha(\rho_I,\alpha^\vee)} - t^{d_\alpha(\rho_I,\alpha^\vee)}},$$

where $\rho_I = (1/2)\sum_{\alpha\in\Phi_I^+} \alpha$, the Weyl weight in Φ_I, and $c(\lambda) = \prod_{\alpha\in\Phi^+\setminus\Phi_I^+} t^{-d_\alpha(\lambda,\alpha^\vee)}$.

PROOF. Let $\rho'_I = \rho - \rho_I$. Note that $w(\rho'_I) = \rho'_I$ for all $w \in W_I$, the Weyl group of Φ_I. Let $D_I(\mu) = \sum_{w\in W_I} (-1)^{\ell(w)} e^{w(\mu)}$ for $\mu \in X$ dominant with respect to I. A calculation similar to that in the proof of Lemma 3.1 gives

$$\psi_\rho\big(D_I(\mu)\big) = \prod_{\alpha\in\Phi^+\setminus\Phi_I^+} t^{-d_\alpha(\mu,\alpha^\vee)} \prod_{\alpha\in\Phi_I^+} (t^{-d_\alpha(\mu,\alpha^\vee)} - t^{d_\alpha(\mu,\alpha^\vee)}).$$

Since $\mathrm{ch}\, V^q I(\lambda) = D_I(\lambda + \rho_I)/D_I(\rho_I)$, the formula in the proposition follows. ∎

COROLLARY 4.2. *Let $\lambda \in X$ be dominant with respect to I. Then $\dim_q V^{q,I}(\lambda) \neq 0$ iff λ is l-regular with respect to Φ_I (i.e., $(\lambda + \rho_I, \alpha^\vee) \not\equiv 0 \pmod{l}$ for all $\alpha \in \Phi_I^+$).*

REMARK 4.3. Let $c_q(\lambda) = \prod_{\alpha\in\Phi^+\setminus\Phi_I^+} q^{-d_\alpha(\lambda,\alpha^\vee)}$, the specialization of $c(\lambda)$ under the assignment $t \longmapsto q$. Clearly, $c_q \colon X \longrightarrow k$ is constant on orbits for the ordinary action of the affine Weyl group $W_{I,l}$ of Φ_I. The function is also constant on orbits for the dot action of $W_{I,l}$. In fact, for $w \in W_{I,l}$ and $\lambda \in X$, $c_q(w\cdot\lambda) = c_q\big(w(\lambda+\rho)\big)c_q(-\rho) = c_q(\lambda+\rho)c_q(-\rho) = c_q(\lambda)$.

REMARK 4.4. One may define a quantum dimension $\dim_{q',I}$ for an object in $\mathcal{G}_{q,I}$ with respect to Φ_I. That is, regard the object as an X_I-graded vector space, X_I being the weight lattice of Φ_I, and define the quantum dimension by referring to X_I and parameter q' instead of X and parameter q. It is easy to determine the relation between these two quantum dimensions: for any indecomposable object V in $\mathcal{G}_{q,I}$, $\dim_q V = c_q(\lambda)\dim_{q',I} V$, λ being the highest weight of a composition factor of V. These two quantum dimensions serve equally for our purpose. In fact, because of the above relation, we may develop a theory for the category $\mathcal{G}_{q,I}$ as we did in §§3–4 by using just $\dim_q$, with small modifications. For example, if V is as before, then $\dim_q V = c_q(\lambda)^2 \dim_q V^*$.

Next, we "quantize" some methods and results in [15]. Let $\mathfrak{X}_I$ be the quotient group $X/(\mathbb{Z}I)$, and denote the coset $\lambda + \mathbb{Z}I$ for $\lambda \in X$ by $[\lambda]$. Define a ring homomorphism map $r_I \colon \mathbb{Z}X \longrightarrow k\mathfrak{X}_I$ by $r_I(e^\lambda) = q^{-2\,\mathrm{wht}(\lambda)} e^{[\lambda]}$.

Any $V \in \mathrm{Ob}(\mathcal{G}_{q,I})$ decomposes as

$$V = \bigoplus_{[\lambda]\in\mathfrak{X}_I} V_{[\lambda]} \text{ with } V_{[\lambda]} = \bigoplus_{\nu\in[\lambda]} V_\nu.$$

LEMMA 4.5. *For any $V \in \mathrm{Ob}(\mathcal{G}_{q,I})$, we have*

$$r_I\,\mathrm{ch}(V) = \sum_{[\lambda]\in\mathfrak{X}_I} (\dim_q V_{[\lambda]}) e^{[\lambda]}.$$

In particular, if V is indecomposable with $\lambda \in X$ as one of its weights, then

$$r_I\,\mathrm{ch}(V) = (\dim_q V) e^{[\lambda]}.$$

Furthermore, if $r_I\,\mathrm{ch}(V) \neq 0$ for $V \in \mathrm{Ob}(\mathcal{G}_{q,I})$, then the trivial object k is a $\mathcal{G}_{q,I}$-direct summand of $V \otimes V^*$.

PROOF. The formulas follow from the definition of r_I and the decomposition of V. Now, suppose $r_I\,\mathrm{ch}(V) \neq 0$. Then there is $\lambda \in X$ with $\dim_q V_{[\lambda]} \neq 0$. Therefore, k is a $\mathcal{G}_{q,I}$-direct summand of $V_{[\lambda]} \otimes V^*_{[\lambda]}$, hence a $\mathcal{G}_{q,I}$-direct summand of $V \otimes V^*$. ∎

We will apply Lemma 4.5 to $V = V^q(\lambda)$ for $\lambda \in X_+$. First, recall the well-known formula:

$$\chi'_I\,\mathrm{ch}\,V^q(\lambda) = \sum_{w\in W^I} (-1)^{\ell(w)}\,\mathrm{ch}\Big(V^{q,I}(w\cdot\lambda)\Big). \tag{4.6}$$

Here W^I is the set of distinguished coset representatives of W over W_I, *i.e.*, the set of elements $w \in W$ with $\ell(ww') = \ell(w) + \ell(w')$ for all $w' \in W_I$, see [12]. Also,

$$\chi'_I = \prod_{\alpha\in\Phi^+\setminus\Phi^+_I} (1 - e^{-\alpha}).$$

Applying r_I to formula 4.6 and using Lemma 4.5, we obtain that

$$r_I(\chi'_I) r_I\Big(\mathrm{ch}\Big(V^q(\lambda)\Big)\Big) = \sum_{w\in W^I} (-1)^{\ell(w)} \dim_q V^{q,I}(w\cdot\lambda) e^{[w\cdot\lambda]}. \tag{4.7}$$

PROPOSITION 4.8. *Let $\lambda \in X^+$. If there is an element $w \in W^I$ such that $w\cdot\lambda$ is l-regular with respect to Φ_I with*

$$\{w' \in W^I \mid [w'\cdot\lambda] = [w\cdot\lambda] \text{ and } w'\cdot\lambda \text{ is } l\text{-regular with respect to } \Phi_I\} = \{w\}, \tag{4.8.1}$$

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then k is a $\mathcal{G}_{q,I}$-direct summand of $V^q(\lambda) \otimes V^q(\lambda)^*$.

PROOF. Thanks to Corollary 4.2, the coefficient of $e^{[w\cdot\lambda]}$ on the right-hand side of the formula 4.7 is $(-1)^{\ell(w)} \dim_q V^{q,I}(w\cdot\lambda) \neq 0$, under the assumption. Thus, the right-hand side of formula 4.7 is nonzero. So $r_I \operatorname{ch} V^q(\lambda) \neq 0$. Now use Lemma 4.5 to obtain the result. ∎

REMARK 4.9. In type A_{n-1}, some observations of [15, §3] are clearly applicable: Namely, let $\lambda \in X_+$, and let $\pi(\lambda) = (m_1 \geq \cdots \geq m_r > 0)$ be a partition of n such that the subroot system $\{\alpha \in \Phi \mid (\lambda + \rho, \alpha^\vee) \equiv 0 \pmod{l}\}$ has type $A_{m_1-1} \times \cdots \times A_{m_r-1}$. If $\pi(\lambda)' = (\xi_1 \geq \cdots \geq \xi_s > 0)$ is the partition dual to $\pi(\lambda)$, choose $I \subset \Pi$ of type $A_{\xi_1-1} \times \cdots \times A_{\xi_s-1}$. Then there is a $w \in W^I$ with $\dim_q V^{q,I}(w\cdot\lambda) \neq 0$ and such that (4.8.1) holds.

5. **Support varieties in quantum case.** In this section, we assume that $l > h$ and k is algebraically closed. The results in the previous sections, together with methods adapted from the theory of algebraic groups in prime characteristic ([7], [8], [14], [15] and [16]), yield some facts about (homological) support varieties in quantum case. We need the following result of Ginzberg and Kumar [10, §3].

THEOREM 5.1. *We have $H^i(\mathcal{G}_q^1, k) = 0$ for odd i. Also, there is a natural graded $\mathcal{G}$-algebra isomorphism*

$$H^{2\bullet}(\mathcal{G}_q^1, k) \cong k[\mathcal{N}],$$

where $\mathcal{N}$ is the variety of ad-nilpotent elements of $\mathfrak{g}$.

For $V \in \mathrm{Ob}(\mathcal{G}_q^1)$, $\mathrm{End}_k(V) = V \otimes V^*$ is a $\mathcal{G}_q^1$-algebra (see Lemma A.5), and hence we have a natural graded algebra homomorphism $\psi_V: H^\bullet(\mathcal{G}_q^1, k) \longrightarrow H^\bullet(\mathcal{G}_q^1, V \otimes V^*)$. The ideal $\mathfrak{I}_V \equiv \operatorname{Ker} \psi_V$ determines a closed subvariety $\mathrm{Supp}_q(V)$ of $\mathcal{N}$, called the *support variety* of V. The following lemma gathers certain elementary facts about support varieties.

LEMMA 5.2. *(1) $\mathrm{Supp}_q(V)$ is a cone for any $V \in \mathrm{Ob}(\mathcal{G}_q^1)$.*

(2) For $V \in \mathrm{Ob}(\mathcal{G}_q^1)$, $\mathrm{Supp}_q(V) = \bigcup_{\lambda \in X/lX} \mathcal{V}(\mathfrak{R}_{V\otimes L_1^q(\lambda)})$, where $\mathfrak{R}_U$, for $U \in \mathrm{Ob}(\mathcal{G}_q^1)$, stands for the left annihilator of the $H^\bullet(\mathcal{G}_q^1, k)$-module $H^\bullet(\mathcal{G}_q^1, U)$, and $\mathcal{V}(\mathfrak{R}_U)$ denotes the subvariety of $\mathcal{N}$ determined by $\mathfrak{R}_U$.

(3) If $V \in \mathrm{Ob}(\mathcal{G}_q)$, then $\mathrm{Supp}_q(V)$ is a G-variety.

(4) If $0 \longrightarrow V_1 \longrightarrow V_2 \longrightarrow V_3 \longrightarrow 0$ is an exact sequence in $\mathcal{G}_q^1$, then, for any $\sigma \in \mathfrak{S}_3$, $\mathrm{Supp}_q(V_{\sigma(1)}) \subset \mathrm{Supp}_q(V_{\sigma(2)}) \cup \mathrm{Supp}_q(V_{\sigma(3)})$. Therefore, if any two $\mathrm{Supp}_q(V_i)$'s are proper subvarieties of $\mathcal{N}$, so is the third.

(5) If $V_1, V_2 \in \mathrm{Ob}(\mathcal{G}_q^1)$, then $\mathrm{Supp}_q(V_1 \oplus V_2) = \mathrm{Supp}_q(V_1) \cup \mathrm{Supp}_q(V_2)$.

(6) If $V_1, V_2 \in \mathrm{Ob}(\mathcal{G}_q^1)$, then $\mathrm{Supp}_q(V_1 \otimes V_2) \subset \mathrm{Supp}_q(V_1)$. Moreover, if $\dim_q V_2 \neq 0$, then $\mathrm{Supp}_q(V_1 \otimes V_2) = \mathrm{Supp}_q(V_1)$.

PROOF. (1) is trivial, and (2) follows from Proposition A.8.

(3) In this case, $H^\bullet(\mathcal{G}_q^1, V \otimes V^*)$ is a $\mathcal{G}$-algebra and ψ_V is a $\mathcal{G}$-algebra homomorphism.

(4) For any $\lambda \in X/lX$, by tensoring the exact sequence with $L_1^q(\lambda)$ and taking cohomology, one arrives at an exact triangle of $H^\bullet(\mathcal{G}_q^1, k)$-modules

$$\begin{tikzcd} H^\bullet\big(\mathcal{G}_q^1, V_1 \otimes L_1^q(\lambda)\big) \arrow[rr] & & H^\bullet\big(\mathcal{G}_q^1, V_2 \otimes L_1^q(\lambda)\big) \arrow[dl] \\ & H^\bullet\big(\mathcal{G}_q^1, V_3 \otimes L_1^q(\lambda)\big) \arrow[ul] & \end{tikzcd}$$

This gives

$$\sqrt{\mathfrak{R}_{V_{\sigma(1)}\otimes L_1^q(\lambda)}} \supset \sqrt{\mathfrak{R}_{V_{\sigma(2)}\otimes L_1^q(\lambda)}} \cap \sqrt{\mathfrak{R}_{V_{\sigma(3)}\otimes L_1^q(\lambda)}},$$

arguing as in the proof of A.8. Therefore,

$$\mathcal{V}(\mathfrak{R}_{V_{\sigma(1)}\otimes L_1^q(\lambda)}) \subset \mathcal{V}(\mathfrak{R}_{V_{\sigma(2)}\otimes L_1^q(\lambda)}) \cup \mathcal{V}(\mathfrak{R}_{V_{\sigma(3)}\otimes L_1^q(\lambda)}).$$

The first assertion of (4) now follows from (2), and the second follows from the fact that $\mathcal{N}$ is irreducible.

(5) Clearly, $\psi_{V_1\oplus V_2} = \psi_{V_1} \oplus \psi_{V_2}$, so, $\mathfrak{I}_{V_1\oplus V_2} = \mathfrak{I}_{V_1} \cup \mathfrak{I}_{V_2}$, and hence the result.

(6) The first claim is clear from Proposition A.7. The second follows from Theorem 3.3 and Proposition A.7. ∎

REMARK 5.3. One may expect that $\mathrm{Supp}_q(V^*) = \mathrm{Supp}_q(V)$ and that $\mathrm{Supp}_q(V\otimes U) \subset \mathrm{Supp}_q(V) \cap \mathrm{Supp}_q(U)$ for any $U, V \in \mathrm{Ob}(\mathcal{G}_q^1)$. This will follow provided $U \otimes V \cong V \otimes U$ for any $U, V \in \mathrm{Ob}(\mathcal{G}_q^1)$. If $V, U \in \mathrm{Ob}(\mathcal{G}_q)$, there exists a functorial isomorphism $V \otimes U \cong U \otimes V$, at least in the case Φ has type A. In fact, Hayashi [11] gives a beautiful proof for the (complete) "Schur algebra" corresponding to a Yang-Baxter operator to be quasi-triangular, yielding a functorial isomorphism $V \otimes U \cong U \otimes V$ for comodules U and V over the matrix bialgebra defined by the operator. This result applies to quantum linear groups (of type A), and to quantum enveloping algebras of type A, because of the duality there between the quantized functional algebras and the quantum enveloping algebras. Although Hayashi's result also applies to other quantized classical groups, there seems to be no duality available (in the case q is a root of unity). On the other hand, by Drinfeld [6], any (complete) quantum enveloping algebra is quasi-triangular if q is generic, thus the category $\mathcal{G}_q$ is a tensor category in this case. Kazhdan and Lusztig claimed in [19] that Drinfeld's proof can be extended to cover the case q is a root of unity. If so, a functorial isomorphism $U \otimes V \cong V \otimes U$ for $U, V \in \mathrm{Ob}(\mathcal{G}_q)$ always exist, and therefore we have $\mathrm{Supp}_q(V^*) = \mathrm{Supp}_q(V)$ and $\mathrm{Supp}_q(V\otimes U) \subset \mathrm{Supp}_q(V) \cap \mathrm{Supp}_q(U)$ for any $U, V \in \mathrm{Ob}(\mathcal{G}_q)$ (but not necessarily for $U, V \in \mathrm{Ob}(\mathcal{G}_q^1)$).

Recall for $\lambda, \mu \in \bar{C}_l$, the translation functor T_λ^μ is defined as follows. Let ν be the unique dominant weight in the W-orbit (under the ordinary action) of $\mu - \lambda$, and $M = M(\lambda, \mu)$ an indecomposable object in $\mathcal{G}_q$ such that ν is the unique maximal weight of M and the weight space M_ν is 1-dimensional. For example, take $M = L^q(\nu)$. Then for any $V \in \mathrm{Ob}(\hat{\mathcal{G}}_q^1)$ belonging to λ (*i.e.*, all composition factors of V have highest weights in $W_l \cdot \lambda$), $T_\lambda^\mu(V)$ is the largest submodule of $V \otimes M$ belonging to μ (which is a direct summand). The functor T_λ^μ is independent of the choice of M. The theory developed in [17, II§7] for algebraic groups can be adapted to the context of quantum groups formally.

From the definition of T_λ^μ and Lemma 5.2(4)(5), the following result is obvious.

COROLLARY 5.4. *Let* $V \in \mathrm{Ob}(\mathcal{G}_q)$ *belonging to* λ. *Then* $\mathrm{Supp}_q\big(T_\lambda^\mu(V)\big) \subset \mathrm{Supp}_q(V)$.

The following is an easy consequence of the theory developed in §§2–3.

THEOREM 5.5. *For any* $V \in \mathrm{Ob}(\mathcal{G}_q^1)$ *with* $\dim_q V \neq 0$ *we have* $\mathrm{Supp}_q(V) = \mathcal{N}$. *In particular, if* $\lambda \in X_+$ *is l-regular, then*

$$\mathrm{Supp}_q\big(V^q(\lambda)\big) = \mathrm{Supp}_q\big(A^q(\lambda)\big) = \mathcal{N}.$$

Moreover, if the quantum Lusztig conjecture is true for the category $\mathcal{G}_q$, *then for any l-regular dominant weight* λ, *we have*

$$\mathrm{Supp}_q\big(L^q(\lambda)\big) = \mathcal{N}.$$

PROOF. By Theorem 3.3, the natural homomorphism $k \longrightarrow V \otimes V^*$ is a split injection. This forces ψ_V to be a split injection also. The first statement is clear. Now the other conclusions follow from Proposition 2.2 and Theorem 2.7. ∎

To prove the converse of the Theorem 5.5, namely, that if $\lambda \in X_+$ is not l-regular, then $\mathrm{Supp}_q\big(V^q(\lambda)\big)$, $\mathrm{Supp}_q\big(A^q(\lambda)\big)$ and $\mathrm{Supp}_q\big(L^q(\lambda)\big)$ are proper subvarieties of $\mathcal{N}$, we need to work with the following categories (and the categories introduced in §3):

(1) The category $\mathcal{B}$—the category of rational B-modules, B being the Borel subgroup of G corresponding to negative roots. This is also a category of $\mathfrak{b}$-modules, $\mathfrak{b}$ being the Borel subalgebra of $\mathfrak{g}$ corresponding to negative roots.

(2) The category $\mathcal{B}_q$—the category of rational B_q-modules, B_q being the Borel subgroup of B_q corresponding to negative roots, or the category of the integral modules of type 1 over the Borel subalgebra $\mathfrak{U}_q^0\mathfrak{U}_q^-$.

(3) The category $\mathcal{B}_q^1$—the infinitesimal version of $\mathcal{B}_q$.

(4) The category $\tilde{\mathcal{G}}_q^1$—the category of rational $(G_q)_1B$-modules, or the category of integral modules of type 1 over the subalgebra of $\mathfrak{U}_q$ generated by $\mathbf{u}_q$ and $\mathfrak{U}_q^0\mathfrak{U}_q^-$.

We have the following sequences of categories with F the Frobenius twist functor and R the restriction functor. Here 1 stands for the category of k-vector spaces. These sequences are "exact" in the sense that the arrow-reversing sequences of quantum groups and quantum enveloping algebras are exact.

$$1 \longleftarrow \mathcal{G}_q^1 \xleftarrow{R} \mathcal{G}_q \xleftarrow{F} \mathcal{G} \longleftarrow 1,$$
$$1 \longleftarrow \mathcal{B}_q^1 \xleftarrow{R} \mathcal{B}_q \xleftarrow{F} \mathcal{B} \longleftarrow 1,$$
$$1 \longleftarrow \mathcal{G}_q^1 \xleftarrow{R} \tilde{\mathcal{G}}_q^1 \xleftarrow{F} \mathcal{B} \longleftarrow 1.$$

Let $1 \longleftarrow \mathcal{K} \xleftarrow{R} \mathcal{X} \xleftarrow{F} \mathcal{Y} \longleftarrow 1$ be one of the above sequences. As in §3, for $U \in \mathrm{Ob}(\mathcal{Y})$, $F(U)$ is denoted by $U^{(l)}$. Conversely, if $V \in \mathrm{Ob}(\mathcal{X})$ has the form $F(U)$ for $U \in \mathrm{Ob}(\mathcal{Y})$, then U, which is uniquely determined by V, will be denoted as $V^{(-l)}$. In

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particular, for $V \in \mathrm{Ob}(\mathcal{X})$, $H^i(\mathcal{K}, V) \equiv H^i\big(\mathcal{K}, R(V)\big)$ has the form $F(U)$ for $U \in \mathrm{Ob}(\mathcal{Y})$ (see [21, §§2.9–2.10] and [10, (5.2.1)]), so that $H^i(\mathcal{K}, V)^{(-l)}$ is defined. Also, since

$$\mathrm{Hom}_{\mathcal{Y}}\big(U, H^0(\mathcal{K}, V)^{(-l)}\big) \cong \mathrm{Hom}_{\mathcal{X}}\big(U^{(l)}, H^0(\mathcal{K}, V)\big) \cong \mathrm{Hom}_{\mathcal{X}}\big(U^{(l)}, V\big)$$

naturally for $U \in \mathrm{Ob}(\mathcal{Y})$ and $V \in \mathrm{Ob}(\mathcal{X})$, the functor $V \longmapsto H^0(\mathcal{K}, V)^{(-l)}$ is the right adjoint of the functor $U \longmapsto F(U)$. It follows that $H^i(\mathcal{K}, V)^{(-l)} \cong R^i \mathrm{Ind}_{\mathcal{X}}^{\mathcal{Y}} V$, functorially.

The conclusions (1) and (2) of the following lemma appear in [10, §2.5].

LEMMA 5.6. *(1) We have $H^i(\mathcal{B}_q^1, k) = 0$ for odd i. Also, there is a natural graded $\mathcal{B}$-algebra isomorphism*

$$H^{2\bullet}(\mathcal{B}_q^1, k)^{(-l)} \cong k[\mathfrak{n}],$$

where $\mathfrak{n} = \mathcal{N} \cap \mathfrak{b}$, the subalgebra of ad-nilpotent elements in $\mathfrak{b}$.

(2) If $\lambda \in X$ is not of the form $w \cdot 0 + l\nu$ for $w \in W$ and $\nu \in X$, then $H^\bullet(\mathcal{B}_q^1, \lambda) = 0$.

(3) For $w \in W$ and $\nu \in X$, we have a $\mathcal{B}$-isomorphism

$$H^i(\mathcal{B}_q^1, w \cdot 0 + l\nu)^{(-l)} \cong H^{i-\ell(w)}(\mathcal{B}_q^1, k)^{(-l)} \otimes \nu.$$

PROOF. We only give a proof for (3). Obviously, it is enough to consider the case $\nu = 0$. Thanks to [10, §2.5], $H^\bullet(\mathcal{B}_q^1, w \cdot 0)^{(-l)}$ is a free $H^\bullet(\mathcal{B}_q^1, k)^{(-l)}$-module with a generator in degree $\ell(w)$. (This implies the existence of an isomorphism of vector spaces as in (3)). Thus, $H^{\ell(w)}(\mathcal{B}_q^1, w \cdot 0)^{(-l)}$ is 1-dimensional with weight ξ, say, and, since the cup product is compatible with the action of B,

$$H^j(\mathcal{B}_q^1, w \cdot 0)^{(-l)} \cong H^{j-\ell(w)}(\mathcal{B}_q^1, k)^{(-l)} \otimes \xi.$$

We have to show $\xi = 0$.

Since $\mathrm{Ind}_{\mathcal{B}_q}^{\mathcal{G}} \cong \mathrm{Ind}_{\mathcal{B}}^{\mathcal{G}} \circ \mathrm{Ind}_{\mathcal{B}_q}^{\mathcal{B}} \cong \mathrm{Ind}_{\mathcal{G}_q}^{\mathcal{G}} \circ \mathrm{Ind}_{\mathcal{B}_q}^{\mathcal{G}_q}$, we have the following two spectral sequences

$$E_2^{ij} = R^i \mathrm{Ind}_{\mathcal{B}}^{\mathcal{G}}\big(H^j(\mathcal{B}_q^1, w \cdot 0)^{(-l)}\big) \Rightarrow R^{i+j} \mathrm{Ind}_{\mathcal{B}_q}^{\mathcal{G}} w \cdot 0,$$

$${}'E_2^{ij} = H^i(\mathcal{G}_q^1, R^j \mathrm{Ind}_{\mathcal{B}_q}^{\mathcal{G}_q} w \cdot 0)^{(-l)} \Rightarrow R^{i+j} \mathrm{Ind}_{\mathcal{B}_q}^{\mathcal{G}} w \cdot 0.$$

Since $H^j(\mathcal{B}_q^1, w \cdot 0)^{(-l)} = 0$ for $j < \ell(w)$, the first spectral sequence yields a $\mathcal{G}$-isomorphism

$$\mathrm{Ind}_{\mathcal{B}}^{\mathcal{G}} \xi \cong R^{\ell(w)} \mathrm{Ind}_{\mathcal{B}_q}^{\mathcal{G}} w \cdot 0.$$

On the other hand, we have (see [3, (6.7)], [21, (10.2.3)])

$$R^j \mathrm{Ind}_{\mathcal{B}_q}^{\mathcal{G}_q} w \cdot 0 \cong \begin{cases} k, & j = \ell(w), \\ 0, & \text{otherwise.} \end{cases}$$

Thus, the second spectral sequence degenerates to a $\mathcal{G}$-isomorphism

$$H^i(\mathcal{G}_q^1, k)^{(-l)} \cong R^{i+\ell(w)} \mathrm{Ind}_{\mathcal{B}_q}^{\mathcal{G}} w \cdot 0.$$

Combining the isomorphisms obtained from these spectral sequences, we see that

$$\mathrm{Ind}_{\mathcal{B}}^{\mathcal{G}} \xi \cong H^0(\mathcal{G}_q^1, k)^{(-l)} = k,$$

as $\mathcal{G}$-objects. This forces $\xi = 0$, as required. ∎

Now we are ready to prove the following result.

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PROPOSITION 5.7. *Suppose Φ is indecomposable. If λ be a dominant weight in $\bar{C}_l \backslash C_l$, then* $\mathrm{Supp}_q\big(V^q(\lambda)\big) = \mathrm{Supp}_q\big(A^q(\lambda)\big) = \mathrm{Supp}_q\big(L^q(\lambda)\big)$ *is a proper subvariety of* $\mathcal{N}$.

PROOF. Using Lemma 5.6, the proof in [14, §§4.8–4.11] for the corresponding algebraic group result can be translated *mutatis mutandis*. Since this is the key point of the next theorem, we sketch the argument.

By hypothesis, $\{\alpha \in \Phi \mid (\lambda + \rho, \alpha^\vee) = 0 \pmod{l}\} = \{\alpha_0, -\alpha_0\}$, where α_0 is the highest short root. Let $m = \big(\ell(s_{\alpha_0}) + 1\big)/2$. Let also $\tilde{A}_1^q(\nu) = \mathrm{Ind}_{\mathcal{B}_q}^{\tilde{\mathcal{G}}_q^1} \nu$ for $\nu \in X$, $M_1 = T_\lambda^0 \tilde{A}_1^q(\lambda)$ and $M = T_\lambda^0 A^q(\lambda)$. We split the proof into 5 steps.

STEP 1. For $i < 2m - 1$, we have[5]

$$H^\imath(\mathcal{G}_q^1, M_1)^{(-l)} \cong \begin{cases} k[\mathfrak{n}]_{\imath/2}, & i \text{ even}, \\ 0, & i \text{ odd}; \end{cases} \tag{5.7.1}$$

and, for $i \geq m$ there exists an exact $\mathcal{B}$-sequence
(5.7.2)

$$0 \longrightarrow H^{2\imath-1}(\mathcal{G}_q^1, M_1)^{(-l)} \longrightarrow k[\mathfrak{n}]_{\imath-m} \otimes \alpha_0 \longrightarrow k[\mathfrak{n}]_\imath \longrightarrow H^{2\imath}(\mathcal{G}_q^1, M_1)^{(-l)} \longrightarrow 0.$$

This follows from the $\mathcal{B}$-isomorphism $H^\imath\big(\mathcal{G}_q^1, \tilde{A}_1^q(\nu)\big)^{(-l)} \cong H^\imath(\mathcal{B}_q^1, \nu)^{(-l)}$ ($\nu \in X$), Lemma 5.6 and the long exact sequence of cohomology corresponding to the short exact $\tilde{\mathcal{G}}_q^1$-sequence $0 \longrightarrow \tilde{A}_1^q(0) \longrightarrow M_1 \longrightarrow \tilde{A}_1^q(s_{\alpha_0} \cdot 0 + l\alpha_0) \longrightarrow 0$ (compare [17, (II.9.19)]).

STEP 2. (5.7.2) for $i = m$ is actually a short exact sequence

$$0 \longrightarrow \alpha_0 \longrightarrow k[\mathfrak{n}]_m \longrightarrow H^{2m}(\mathcal{G}_q^1, M_1)^{(-l)} \longrightarrow 0. \tag{5.7.3}$$

By (5.7.2), it is enough to show

$$\mathrm{Hom}_{\mathcal{B}}\big(\alpha_0, H^{2m-1}(\mathcal{G}_q^1, M_1)^{(-l)}\big) \cong H^0\Big(\mathcal{B}, H^{2m-1}\big(\mathcal{G}_q^1, M_1 \otimes (-l\alpha_0)\big)^{(-l)}\Big) = 0.$$

This is the $E_2^{0,2m-1}$-term in the Hochschild-Serre spectral sequence

$$E_2^{\imath\jmath} = H^\imath\Big(\mathcal{B}, H^\jmath\big(\mathcal{G}_q^1, M_1 \otimes (-l\alpha_0)\big)^{(-l)}\Big) \Rightarrow H^{\imath+\jmath}\big(\tilde{\mathcal{G}}_q^1, M_1 \otimes (-l\alpha_0)\big). \tag{5.7.4}$$

By (5.7.1), we have $E_2^{\imath,2\jmath} \cong H^\imath\big(\mathcal{B}, k[\mathfrak{n}]_\jmath \otimes (-\alpha_0)\big)$ and $E_2^{\imath,2\jmath-1} = 0$ for $j < m$. Then, by [2, (6.7(2)(3))] (the argument there works for characteristic 0), $E_2^{\imath\jmath} = 0$ for $2m - 2i < j < 2m$ and $E_2^{1,2m-2} = k$. Thus, the spectral sequence (5.7.4) gives an exact sequence

$$0 \longrightarrow k \longrightarrow H^{2m-1}\big(\tilde{\mathcal{G}}_q^1, M_1 \otimes (-l\alpha_0)\big) \longrightarrow E_2^{0,2m-1} \longrightarrow 0. \tag{5.7.5}$$

Since $M_1 \otimes (-l\alpha_0) \cong T_\lambda^0 \tilde{A}_q^1(\lambda - l\alpha_0) = T_\lambda^0 \tilde{A}_q^1(s_{\alpha_0} \cdot \lambda)$, by the Borel-Weil-Bott theorem for small weights (see [3, (6.7)], [21, (10.2.3)]), we obtain

$$R^\jmath\, \mathrm{Ind}_{\tilde{\mathcal{G}}_q^1}^{\mathcal{G}_q}\big(M_1 \otimes (-l\alpha_0)\big) \cong T_\lambda^0 R^\jmath\, \mathrm{Ind}_{\mathcal{B}_q}^{\mathcal{G}_q}(s_{\alpha_0} \cdot \lambda) \cong \begin{cases} T_\lambda^0 A^q(\lambda) = M, & j = 2m - 1, \\ 0, & \text{otherwise}. \end{cases}$$

5 $k[\mathfrak{n}]_\imath$ denotes the $\imath$-th homogeneous component of the graded algebra $k[\mathfrak{n}]$. A similar notational convention is adopted for $k[\mathcal{N}]$.

Therefore, the spectral sequence arising from the reciprocity for induction yields an isomorphism $H^i\big(\tilde{\mathcal{G}}_q^1, M_1 \otimes (-l\alpha_0)\big) \cong H^{i-2m+1}(\mathcal{G}_q, M)$. Since we have an exact $\mathcal{G}_q$-sequence $0 \longrightarrow k \longrightarrow M \longrightarrow A^q(s_{\alpha_0} \cdot 0 + l\alpha_0) \longrightarrow 0$ (compare [17, (II.7.19)]), $H^i(\mathcal{G}_q, M) = 0$ for $i > 0$ (see [21, (10.4.6)]; the argument works for quantum enveloping algebras). Thus,

$$H^i\big(\tilde{\mathcal{G}}_q^1, M_1 \otimes (-l\alpha_0)\big) = \begin{cases} k, & i = 2m-1, \\ 0, & \text{otherwise.} \end{cases}$$

This, together with (5.7.5), shows $H^0\Big(\mathcal{B}, H^{2m-1}\big(\mathcal{G}_q^1, M_1 \otimes (-l\alpha_0)\big)^{(-l)}\Big) = 0$.

STEP 3. There is a $\mathcal{G}$-isomorphism

$$\operatorname{Ind}_{\mathcal{B}}^{\mathcal{G}}\big(H^j(\mathcal{G}_q^1, M_1)^{(-l)}\big) \cong H^j(\mathcal{G}_q^1, M)^{(-l)}, \quad \text{for } i \le 2m. \tag{5.7.6}$$

Since the induction from $\mathcal{B}$ to $\tilde{\mathcal{G}}_q^1$ is exact, we have

$$R^i \operatorname{Ind}_{\tilde{\mathcal{G}}_q^1}^{\mathcal{G}_q} M_1 \cong T_\lambda^0 R^i \operatorname{Ind}_{\mathcal{B}_q}^{\mathcal{G}_q} \lambda \cong \begin{cases} M, & i = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, one of the spectral sequences associated to M_1 and $\operatorname{Ind}_{\tilde{\mathcal{G}}_q^1}^{\mathcal{G}} \cong \operatorname{Ind}_{\mathcal{B}}^{\mathcal{G}} \circ \operatorname{Ind}_{\tilde{\mathcal{G}}_q^1}^{\mathcal{B}} \cong \operatorname{Ind}_{\mathcal{G}_q}^{\mathcal{G}} \circ \operatorname{Ind}_{\tilde{\mathcal{G}}_q^1}^{\mathcal{G}_q}$ degenerates, and the other one gives

$$E_2^{ij} = R^i \operatorname{Ind}_{\mathcal{B}}^{\mathcal{G}}\big(H^j(\mathcal{G}_q^1, M_1)^{(-l)}\big) \Rightarrow H^{i+j}(\mathcal{G}_q^1, M)^{(-l)}.$$

We obtain (5.7.6) from this spectral sequence by using (5.7.1) and the following fact from the theory of algebraic groups (see, for example, [17, (II.12.12)])

$$H^i \operatorname{Ind}_{\mathcal{B}}^{\mathcal{G}} k[\mathfrak{n}] = \begin{cases} k[\mathcal{N}], & i = 0, \\ 0, & \text{otherwise.} \end{cases} \tag{5.7.7}$$

STEP 4. There is a short exact $\mathcal{G}$-sequence

$$0 \longrightarrow L(\alpha_0) \longrightarrow k[\mathcal{N}]_m \longrightarrow H^{2m}(\mathcal{G}_q^1, M)^{(-l)} \longrightarrow 0. \tag{5.7.8}$$

This is obtained by inducing (5.7.3) to $\mathcal{G}$, using (5.7.6), (5.7.7) and the Borel-Weil-Bott Theorem.

STEP 5. The exact sequence (5.7.8) means that $\dim k[\mathcal{N}]_m > \dim H^{2m}(\mathcal{G}_q^1, M)^{(-l)}$, so any homomorphism from $k[\mathcal{N}]_m$ to $H^{2m}(\mathcal{G}_q^1, M)^{(-l)}$ has nontrivial kernel.

Since $M = T_\lambda^0 A^q(\lambda)$ is the direct summand of $A^q(\lambda) \otimes V^q(-w_0\lambda) = A^q(\lambda) \otimes A^q(\lambda)^*$ belonging to 0, the canonical homomorphism $k \longrightarrow A^q(\lambda) \otimes A^q(\lambda)^*$ factors through M, so $\psi_{A^q(\lambda)}$ factors through $H^\bullet(\mathcal{G}_q^1, M)$. Thus, $\operatorname{Ker} \psi_{A^q(\lambda)} \neq 0$, proving the proposition for $A^q(\lambda)$, and also for $V^q(\lambda)$ and $L^q(\lambda)$, since $V^q(\lambda) = L^q(\lambda) = A^q(\lambda)$. ■

THEOREM 5.8. *Let $\lambda \in X_+$ be not l-regular. Then* $\operatorname{Supp}_q\big(A^q(\lambda)\big)$, $\operatorname{Supp}_q\big(V^q(\lambda)\big)$ *and* $\operatorname{Supp}_q\big(L^q(\lambda)\big)$ *are proper subvarieties of* $\mathcal{N}$.

PROOF. Suppose for a moment that Φ is indecomposable. Let C be the l-alcove having λ in its upper closure. We proceed by induction on the distance $d(C)$ from C to the bottom l-alcove C_l Recall that $d(C)$ ıs the number of l-walls separatıng C and C_l If $d(C) = 0$, $C = C_l$, and Propositıon 5 7 applıes So assume $d(C) > 0$ Choose a wall H of C separatıng C and C_l, and choose a weıght $\mu \in \bar{C}$ ın the facet of codımensıon 1 contaıned ın H Let λ_0 (resp μ_0) be the W_l-conȷugate of λ (resp μ) ın $\bar{C}_l$ Then we have an exact sequence (compare [17, (II 7 13)])

$$0 \longrightarrow A^q(s \ \ \lambda) \longrightarrow T_{\mu_0}^{\lambda_0} A^q(\mu) \longrightarrow A^q(\lambda) \longrightarrow 0,$$

where $s = s_H$ ıs the affine reflectıon wıth respect to H The ınductıon hypothesıs can be applıed to μ and $s \ \ \lambda$ (It may happen that $A^q(s \ \ \lambda) = 0$) Thus, by Corollary 5 4 and Lemma 5 2(4), $\mathrm{Supp}_q\big(A^q(\lambda)\big)$ ıs proper

A sımılar argument applıes to $V^q(\lambda)$ For ırreducıble modules, we can use ınductıon on the standard partıal orderıng on X We have an exact sequence

$$0 \longrightarrow L^q(\lambda) \longrightarrow A^q(\lambda) \longrightarrow M^q(\lambda) \longrightarrow 0$$

The hıghest weıghts of composıtıon factors of $M^q(\lambda)$ are strıctly smaller than λ The ınductıon, together wıth Lemma 5 2(4), ensures that $\mathrm{Supp}_q\big(M^q(\lambda)\big)$ ıs proper Then, by Lemma 5 2(4) agaın, $\mathrm{Supp}_q\big(L^q(\lambda)\big)$ ıs also proper

If Φ ıs decomposable, the algebra $H^\bullet(\mathcal{G}_q^1, k)$ decomposes ınto the tensor product of ıdeals correspondıng to ındecomposable components of Φ, sımılarly for the algebra $H^\bullet(\mathcal{G}_q^1, V \otimes V^*)$ for $V \in \mathrm{Ob}(\mathcal{G}_q^1)$ The homomorphısm ψ_V decomposes correspondıngly ınto a tensor product If $V = V^q(\lambda)$, $A^q(\lambda)$ or $L^q(\lambda)$ wıth λ not l-regular, then one of the components of ψ_V has nontrıvıal kernel, by the above Thus, ψ_V has nontrıvıal kernel and $\mathrm{Supp}_q(V)$ ıs proper ■

Note that $\mathcal{N}$ has a unıque dense G-orbıt, the orbıt $\mathcal{N}_{\mathrm{reg}}$ of regular ad-nılpotent elements Thus, for $V \in \mathrm{Ob}(\mathcal{G}_q)$, $\mathrm{Supp}_q(V)$ ıs proper ıff $\mathrm{Supp}_q(V) \subset \mathcal{N} \setminus \mathcal{N}_{\mathrm{reg}}$

For $I \subset \Pi$, one may define the support varıety $\mathrm{Supp}_{q\ I}(V)$ for $V \in \mathcal{G}_{q\ I}^1$ (see §4) Thıs ıs a subvarıety of $\mathcal{N}_I$, the varıety of ad-nılpotent elements ın the Levı subalgebra $\mathfrak{g}_I$ of $\mathfrak{g}$ In partıcular, ıf $V \in \mathrm{Ob}(\mathcal{G}_q^1)$, vıewıng as an object of $\mathcal{G}_{q\ I}^1$, $\mathrm{Supp}_{q\ I}(V)$ ıs canonıcally a subvarıety of $\mathrm{Supp}_q(V)$

Usıng the results of §4, we have the followıng analogue of [15, §3] Its proof ıs clear from the defınıtıons above, together wıth the results of §2 and §4

PROPOSITION 5 9 *Let $\lambda \in X_+$ and let $I \subset \Pi$ satısfy the condıtıon of Theorem 5 8 Then $G \ \ \mathcal{N}_I \subset \mathrm{Supp}_q\big(V^q(\lambda)\big)$ and $G \ \ \mathcal{N}_I \subset \mathrm{Supp}_q\big(A^q(\lambda)\big)$ Moreover, ıf the quantum Lusztıg conȷecture ıs true for $\mathcal{G}_{q\ I}$, we also have $G \ \ \mathcal{N}_I \subset \mathrm{Supp}_q\big(L^q(\lambda)\big)$*

Fınally, we gıve the followıng exact descrıptıon of $\mathrm{Supp}_q\big(V^q(\lambda)\big)$ and $\mathrm{Supp}_q\big(A^q(\lambda)\big)$ for λ ın a facet of codımensıon 1 ın the case Φ has type A For the algebraıc group case, see [15] and [17]

THEOREM 5 10 *Assume that Φ ıs of type $\mathrm{A}_{n \ 1}$ Let λ be a domınant weıght ın a facet of codımensıon 1 Then*

$$\mathrm{Supp}_q\big(V^q(\lambda)\big) = \mathrm{Supp}\big(A^q(\lambda)\big) = \mathcal{N} \setminus \mathcal{N}_{\mathrm{reg}}$$

Moreover, if the quantum Lusztig conjecture is true for root system of type A_{n-2}, *we also have*

$$\mathrm{Supp}\big(L^q(\lambda)\big) = \mathcal{N} \setminus \mathcal{N}_{\mathrm{reg}}.$$

PROOF. In this case, the subroot system $\{\alpha \in \Phi \mid (\lambda+\rho, \alpha^\vee) \equiv 0 \pmod{l}\}$ has type A_1. Thus, by Remark 4.9, condition (4.8.1) is satisfied by a $w \in W$ and a subset I of Π of type A_{n-2}. Thus, by Proposition 5.9, $G \cdot \mathcal{N}_I \subset \mathrm{Supp}_q\big(A^q(\lambda)\big)$. Since $G \cdot \mathcal{N}_I$ contains the unique dense G-orbit of $\mathcal{N} \setminus \mathcal{N}_{\mathrm{reg}}$, and $\mathrm{Supp}_q\big(A^q(\lambda)\big) \subset \mathcal{N} \setminus \mathcal{N}_{\mathrm{reg}}$, by Theorem 5.8. The result for $A^q(\lambda)$ follows.

The same argument works for $V^q(\lambda)$, and, if the quantum Lusztig conjecture is true for Φ_I, it works for $L^q(\lambda)$, too. ∎

Appendix. Some homological foundations. In this appendix $\mathfrak{A}$ is a Hopf algebra over k. Let $\mathcal{M}$ be a category of (left) $\mathfrak{A}$-modules or a category of (right) $\mathfrak{A}$-comodules, satisfying the following hypotheses:

(A.1) It is a full abelian subcategory of the category of all $\mathfrak{A}$-modules or $\mathfrak{A}$-comodules;

(A.2) It is closed under taking the dual of finite dimensional object and taking tensor products;

(A.3) It has enough injectives, so that we can develop a cohomological theory in it;

(A.4) If we consider $\mathfrak{A}$-modules, every object in $\mathcal{M}$ is locally finite, *i.e.*, is the union of its subobjects of finite dimension over k.

An object A in $\mathcal{M}$ is called an *$\mathcal{M}$-algebra* if A is also an algebra and if the unit map $\kappa\colon k \longrightarrow A$ and the multiplication map $\mu\colon A \otimes A \longrightarrow A$ are $\mathcal{M}$-morphisms. An $\mathcal{M}$-morphism between two $\mathcal{M}$-algebras is called an *$\mathcal{M}$-algebra homomorphism* if it is also an algebra homomorphism. The simplest $\mathcal{M}$-algebra is the trivial object k in $\mathcal{M}$, and the unit map $\kappa\colon k \longrightarrow A$ for any $\mathcal{M}$-algebra A is an $\mathcal{M}$-algebra homomorphism.

As in [22, Appendix], given an $\mathcal{M}$-algebra A, the cohomology group $H^\bullet(\mathcal{M}, A) \equiv \mathrm{Ext}^\bullet(k, A)$ becomes a graded k-algebra under the cup product, which is induced by the Künneth isomorphism $H^i(\mathcal{M}, A) \otimes H^j(\mathcal{M}, A) \cong H^{i+j}(\mathcal{M}, A \otimes A)$ and the product homomorphism $A \otimes A \longrightarrow A$. More generally, if A is an $\mathcal{M}$-algebra and $V \in \mathrm{Ob}(\mathcal{M})$ has a left A-module structure compatible with the $\mathcal{M}$-structures, then $H^\bullet(\mathcal{M}, V)$ is a left $H^\bullet(\mathcal{M}, A)$-module. In particular, for any $V \in \mathrm{Ob}(\mathcal{M})$, $H^\bullet(\mathcal{M}, V)$ is a left $H^\bullet(\mathcal{M}, k)$-module. Also, if A and B are $\mathcal{M}$-algebras, then any $\mathcal{M}$-algebra homomorphism $\psi\colon A \longrightarrow B$ will induce a graded algebra homomorphism $\psi_*\colon H^\bullet(\mathcal{M}, A) \longrightarrow H^\bullet(\mathcal{M}, B)$. In particular, the unit map $\kappa\colon k \longrightarrow A$ gives rise to a graded algebra homomorphism $\kappa_*\colon H^\bullet(\mathcal{M}, k) \longrightarrow H^\bullet(\mathcal{M}, A)$.

We are mainly interested in a special class of $\mathcal{M}$-algebras, described in the following lemma, whose easy proof is left to the reader. The assertion (2), in the context of comodules, appears in [22, (A.4.2)] as an example. Also, the lemma, restricted to quantum enveloping algebras, is contained in [1, (3.6)].

LEMMA A.5. *Let $V \in \mathrm{Ob}(\mathcal{M})$ be finite dimensional over k with basis $\{v_i\}$. Then:*

(1) The natural map $\pi: V^* \otimes V \longrightarrow k$, sending $f \otimes v$ to $f(v)$ for $f \in V^*$ and $v \in V$, is an $\mathcal{M}$-morphism;

(2) $A = \mathrm{End}_k(V) \cong V \otimes V^*$ is an $\mathcal{M}$-algebra with multiplication identity $1_A = \sum_i v_i \otimes f_i$, where $\{f_i\}$ is the basis of V^* dual to the given basis of V.

REMARK A.6. Note that Lemma A.5 is not trivial—if we interchange V and V^*, the results may not hold! In fact, if we still have these results, then the trivial object k will be a direct summand of $V \otimes V^*$, provided $\dim V$ is not 0 in k. If, in addition, V is an injective object in $\mathcal{M}$, then it follows that k (and therefore any object in $\mathcal{M}$) is injective (see [22, (2.8.2)]). Hence any object in $\mathcal{M}$ is completely reducible. We have counterexamples to show this is not true. (For example, the category of rational modules over quantum group $G_q = \mathrm{GL}_q(n)$ or $\mathrm{SL}_q(n)$ with char $k = 0$ and q is a primitive l-th root of unity for an odd integer l. See [21, (9.10.4)].)

If the antipode of $\mathfrak{A}$ has order 2, then the canonical map $V \longrightarrow V^{**}$ is an $\mathcal{M}$-morphism. It follows from Lemma A.5 that the canonical map $\pi': V \otimes V^* = (V^*)^* \otimes V^* \longrightarrow k$ is an $\mathcal{M}$-morphism. From this observation, an interesting (perhaps well-known) fact follows easily: If the category of comodules or the category of locally finite modules for a k-Hopf algebra with $\gamma^2 = \mathrm{id}$ (*e.g.*, a commutative or cocommutative Hopf algebra) has a finite dimensional injective object I with $\dim I$ nonzero in k, then any object in the category is completely reducible. In particular, if a finite dimensional Hopf algebra with $\gamma^2 = \mathrm{id}$ has dimension nonzero in k (*e.g.*, char $k = 0$), then any comodule or modules over the Hopf algebra is completely reducible. The complete reducibility of ordinary representations of a finite group is a special case of the above assertion.

Denote by $\mathfrak{I}_V$ the kernel of the graded algebra homomorphism $\psi_V: H^\bullet(\mathcal{M}, k) \longrightarrow H^\bullet(\mathcal{M}, V \otimes V^*)$ induced by the unit map $\kappa: k \longrightarrow V \otimes V^*$ given in Lemma A.5(2). Then we have the following result.

PROPOSITION A.7. *Let $V, U \in \mathrm{Ob}(\mathcal{M})$ be finite dimensional. Then $\mathfrak{I}_V \subset \mathfrak{I}_{V \otimes U}$. Moreover, if there exists an $\mathcal{M}$-homomorphism $\eta: V \otimes V^* \longrightarrow k$ with $\eta \circ \kappa: k \longrightarrow k$ nonzero, then $\mathfrak{I}_V = \mathfrak{I}_{V \otimes U}$.*

PROOF. Let $\kappa: k \longrightarrow U \otimes U^*$, $\kappa': k \longrightarrow V \otimes V^*$ and $\kappa'': k \longrightarrow V \otimes U \otimes U^* \otimes V^*$ be the unit maps. Clearly, $\kappa'' = (\mathrm{id}_V \otimes \kappa \otimes \mathrm{id}_{V^*}) \circ \kappa'$, identifying $V \otimes V^*$ with $V \otimes k \otimes V^*$. Thus, $\psi_{V \otimes U}$ factors through ψ_V. This implies the first assertion.

For the second assertion, note that $\eta \circ \kappa = a \cdot \mathrm{id}$ for a nonzero $a \in k$. Replacing η by $a^{-1}\eta$, one may assume $a = 1$. Then $\kappa' = (\mathrm{id}_V \otimes \eta \otimes \mathrm{id}_{V^*}) \circ (\mathrm{id}_V \otimes \kappa \otimes \mathrm{id}_{V^*}) \circ \kappa'$, *i.e.*, $\kappa' = (\mathrm{id}_V \otimes \eta \otimes \mathrm{id}_{V^*}) \circ \kappa''$. Thus, $\psi_V = (\mathrm{id}_V \otimes \eta \otimes \mathrm{id}_{V^*})_* \circ \psi_{V \otimes U}$, forcing $\mathfrak{I}_{V \otimes U} \subset \mathfrak{I}_V$. Therefore, $\mathfrak{I}_V = \mathfrak{I}_{V \otimes U}$. ∎

Note that in Proposition A.7, η may not be an algebra homomorphism.

Denote by $\mathfrak{R}_U$ the left annihilator of $H^\bullet(\mathcal{M}, U)$ in $H^\bullet(\mathcal{M}, k)$. This is an ideal of $H^\bullet(\mathcal{M}, k)$.

PROPOSITION A.8. *Assume that $H^{\bullet}(\mathcal{M},k)$ is commutative. Let V be a finite dimensional object in $\mathcal{M}$. Then*

$$\Im_V \subset \bigcap_{U\in \mathrm{Ob}(\mathcal{M})} \Re_{V\otimes U} \subset \bigcap_{\substack{L\in \mathrm{Ob}(\mathcal{M})\\ \text{irreducible}}} \Re_{V\otimes L} \subset \sqrt{\Im_V}.$$

PROOF. Let $A = V\otimes V^*$. Viewing $V\otimes U$ canonically as a left A-module with the $\mathcal{M}$-morphism $\zeta_U \equiv \mathrm{id}_V \otimes\pi\otimes \mathrm{id}_U$ (see Lemma A.5(1)) as its structure map, the k-action on $V\otimes U$ factors through the action of A. By taking cohomology, the cup product $H^i(\mathcal{M},A)\otimes H^j(\mathcal{M},V\otimes U) \longrightarrow H^{i+j}(\mathcal{M},A\otimes V\otimes U)$ and the morphism $(\zeta_U)_*\colon H^{i+j}(\mathcal{M},A\otimes V\otimes U) \longrightarrow H^{i+j}(\mathcal{M},V\otimes U)$ define an $H^{\bullet}(\mathcal{M},A)$-module structure on $H^{\bullet}(\mathcal{M},V\otimes U)$, and the $H^{\bullet}(\mathcal{M},k)$-action on $H^{\bullet}(\mathcal{M},V\otimes U)$ factors through the action of $H^{\bullet}(\mathcal{M},A)$. It follows that $\Im_V \subset \Re_{V\otimes U}$, hence $\Im_V \subset \bigcap_U \Re_{V\otimes U}$.

The inclusion $\bigcap_U \Re_{V\otimes U} \subset \bigcap_L \Re_{V\otimes L}$ is trivial.

Since $\Im_V = \Re_{V\otimes V^*}$, we complete the proof by showing the inclusion $\bigcap_L \Re_{V\otimes L} \subset \sqrt{\Re_{V\otimes U}}$ for any finite dimensional object U of $\mathcal{M}$ by induction on $\dim U$. If U is irreducible, this is trivial. In general, let U_1 be a proper subobject of U and $U_2 = U/U_1$. The exact sequence of A-modules $0 \longrightarrow V\otimes U_1 \longrightarrow V\otimes U \longrightarrow V\otimes U_2 \longrightarrow 0$ leads to an exact triangle of $H^{\bullet}(\mathcal{M},k)$-modules:

$$\begin{tikzcd}
H^{\bullet}(\mathcal{M},V\otimes U_1) \arrow[rr] & & H^{\bullet}(\mathcal{M},V\otimes U) \arrow[dl] \\
& H^{\bullet}(\mathcal{M},V\otimes U_2) \arrow[ul] &
\end{tikzcd}$$

Let $a \in \bigcap_L \Re_{V\otimes L}$ and $x \in H^{\bullet}(\mathcal{M},V\otimes U)$. By induction applied to U_2, $a^r x$ is in the image of $H^{\bullet}(\mathcal{M},V\otimes U_1)$ for large r. Then, by induction applied to U_1, $a^s x = 0$ for some $s \geq r$. Thus, $a \in \sqrt{\Re_{V\otimes U}}$, as required. ∎

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