ON THE ABSOLUTE SUMMABILITY BY BOREL'S INTEGRAL METHOD OF THE DERIVED FOURIER SERIES AND ITS CONJUGATE SERIES

by R. M. SHARMA
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Summary. Mohanty (1) and (3) considered the absolute summability of conjugate series and Fourier series by Borel's integral method by proving the following theorems.

**Theorem A.** If \( \psi(t) \log \frac{k}{t} \) is of bounded variation in \((0, \pi)\), then \( \sum_{n=1}^{\infty} B_n(\theta) \) is summable \( \left| B' \right| \).

**Theorem B.** If \( g(t) \) is of bounded variation in \((0, \pi)\), then the series \( \sum_{n=1}^{\infty} A_n(\theta) \) is summable \( \left| B' \right| \).

The present author considered the absolute summability of derived Fourier series and its conjugate series by Borel's integral method. Theorems proved by the present author are

**Theorem 1.** If

(i) \( \psi(+0) = 0 \)

and

(ii) \( \int_{0}^{b} t^{-2} |d\psi(t)| < \infty; 0 < \delta < 1. \)

then the series \( \sum_{n=1}^{\infty} nB_n(\theta) \) is summable \( \left| B' \right| \).

**Theorem 2.** If

(i) \( \phi(+0) = O(1) \)

and

(ii) \( \int_{0}^{b} t^{-2} |d\phi(t)| < \infty; 0 < \delta < 1. \)

then the series \( \sum_{n=1}^{\infty} nA_n(\theta) \) is summable \( \left| B' \right| \).

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1. Definition

A series \( \sum_{n=0}^{\infty} a_n \) is said to be summable \((B')\) to sum \(A\) if

\[
\int_0^\infty e^{-x} \sum_{n=0}^{\infty} \frac{a_n x^n}{n!} \, dx = \lim_{x \to \infty} \int_0^x e^{-x} \sum_{n=0}^{\infty} \frac{a_n x^n}{n!} \, dx = A.
\]

If the above integral is absolutely convergent, we say that the series \( \sum_{n=0}^{\infty} a_n \) is absolutely summable by Borel’s integral method \((2)\) or summable \(|B'|\).

2. Let \( f(t) \) be Lebesgue integrable in \((-\pi, \pi)\) and periodic with period \(2\pi\) and let

\[
f(t) \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} A_n(t). \tag{2.1}
\]

The allied series of \((2.1)\) at \(t = \theta\) is

\[
\sum_{n=1}^{\infty} (b_n \cos n\theta - a_n \sin n\theta) = \sum_{n=1}^{\infty} B_n(\theta)
\]

and the derived Fourier series is

\[
\sum_{n=1}^{\infty} nB_n(\theta). \tag{2.2}
\]

The conjugate series of \((2.2)\) is

\[
\sum_{n=1}^{\infty} nA_n(\theta). \tag{2.3}
\]

We write

\[
\phi(t) = \frac{1}{2} \{f(\theta + t) + f(\theta - t)\} \tag{2.4}
\]

\[
\psi(t) = \frac{1}{2} \{f(\theta + t) - f(\theta - t)\} \tag{2.5}
\]

and

\[
g(t) = \phi(t) \log \frac{k}{t}. \tag{2.6}
\]

Mohanty \((1)\) and \((3)\) proved

**Theorem A.** If \( \psi(t) \log \frac{k}{t} \) is of bounded variation in \((0, \pi)\), then \( \sum_{n=1}^{\infty} B_n(\theta) \) is summable \(|B'|\).

**Theorem B.** If \( g(t) \) is of bounded variation in \((0, \pi)\) then the series \( \sum_{n=1}^{\infty} A_n(\theta) \) is summable \(|B'|\).

3. The object of the present paper is to prove the following two theorems.

**Theorem 1.** If \((i)\) \( \psi(+0) = 0 \)

and \((ii)\)

\[
\int_0^{\delta} t^{-2} |d\psi(t)| < \infty; \quad 0 < \delta < 1,
\]

then the series \( \sum_{n=1}^{\infty} nB_n(\theta) \) is summable \(|B'|\).
Theorem 2. If (i) $\phi(+0) = O(1)$
and (ii) $\int_0^\delta t^{-2} |d\phi(t)| < \infty$; $0 < \delta < 1$
then the series $\sum_{n=1}^\infty nA_n(\theta)$ is summable $|B'|$.

4. In order to simplify the proof we require the following estimates for the function
\[ g_1(x, t) = \int_t^\delta e^{x \cos u} \sin (u + x \sin u) du; \ 0 < t < \delta < 1, \ x > 0. \]
\[ = O(e^x) \quad (4.1) \]
\[ = O(x^{-1} e^{x \cos t}) \quad (4.2) \]
\[ = O(x^{-1} e^x) \quad (4.3) \]
\[ g_2(x, t) = \int_t^\delta e^{x \cos u} \cos (u + x \sin u) du; \ 0 < t < \delta < 1, \ x > 0. \]
\[ = O(e^x) \quad (4.4) \]
\[ = O(x^{-1} e^{x \cos t}) \quad (4.5) \]
\[ = O(x^{-1} e^x) \quad (4.6) \]

Proof. Let $0 < t < \delta < 1, \ x > 0$ and $\epsilon_1, \ \epsilon_2$ be either 0 or 1. Define
\[ h(x, t) = \int_t^\delta e^{x \cos u} \sin (u + \frac{1}{2} \pi \epsilon_1) \sin (x \sin u + \frac{1}{2} \pi \epsilon_2) du, \]
using the second mean value theorem for integrals twice
\[ h(x, t) = e^{x \cos t} \int_t^\delta \sin (u + \frac{1}{2} \pi \epsilon_1) \sin (x \sin u + \frac{1}{2} \pi \epsilon_2) du; \ (t < s < \delta) \]
\[ = x^{-1} e^{x \cos t} \int_t^\delta x \cos u \sin (x \sin u + \frac{1}{2} \pi \epsilon_2) \sec u \sin (x \sin u + \frac{1}{2} \pi \epsilon_1) du \]
\[ = x^{-1} e^{x \cos t} \left[ \epsilon_1 + (1 - \epsilon_1) \tan s \right] \int_r^s x \cos u \sin (x \sin u + \frac{1}{2} \pi \epsilon_2) du \]
\[ = 0(x^{-1} e^{x \cos t}) \]
from which properties (4.2), (4.3), (4.5) and (4.6) follow at once.

5. Proof of Theorem 1
\[ \sum_{n=1}^\infty nB_n(\theta) \text{ is summable } |B'| \text{ if } \]
\[ I = \int_0^\infty e^{-x} \left| \sum_{n=1}^\infty \frac{nB_n(\theta)}{n!} x^n \right| dx < \infty. \]
Now

\[
I = 2\pi^{-1} \left( \int_0^\infty e^{-x} \left| \sum_{n=1}^\infty \frac{\psi(t) \sin nt}{(n-1)!} \right| dx \right) 
\]

\[
= 2\pi^{-1} \left( \int_0^\infty e^{-x} \left| \int_0^\infty \psi(t) \cdot \sum_{n=1}^\infty \frac{x^n \sin nt}{(n-1)!} dt \right| dx \right) 
\]

\[
= 2\pi^{-1} \left( \int_0^\infty e^{-x} \left| \int_0^\infty \psi(t) \cdot xe^{\cos t} \cdot \sin (t+x \sin t) dt \right| dx \right) 
\]

\[
\leq 2\pi^{-1} \left( \int_0^\infty xe^{-x} \left| \int_0^\delta \psi(t) \cdot xe^{\cos t} \cdot \sin (t+x \sin t) dt \right| dx \right) 
\]

\[
+ 2\pi^{-1} \left( \int_0^\infty xe^{-x} \left| \int_\delta^\infty \psi(t) \cdot xe^{\cos t} \cdot \sin (t+x \sin t) dt \right| dx \right) ; \quad (0 < \delta < 1) 
\]

\[
\leq I_1 + I_2, \quad \text{say} \quad (5.1) 
\]

We have

\[
I_2 \leq 2\pi^{-1} \int_0^\infty xe^{-x} dx \int_0^\delta \left| \psi(t) \right| e^x \cos \delta dt 
\]

\[
\leq 2\pi^{-1} \int_0^\infty xe^{-x} \cdot e^x \cos \delta dx \int_0^\delta \left| \psi(t) \right| dt, \quad (\delta < \zeta < \pi) 
\]

\[
\leq 2\pi^{-1} \left\{ \left[ \frac{-xe^{-2x\sin^2 \frac{1}{2}\delta}}{2 \sin^2 \frac{1}{2}\delta} \right]_0^\infty + \int_0^\infty \frac{e^{-2x\sin^2 \frac{1}{2}\delta}}{2 \sin^2 \frac{1}{2}\delta} dx \right\} \int_0^\delta \left| \psi(t) \right| dt 
\]

\[
\leq \frac{1}{2} \pi^{-1} \cosec^2 \frac{1}{2}\delta \cdot \int_0^\infty \left| \psi(t) \right| dt < \infty. \quad (5.2) 
\]

Now

\[
\int_0^\delta \psi(t) e^{\cos t} \cdot \sin (t+x \sin t) dt 
\]

\[
= - \left\{ \psi(t) \int_t^\delta e^{\cos u} \cdot \sin (u+x \sin u) du \right\}_0^\delta + \int_0^\delta d\psi(t) \int_t^\delta e^{\cos u} \cdot \sin (u+x \sin u) du 
\]

\[
= \int_0^\delta d\psi(t) g_1(x, t), \quad \text{say} \quad (5.3) 
\]

by condition (i) of the theorem. Therefore

\[
I_1 = 2\pi^{-1} \left( \int_0^\infty xe^{-x} \left| \int_0^\delta d\psi(t) g_1(x, t) \right| dx \right) 
\]

\[
\leq 2\pi^{-1} \left( \int_0^\delta \left| d\psi(t) \right| \int_0^\infty xe^{-x} \left| g_1(x, t) \right| dx \right) 
\]

\[
\leq 2\pi^{-1} \int_0^\delta \left| d\psi(t) \right| \cdot J, \quad \text{say} \quad (5.4) 
\]
Now
\[ J = \int_{0}^{1} xe^{-x} g_1(x, t) \, dx + \int_{1}^{t^{-1}} xe^{-x} g_1(x, t) \, dx + \int_{t^{-1}}^{\infty} xe^{-x} g_1(x, t) \, dx \]
\[ = J_1 + J_2 + J_3, \text{ say} \] (5.5)

By (4.1) we have
\[ J_1 = \int_{0}^{1} xe^{-x} \cdot O(e^x) \, dx \]
\[ = O(1) \] (5.6)

By (4.3) we have
\[ J_2 = \int_{1}^{t^{-1}} xe^{-x} \cdot O(x^{-1}e^x) \, dx \]
\[ = O(t^{-1}) \] (5.7)

By (4.2) we have
\[ J_3 = \int_{t^{-1}}^{\infty} xe^{-x} \cdot O(x^{-1}e^x \cos t) \, dx \]
\[ = O \left( \int_{t^{-1}}^{\infty} e^{-2x\sin^2 \frac{t}{2}} \, dx \right) \]
\[ = O(t^{-2}). \] (5.8)

There is therefore an \( A \) with
\[ I \leq I_1 + I_2 \leq A + A \int_{0}^{d} t^{-2} \, |d\psi(t)| < \infty. \]

Thus the theorem is proved

6. Proof of Theorem 2

\[ \sum_{n=1}^{\infty} nA_n(\theta) \] is summable \( B' \) if
\[ I' = \int_{0}^{\infty} e^{-x} \left| \sum_{n=1}^{\infty} \frac{nA_n(\theta)}{n!} x^n \right| \, dx < \infty. \]

Now
\[ I' = 2\pi^{-1} \int_{0}^{\infty} e^{-x} \left| \sum_{n=1}^{\infty} \frac{\phi(t) \cos nt \, dt}{(n-1)!} x^n \right| \, dx \]
\[ \leq 2\pi^{-1} \int_{0}^{\infty} xe^{-x} \left| \int_{0}^{d} \phi(t)e^{x \cos t} \cdot \cos (t + x \sin t) \, dt \right| \, dx \]
\[ + 2\pi^{-1} \int_{0}^{\infty} xe^{-x} \left| \int_{0}^{d} \phi(t)e^{x \cos t} \cdot \cos (t + x \sin t) \, dt \right| \, dx \]
\[ \leq I_1 + I_2; \] (6.1)
say, where $0 < \delta < 1$ and

$$I_2' \leq 2\pi^{-1} \int_0^\infty xe^{-x} \cdot e^x \cos \delta dx \int_0^\delta |\phi(t)| \, dt; \ (\delta < \zeta < \pi) < \infty \quad (6.2)$$

Now

$$\int_0^\delta \phi(t)e^t \cdot \cos (t + x \sin t) \, dt = O(1) + \int_0^\delta d\phi(t)g(x, t)$$

by condition (i) of the theorem. Therefore

$$I_1' \leq 2\pi^{-1} \int_0^\delta |d\phi(t)| \cdot \int_0^\infty xe^{-x} \cdot |g_2(x, t)| \, dx + A$$

$$\leq 2\pi^{-1} \int_0^\delta |d\phi(t)| \cdot J' + A, \ \text{say} \quad (6.3)$$

Proceeding as in the proof of Theorem 1 and using (4.4), (4.5) and (4.6) we have

$$J' = O(t^{-2}) \quad (6.4)$$

Therefore

$$I' \leq A + A \int_0^\delta t^{-2} |d\phi(t)| < \infty.$$

Thus the theorem is proved.

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REFERENCES

