Examples of linear multi-box splines

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Abstract
Let $S_1 = S_1(v_0, \ldots, v_{r+1})$ be the space of compactly supported $C^0$ piecewise linear functions on a mesh $M$ of lines through $\mathbb{Z}^2$ in directions $v_0, \ldots, v_{r+1}$, possibly satisfying some restrictions on the jumps of the first order derivative. A sequence $\phi = (\phi_1, \ldots, \phi_r)$ of elements of $S_1$ is called a multi-box spline if every element of $S_1$ is a finite linear combination of shifts of (the components of) $\phi$. We give some examples for multi-box splines and show that they are stable. It is further shown that any multi-box spline is not always symmetric

1. Introduction
Multi-box splines were introduced by Goodman [2, 3, 5]. They are $C^0$ piecewise polynomials of degree $n = 1$, which, unlike box splines, allow both stability and reproduction of arbitrary polynomials of degree $n = 1$.

We shall first define the multi-box splines. Let $r \geq 1$ and $v_0, \ldots, v_{r+1}$ be pairwise linearly independent vectors in $\mathbb{Z}^2$, where without loss of generality we suppose that for $j = 0, \ldots, r+1$, the components of $v_j$ are coprime. We shall denote by $S_1 = S_1(v_0, \ldots, v_{r+1})$ the space of all functions $f : \mathbb{R}^2 \to \mathbb{R}$ with continuous Fourier transforms of form

$$\hat{f}(u) = \sum_{|\alpha| = r-1} P_\alpha(e^{-iu})u^\alpha, \quad u \in \mathbb{R}^2,$$

where for any multi-index $\alpha \in \mathbb{N}^2$ of order $|\alpha| = r - 1$, $P_\alpha$ is a Laurent polynomial with real coefficients [4, 5]. Here and elsewhere, for $u, v \in \mathbb{R}^2$, $uv$ denotes their scalar product.

Before giving some properties, we discuss the possible symmetry of multi-box splines $\phi$ for all $r \geq 1$. We say that $\phi = (\phi_1, \ldots, \phi_r)$ is symmetric if for $j = 1, \ldots, r$, there are $\sigma_j = \pm 1$, that is, $\phi_j$ is even or odd about $\frac{1}{2}\alpha_j$, $\alpha_j \in \{0, 1\}^2$, with

$$\phi_j(-\cdot) = \sigma_j \phi_j(\cdot + \alpha_j).$$

Theorem 1 [2]. In (1.1), let

$$P_\alpha(z) = \sum_{j \in \mathbb{Z}^2} c_{j,\alpha} z^j.$$

Let $V$ denote the set of all non-zero coefficients $c_{j,\alpha}$, $|\alpha| = r - 1$. Then if $f$ is a spline function with compact support and is given by (1.1), it has a support in the convex hull of $V$. Conversely if $f$ in $S_1$ has its support in a convex closed region $R$ and $W$ denotes $R$ intersection with $\mathbb{Z}^2$ (that is all integer points in $R$), then $f$ has the form (1.1) with the set $V$ of non-zero coefficients lying in $W$.

Theorem 2 [5]. A function $f$ lies in $S_1$ if and only if it is a $C^0$ spline function of degree 1 over $M(v_0, \ldots, v_{r+1})$ with compact support such that the jump of any first order derivative across any line in $M = M(v_0, \ldots, v_{r+1})$ can change only at points of $\mathbb{Z}^2$. 

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Figure 1. The symmetric hexagonal mesh of the multi-box spline.

For many choices of $v_0, \ldots, v_{r+1}$, Theorem 2 is automatically satisfied and so $S_1$ comprises all $C^0$ spline functions of 1st degree $n = 1$, over $M$ with compact support.

**Theorem 3** [5]. If at most two lines in $M(v_0, \ldots, v_{r+1})$ intersect at points not in $\mathbb{Z}^2$, then $S_n(v_0, \ldots, v_{r+1})$ comprises all $C^0$ spline functions of degree $n = 1$ over $M(v_0, \ldots, v_{r+1})$ with compact support.

Now we want to show when, and for which conditions, $\psi \in S_1$ can be a local generator for $S_1$, where $S_1$ denotes all row vectors comprising $r$ elements of $S_1$. So we shall introduce the following theorem.

**Theorem 4** [5]. The space $S_1 = S_1(v_0, \ldots, v_{r+1})$ has a local generator $\phi = (\phi_1, \ldots, \phi_r)$. Moreover $\psi = (\psi_1, \ldots, \psi_r) \in S_1$ is local generator for $S_1$ if and only if

$$\hat{\psi}(u) = \frac{\hat{u}M(e^{-iu})}{(iuv_0) \cdots (iuv_{r+1})}, \quad u \in \mathbb{R}^2,$$

where $\hat{u} := (u_1^{-1}, u_1^{-2}u_2, \ldots, u_2^{-1})$ and $M$ is an $r \times r$ matrix of Laurent polynomials with:

$$\det M(z) = cz^k \prod_{j=0}^{r+1} (1 - z^{v_j}), \quad z = (z, \omega) \in (\mathbb{C} \setminus \{0\})^2,$$

for some $k = (k_1, k_2) \in \mathbb{Z}^2$, $c \in \mathbb{R}$, $c \neq 0$, where $z^k = z^{k_1}z^{k_2}$.

For the case $r = 1$, Theorem 4 states that the function $\psi$ is a local generator for $S_1$ and therefore it is a shift of a multiple of the box spline $B_1$ (see [5]). For this reason any local generator $\phi$ for $S_1(v_0, \ldots, v_{r+1})$ is called a multi-box spline for the case $r = 2$ and extended to $r \geq 2$ in [5].

For $r \geq 2$ the generator is not unique, because there are many choices of $M$ in (1.4).

The following theorem gives us the conditions of one of the most important properties of the multi-box spline, the stability.

**Theorem 5** [4, 5]. For the space $S_1 = S_1(v_0, \ldots, v_{r+1})$ the following are equivalent.

(a) There is a stable local generator $\phi = (\phi_1, \ldots, \phi_r)$ of $S_1$.

(b) Every local generator $\phi = (\phi_1, \ldots, \phi_r)$ of $S_1$ is stable.

(c) At most $r$ lines in the mesh $M(v_0, \ldots, v_{r+1})$ intersect except at points in $\mathbb{Z}^2$.

(d) For each $u \in \mathbb{R}^2 \setminus 2\pi\mathbb{Z}^2$, there are at most $r$ vectors $v_j$ in $\{v_0, \ldots, v_{r+1}\}$ with $e^{iuv_j} = 1$.

In Section 2 we will construct some linear multi-box spline functions on a six-direction hexagonal mesh as illustrated in [1, p. 101] and as shown in Figure 1. We study the main properties of these newly created functions.
Figure 2. The six-direction mesh of the multi-box spline.

Section 3 illustrates one example of a linear multi-box spline in a different mesh as shown in Figure 7 and then studies its main properties.

2. Some linear multi-box splines on a hexagonal mesh

Take $r = 4$, and the pairwise linearly independent vectors $v_0 = (1, 0), v_1 = (0, 1), v_2 = (1, 1), v_3 = (1, -1), v_4 = (2, 1), v_5 = (1, 2)$.

In this section, we consider two meshes (the six-direction mesh with two coordinate directions as shown in Figure 2, and a symmetric six-direction mesh as seen in Figure 1) obtained from each other by the linear transformation of the plane as in equation (2.3).

Based on Theorem 1, every function $f$ of the space $S_1 = S_1(v_0, \ldots, v_5)$ in the first mesh is to be determined by its continuous Fourier transform $\hat{f}$ which is defined by equation (1.1). That is

$$\hat{f}(u, v) = \frac{u^3 P(z, \omega) + u^2 v Q(z, \omega) + uv^2 R(z, \omega) + v^3 S(z, \omega)}{uv(u + v)(u - v)(u + 2v)(2u + v)}.$$  (2.1)

We now define four generators $f_1, \ldots, f_4$ on the six-direction mesh and their counterparts $g_1, \ldots, g_4$ on the hexagonal mesh. Their supports are shown in Figures 3 and 4, respectively.

For the first function and because

$$[v(u + v)(u - v), uv(u + v), u(u + 2v)(u + v), uv(2u + v)] = [u^3, u^2v, uv^2, v^3] \ast M,$$

where $M$ is the $4 \times 4$ matrix and the determinant of $M = 1 \neq 0$, we can replace the numerator of $\hat{f}$ to give $\hat{f}_1$ as follows:

$$\hat{f}_1(u, v) = \frac{v(u + v)(u - v)P(u, v) + uv(u + v)Q(u, v)}{uv(u + v)(u - v)(u + 2v)(2u + v)}$$

$$+ \frac{u(u + v)(u + 2v)R(u, v) + uv(2u + v)S(u, v)}{uv(u + v)(u - v)(u + 2v)(2u + v)},$$  (2.2)

where $z = e^{-iu}, \omega = e^{-iv}$. Here $P, Q, R$ and $S$ are Laurent polynomials with real coefficients, defined by

$$P(z, \omega) = a + a_1 z + a_2 z \omega + a_3 \omega + a_4 z^{-1} + a_5 z^{-1} \omega^{-1} + a_6 \omega^{-1},$$

$$Q(z, \omega) = b + b_1 z + b_2 z \omega + b_3 \omega + b_4 z^{-1} + b_5 z^{-1} \omega^{-1} + b_6 \omega^{-1},$$
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Figure 3. Supports of the functions $f_i$, $i = 1, \ldots, 4$.

Figure 4. Supports of the functions $g_i$, $i = 1, \ldots, 4$.

$$R(z, \omega) = c + c_1 z + c_2 z \omega + c_3 \omega + c_4 z^{-1} + c_5 z^{-1} \omega^{-1} + c_6 \omega^{-1},$$

$$S(z, \omega) = d + d_1 z + d_2 z \omega + d_3 \omega + d_4 z^{-1} + d_5 z^{-1} \omega^{-1} + d_6 \omega^{-1}.$$ 

The coefficients $a, a_1, \ldots, a_6, b, b_1, \ldots, b_6, c, c_1, \ldots, c_6, d, d_1, \ldots, d_6$ will be determined later. In order to have the function $g_1$ in the symmetric hexagonal mesh as in Figure 1, we use the following transformation

$$g_1(x, y) = f_1(x + y / \sqrt{3}, 2y / \sqrt{3}),$$

and by taking the Fourier transform, we find that

$$\hat{g}_1(u, v) = \hat{f}_1(u, \sqrt{3}v / 2 - u / 2). \quad (2.3)$$
By substituting the new values of \( u, v \) from (2.3) for \( \hat{f}_1(u, v) \) in (2.2), we get

\[
\hat{g}_1(u, v) = \frac{(\sqrt{3}v - u)(\sqrt{3}v + u)(\sqrt{3}u - v)P(u, v)}{uv(\sqrt{3}v - u)(\sqrt{3}v + u)(\sqrt{3}u - v)(\sqrt{3}u + v)} + \frac{u(\sqrt{3}v - u)(\sqrt{3}v + u)Q(u, v)}{uv(\sqrt{3}v - u)(\sqrt{3}v + u)(\sqrt{3}u - v)(\sqrt{3}u + v)} + \frac{uv(\sqrt{3}v + u)R(u, v)}{uv(\sqrt{3}v - u)(\sqrt{3}v + u)(\sqrt{3}u - v)(\sqrt{3}u + v)} + \frac{u(\sqrt{3}v - u)(\sqrt{3}u + v)S(u, v)}{uv(\sqrt{3}v - u)(\sqrt{3}v + u)(\sqrt{3}u - v)(\sqrt{3}u + v)},
\]

(2.4)

where \( P \) takes the following forms in the hexagonal mesh

\[
P(u, v) = a + \sum_{j=1}^{6} a_j e^{-iu \cdot p_j},
\]

with similar formulas for \( Q(u, v), R(u, v) \) and \( S(u, v) \).

Here and throughout this section, we write \( u = (u, v) \) and \( p_j \) as defined in [1] as follows:

\[
p_j = \begin{pmatrix} \cos(j - 1)\pi/3 \\ \sin(j - 1)\pi/3 \end{pmatrix},
\]

for \( j = 1, \ldots, 6 \).

We follow the same steps to construct the second function, to give

\[
\hat{f}_2(u, v) = \frac{uv(u + v)P(z, w) + v(u + v)(u + 2v)Q(z, w) + u(u - v)(2u + v)R(z, w)}{uv(u + v)(u - v)(2u + v)} + \frac{(u - v)(u + 2v)(2u + v)S(z, w)}{uv(u + v)(u - v)(2u + v)}. \tag{2.5}
\]

By setting

\[
g_2(x, y) = f_2(x + y/\sqrt{3}, 2y/\sqrt{3}),
\]

we get

\[
\hat{g}_2(u, v) = \frac{u(\sqrt{3}v - u)(\sqrt{3}v + u)P(u, v)}{uv(\sqrt{3}v - u)(\sqrt{3}v + u)(\sqrt{3}u - v)(\sqrt{3}u + v)} + \frac{v(\sqrt{3}v - u)(\sqrt{3}v + u)Q(u, v)}{uv(\sqrt{3}v - u)(\sqrt{3}v + u)(\sqrt{3}u - v)(\sqrt{3}u + v)} + \frac{u(\sqrt{3}u + v)(\sqrt{3}v - u)R(u, v)}{uv(\sqrt{3}v - u)(\sqrt{3}v + u)(\sqrt{3}u - v)(\sqrt{3}u + v)} + \frac{v(\sqrt{3}u + v)(\sqrt{3}u - v)S(u, v)}{uv(\sqrt{3}v - u)(\sqrt{3}v + u)(\sqrt{3}u - v)(\sqrt{3}u + v)}. \tag{2.6}
\]

2.1. First function

The function \( g_1 \) is symmetric as in

\[
g_1(x, y) = g_1(x/2 - \sqrt{3}y/2, \sqrt{3}x/2 + y/2),
\]

and by using the Fourier transform

\[
\hat{g}_1(u, v) = \hat{g}_1(u/2 - \sqrt{3}v/2, \sqrt{3}u/2 + v/2).
\]
Now by using the above transformation and by replacing the values \( u, v \) in (2.4), we get
\[
\hat{g}_1(u, v) = \frac{uv(u + \sqrt{3}v)\tilde{P}(u, v)}{uv(\sqrt{3}v - u)(\sqrt{3}v + u)(\sqrt{3}u + v)} \\
+ \frac{u(u - \sqrt{3}v)(u + \sqrt{3}v)\tilde{Q}(u, v)}{uv(\sqrt{3}v - u)(\sqrt{3}v + u)(\sqrt{3}u + v)} \\
+ \frac{u(u - \sqrt{3}v)(\sqrt{3}u + v)\tilde{R}(u, v)}{uv(\sqrt{3}v - u)(\sqrt{3}v + u)(\sqrt{3}u + v)} \\
+ \frac{(u - \sqrt{3}v)(u + \sqrt{3}v)(\sqrt{3}u - v)\tilde{S}(u, v)}{uv(\sqrt{3}v - u)(\sqrt{3}v + u)(\sqrt{3}u + v)},
\]
(2.7)
where
\[
\tilde{P}(u, v) = a + \sum_{j=1}^{5} a_j e^{-\frac{j-1}{2} \pi i} P_j^{-1} = a + \sum_{j=0}^{5} a_{j+1} e^{-\frac{j}{2} \pi i} P_j,
\]
and similarly \( \tilde{Q}(u, v), \tilde{R}(u, v) \) and \( \tilde{S}(u, v) \).

By comparing (2.4) and (2.7) we get
\[
\begin{cases}
P = -\tilde{S}, & Q = -\tilde{Q}, \\
R = -\tilde{P}, & S = -\tilde{R}.
\end{cases}
\]
We solve in turn each of the above equations.

We shall now apply the continuity conditions for \( \hat{g}_1(u, v) \) in (2.4), which means if the denominator of the fraction is zero, then the numerator must be zero. By applying this observation, we get the following.

- **Condition 1**: \( a_3 = -a_2 \).
  
  This is obtained using the fact that \( P(0, v) = 0 \) (for all \( v \neq 0 \)).

- **Condition 2**: \( 3a_1 + 2b_1 - 6a_2 = 0 \).
  
  This is obtained using the fact that \( 3P(u, 0) + 2Q(u, 0) + 6S(u, 0) = 0 \).

Now \( a_1 \) and \( a_2 \) can be chosen arbitrarily. If we take \( a_1 = -a_2 = 1 \) then this leads to a particularly simple formula for \( \hat{g}_1 \), and \( b_1 = -9/2 \).

Considering all the previous results we find that \( P = R = S \) and
\[
Q = -(9/2)P.
\]
Finally by substituting the values of \( P, R, S \) and \( Q \) in (2.4), we find the first multi-box spline in the hexagonal support as
\[
\hat{g}_1(u, v) = \frac{2P_1(u, v)}{u(\sqrt{3}v - u)(\sqrt{3}v + u)},
\]
(2.8)
where \( P_1 \) takes the following form:
\[
P_1(u, v) = e^{-iu} - e^{-i(u/2+\sqrt{3}v/2)} + e^{-i(-u/2+\sqrt{3}v/2)} - e^{iu} + e^{-i(-u/2-\sqrt{3}v/2)} - e^{-i(u/2-\sqrt{3}v/2)}.
\]

By changing variables \( z = e^{-iu}, \ w = e^{-iv} \) and a simple calculation, we obtain
\[
\hat{f}_1(u, v) = \frac{z^{-1}\omega^{-1}(1 - z)(1 - \omega)(1 - z\omega)}{uv(u + v)}.
\]
(2.9)

### 2.2. Second function

The function \( g_2 \) is symmetric as in
\[
g_2(x, y) = g_2(x, -y), \quad g_2(x, y) = g_2(1 - x, y),
\]
and by using the Fourier transform
\[ \hat{g}_2(u, v) = \hat{g}_2(u, -v), \quad \hat{g}_2(u, v) = e^{-iu} \hat{g}_2(-u, v). \]

Similar to in the previous section, we find \( g_2(u, v) \) as follows:
\[ \hat{g}_2(u, v) = u R(u, v) + \sqrt{3} v S(u, v) \]
where
\[ R(u, v) = e^{-i(u/2 + \sqrt{3}v/2)} - e^{-i(u/2 - \sqrt{3}v/2)}, \quad S(u, v) = 1 - e^{-iu}. \]
Then
\[ \hat{f}_2(u, v) = \frac{u(z\omega - \omega^{-1}) + (u + 2v)(1 - z)}{uv(u + v)(u + 2v)}. \]

2.3. Third function
The third function, \( f_3 \), is defined by its Fourier transform \( \hat{f}_3 \), as follows
\[ f_3(x, y) = f_2(y, y - x). \]
Now we calculate the function \( f_3 \), and by normalization, we get
\[ \hat{f}_3(u, v) = \frac{(u + v)(\omega - z) - (u - v)(1 - z\omega)}{uv(u + v)(u - v)}. \]

2.4. Fourth function
The function \( f_4 \) is defined by
\[ f_4(x, y) = f_2(y - x, -x). \]
Therefore, we find the Fourier transform of \( f_4 \) by normalization as follows:
\[ \hat{f}_4(u, v) = \frac{-v(z^{-1} - \omega z) + (2u + v)(1 - \omega)}{uv(u + v)(2u + v)}. \]

2.5. Summary
We may write \( \hat{f} = (\hat{f}_1, \hat{f}_2, \hat{f}_3, \hat{f}_4) \) in the following matrix form:
\[ \begin{pmatrix} \hat{f}_1(u, v) \\ \hat{f}_2(u, v) \\ \hat{f}_3(u, v) \\ \hat{f}_4(u, v) \end{pmatrix} = M(z, \omega) \times \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix} \times \frac{1}{uv(u + v)(u - v)(2u + v)}, \]
where
\[ \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix} = \begin{pmatrix} (u - v)(u + 2v) \\ u(u - v)(2u + v) \\ (u + v)(u + 2v)(2u + v) \\ v(u - v)(u + 2v) \end{pmatrix}, \]
and
\[ M(z, \omega) = \begin{pmatrix} (z - 1)(\omega - 1)(z\omega - 1) & 0 & 0 & 0 \\ 1 - z & z\omega - \omega^{-1} & 0 & 0 \\ z\omega - 1 & 0 & \omega - z & 0 \\ 1 - \omega & 0 & 0 & z\omega - z^{-1} \end{pmatrix}. \]
Now we calculate the determinant of $M$ which takes the following form:

$$\det M(z, \omega) = z^{-2}\omega^{-1}(1-z)(1-\omega)(1-z\omega)(1-z\omega^2)(1-z\omega^{-1})(1-z\omega^2).$$

By Theorem 4, the function $f$ is a multi-box spline if the determinant of $M(z, w)$ is in the following form:

$$\det M(z) = cz^k r + 1 \prod_{j=0}^{r+1} (1 - z v_j), \quad z \in \mathbb{C}\backslash\{0\}^2,$$

for some $k \in \mathbb{Z}^2$, $c \in \mathbb{R}$, $c \neq 0$. By comparing our determinant with the above form we find that $c = 1$, $k = (-2, -1)$. So $f$ is a multi-box spline.

Note that three lines in the mesh $M = M(v_0, \ldots, v_5)$ intersect at a point in $\mathbb{R}^2\backslash\mathbb{Z}^2$. Now Theorem 3 says that if at most $n+1 = 2$ lines in the mesh $M = M(v_0, \ldots, v_5)$ intersect other than in $\mathbb{Z}^2$, then the space $S_1 = S_1(v_0, \ldots, v_5)$ comprises all of the continuous linear splines over $M = M(v_0, \ldots, v_5)$ with compact support. Since this condition is not satisfied, $S_1$ comprises those continuous linear splines which satisfy the condition that the jump of $Df$ across any line in $M = M(v_0, \ldots, v_5)$ is constant, at any point where three lines in $M$ intersect. See Theorem 2 for the special case of $n = 1$.

In order to derive this jump condition, it will be more convenient to work with the hexagonal mesh. A typical triangle is illustrated in Figure 5, where the jump condition is at the centroid $A$. The values of the function at the given points $A, B, \ldots, G$ are denoted by $a, b, \ldots, g$.

The value of the jump at the point $A$ is given by this equation:

$$3a = 2g + 2e + 2c - (b + f + d).$$

Now we have to explain where the above condition comes from. Here the jump condition is on the first order derivatives.

We take the triangle $\triangle AGF$ with vertices $A(0, \sqrt{3}/3)$, $F(-1/2, \sqrt{3}/2)$, $G(-1/4, \sqrt{3}/4)$, with the corresponding values of our function at these being $a, f, g$.

Let the direction vector $V = (1/2, \sqrt{3}/2)$, which is parallel to the segment $[BD]$, be defined in terms of the two vectors $GA$ and $GF$ as follows:

$$v_1 = \alpha GA + \beta GF,$$

where

$$GA = (1/4, \sqrt{3}/12), \quad GF = (-1/4, \sqrt{3}/4).$$
By resolving the above equation, we find that
\[ \alpha = 3, \quad \beta = 1, \]
and then
\[ v_1 = 3GA + GF. \]
This is equal to
\[ v_1 = 3(A - G) + (F - G). \]
So
\[ D_v f = 3(a - g) + (f - g) = 3a - 4g + f. \] (2.14)
Similarly on the triangle \( \triangle AFE \) with vertices \( A(0, \sqrt{3}/3), \quad F(-1/2, \sqrt{3}/2), \quad E(0, \sqrt{3}/2) \), and the corresponding values of our function at these being \( a, f, e \), we find that
\[ D_v f = 3(e - a) + (e - f) = -3a + 4e - f. \] (2.15)
So the jump is
\[ D_v f - D_v f = 3a - 4g + f - (3a - 4g + f) = 6a - 4g + 2f - 4e. \] (2.16)
Similarly on the triangles \( \triangle ABC \) and \( \triangle ACD \), we find the jump is
\[ 2(c - b) - 2(d - c) = 4c - 2b - 2d. \] (2.17)
By comparing (2.16) and (2.17), we get
\[ 6a - 4g + 2f - 4e = 4c - 2b - 2d. \]
This implies that
\[ 3a = 2(g + e + c) - (b + f + d). \]
By the symmetry of this condition, we will get the same result for the jump condition across the other two mesh lines.

Theorem 5 says that if at most \( r = 4 \) lines in the mesh \( M = M(v_0, \ldots, v_5) \) intersect except at points in \( \mathbb{Z}^2 \), then \( f \) is a stable local generator. Since only three lines in the mesh \( M \) intersect other than in \( \mathbb{Z}^2 \), then \( f \) is stable.

The equation (1.2) which defines the symmetry of any function \( f = (f_1, f_2, f_3, f_4) \) is satisfied and \( f \) is symmetric about \( (0,0), (1,0), (1,1) \) and \( (0,1) \) respectively, therefore the multi-box spline \( f \) is symmetric too.

We can see from (2.9), that the piecewise linear function \( f_1 \) is a particular box-spline, that coincides with a hat function known in the finite element method.

The first function \( f_1 \) can be replaced by another function \( \psi_1 \) which is defined in a smaller support as follows: (see Figure 6)
\[ \psi_1(x, y) = f_1(x, y) + f_2(x, y) + f_2((x, y) + (1, 0)), \quad (x, y) \in \mathbb{R}^2. \]
By taking the Fourier transform of the above equation, we get the following:
\[ \hat{\psi}_1(u, v) = \frac{z^{-1} \omega^{-1} + \omega - \omega^{-1} - z\omega}{uv(u + v)} + \frac{z\omega - \omega^{-1} + \omega - z^{-1} \omega^{-1}}{v(u + v)(u + 2v)}. \] (2.18)
We can easily prove that the function \( \psi = (\psi_1, f_2, f_3, f_4) \) is a multi-box spline and a local generator for \( S_1 \) according to Theorem 4.

3. **Linear multi-box splines on another mesh**

Take \( n = 1, \ r = 4 \), and the pairwise linearly independent vectors \( v_0 = (1, 0), \ v_1 = (0, 1), \ v_2 = (2, 1), \ v_3 = (1, 2), \ v_4 = (2, -1), \ v_5 = (1, -2) \), see Figure 7.
In this case, we firstly need to construct the support of the linear multi-box spline, then form the linear multi-box spline inside it, and show it is a multi-box spline.

By Theorem 4, we have to find generators with good properties like stability and symmetry.

### 3.1. The construction algorithm

We choose five different functions $\phi_1, \phi_2, \psi, \psi_1$ and $\psi_2$ plus the combination of all of them, this will give us the four functions which form our multi-box spline. Each of the functions $\phi_1, \phi_2$ and $\psi$ take the value zero on $\mathbb{Z}^2$, while $\psi_1$ and $\psi_2$ are $\neq 0$ on some points in $\mathbb{Z}^2$.

The combination of $\psi_1$ and $\psi_2$ gives us a new function $\Phi$ which is zero on $\mathbb{Z}^2$. The function $\Phi$ is defined as

$$
\Phi = \psi_1 + \psi_1(\cdot - (1, 0)) - \psi_2 - \psi_2(\cdot - (0, 1)).
$$

The function $\Phi$ is symmetric under the following conditions

$$
\Phi(1 - x, y) = \Phi(x, y), \quad \Phi(x, 1 - y) = \Phi(x, y), \quad \Phi(x, y) = -\Phi(y, x).
$$

We want to express $\Phi$ in terms of shifts of $\phi_1, \phi_2$ and $\psi$, by some unknowns $a, b$ and $c$, as it will be explained in Step 6. By comparing the two expressions of $\Phi$, we can find the coefficients $a, b$ and $c$, and then we may write $\Phi$ in terms of shifts of $\phi_1, \phi_2, \psi$. Then we will construct another function $\phi_3$ as a combination of $\psi_1$ and shifts of $\psi$, and then the function $\phi_4$ as a linear combination of $\psi_2$ and shifts of $\psi$, in such a way that $\psi$ lies in the span of the shifts of $\phi_1, \phi_2, \phi_3, \phi_4$. So $\phi_1, \phi_2, \phi_3, \phi_4$ can be a generator and a multi-box spline for the space $S_1$, which will be proved later by referring to Theorem 3.
3.2. Constructing the linear multi-box spline $\phi$

Step 1. In order to construct the first generator $\phi_1$, we choose the support of $\phi_1$ as illustrated in Figure 8(a). Let $\phi_1$ be defined by its Fourier transform as

$$\hat{\phi}_1(u, v) = \frac{(2u + v)P_1(z, \omega) + (2u - v)Q_1(z, \omega)}{uv(2u - v)(2u + v)},$$  \hspace{1cm} (3.1)

where $z = e^{-iu}$, $\omega = e^{-iv}$. Here $P_1$ and $Q_1$ are Laurent polynomials with real coefficients, defined by

$$P_1(z, \omega) = a + a_1z + a_2z\omega + a_3\omega + a_4z^{-1}\omega + a_5z^{-1},$$
$$Q_1(z, \omega) = b + b_1z + b_2z\omega + b_3\omega + b_4z^{-1}\omega + b_5z^{-1}.$$

We require that the function $\phi_1$ is symmetric as in

$$\phi_1(x, y) = \phi_1(-x, y), \quad \phi_1(x, y) = \phi_1(x, -y + 1),$$

and by using the Fourier transform, we get

$$\hat{\phi}_1(u, v) = \hat{\phi}_1(-u, v), \quad \hat{\phi}_1(u, v) = e^{-iv}\hat{\phi}_1(u, -v).$$  \hspace{1cm} (3.2)

In order to find the coefficients $a$, $a_i$, $b$, $b_i$, we study the continuity conditions as in the previous section, and by taking $a_1 = -1$, then

$$P_1(z, \omega) = -z + \omega z^{-1}, \quad Q_1(z, \omega) = z\omega - z^{-1}, \quad R_1(z, \omega) = 0, \quad S_1(z, \omega) = 0.$$

Thus

$$\hat{\phi}_1(u, v) = \frac{(2u + v)(\omega z^{-1} - z) + (2u - v)(z\omega - z^{-1})}{uv(2u - v)(2u + v)}.$$  \hspace{1cm} (3.3)

We may write $\hat{\phi}_1$ in the following matrix form:

$$(\hat{\phi}_1(u, v)) = M_1(z, \omega) \times \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix} \times \frac{1}{uv(u + v)(u - v)(u + 2v)(2u + v)},$$

where

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix} = \begin{pmatrix} (u - v)(u + 2v)(2u + v) \\ u(u - v)(2u + v) \\ (u + v)(u + 2v)(2u + v) \\ v(u - v)(u + 2v) \end{pmatrix}.$$
and
\[ M_1(z, w) = (P_1(z, \omega) \ Q_1(z, \omega \ R_1(z, \omega) \ S_1(z, \omega)), \]
and also
\[ P_1(z, \omega) = 4\omega z^{-1}(1 - z^2 \omega^{-1}), \]
\[ Q_1(z, \omega) = -4z^{-1}(1 - z^2 \omega), \]
\[ R_1(z, \omega) = 0, \]
\[ S_1(z, \omega) = 0. \]

**Step 2.** The function \( \phi_2 \) is the transformed image of \( \phi_1 \) by an angle of 90 degrees clockwise,
\[ \phi_2(x, y) = \phi_1(-y, x), \]
by taking the Fourier transform
\[ \hat{\phi}_2(u, v) = \hat{\phi}_1(-v, u). \]
Hence
\[ \hat{\phi}_2(u, v) = \frac{(u - 2v)(z\omega - \omega^{-1}) + (u + 2v)(\omega - z\omega^{-1})}{uv(u + 2v)(u - 2v)}, \tag{3.4} \]
see Figure 8(b).
We may write \( \hat{\phi}_2 \) in the following matrix form
\[ (\hat{\phi}_2(u, v)) = M_2(z, \omega) \times \left( \begin{array}{c} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{array} \right) \times \frac{1}{uv(u + v)(u - v)(u + 2v)(2u + v)}, \]
where
\[ \left( \begin{array}{c} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{array} \right) = \left( \begin{array}{c} (u - v)(u + 2v)(2u + v) \\ u(u - v)(2u + v) \\ (u + v)(u + 2v)(2u + v) \\ v(u - v)(u + 2v) \end{array} \right), \]
and
\[ M_2(z, w) = (P_2(z, \omega) \ Q_2(z, \omega \ R_2(z, \omega) \ S_2(z, \omega)), \]
and also
\[ P_2(z, \omega) = 0, \]
\[ Q_2(z, \omega) = 0, \]
\[ R_2(z, \omega) = -4\omega^{-1}(1 - z\omega^2), \]
\[ S_2(z, \omega) = -4z^{-1}(1 - z^{-1}\omega^2). \]

**Step 3.** The function \( \psi_1 \) is to be defined by its Fourier transform as shown in Figure 9(a).
We may replace \( \hat{f}(u, v) \) in (2.1) to give \( \psi_1 \) as follows:
\[ \hat{\psi}_1(u, v) = \frac{(2u + v)(u - 2v)(2u - v)P(z, \omega) + (u + 2v)(u - 2v)(2u - v)Q(z, \omega)}{uv(u + 2v)(u - 2v)(2u - v)(2u + v)}, \]
\[ + \frac{(u + 2v)(2u + v)(2u - v)R(z, \omega) + (u + 2v)(2u + v)(u - 2v)S(z, \omega)}{uv(u + 2v)(u - 2v)(2u - v)(2u + v)}, \tag{3.5} \]
where \( z = e^{-iu}, \omega = e^{-iv} \). Here \( P, Q, R \) and \( S \) are Laurent polynomials with real coefficients, defined by
\[
P(z, \omega) = a_0 + a_1 z + a_2 z \omega + a_3 z^2 + a_4 z^{-1} \omega + a_5 z^{-1} + a_6 \omega^{-1},
\]
and similarly for \( Q(z, \omega), R(z, \omega) \) and \( S(z, \omega) \).

The function \( \phi_1 \) is symmetric as in
\[
\phi_1(-x, y) = \phi_1(x, y), \quad \phi_1(x, y) = \phi(x, 1-y),
\]
by taking the Fourier transform
\[
\hat{\phi}_1(-u, v) = \hat{\phi}_1(u, v), \quad \hat{\phi}_1(u, v) = e^{iv} \hat{\psi}(u, -v).
\]
By applying the continuity conditions, and by taking \( a_3 = 1 \) and \( b_2 = 1 \), \( P, Q, R \) and \( S \) will be defined as
\[
P(z, \omega) = z \omega + \omega^2 - z^{-1} - \omega^{-1}, \quad Q(z, \omega) = z \omega - z^{-1},
\]
\[
R(z, \omega) = \omega^2 + z^{-1} \omega - \omega^{-1} - z, \quad S(z, \omega) = -z + z^{-1} \omega.
\]
So \( \phi_1 \) is defined as
\[
\hat{\psi}_1(u, v) = \frac{z \omega + \omega^2 - z^{-1} - \omega^{-1}}{uv(u + 2v)} + \frac{z \omega - z^{-1}}{uv(2u + v)}
\]
\[
+ \frac{\omega^2 + z^{-1} \omega - \omega^{-1} - z}{uv(u - 2v)} + \frac{z^{-1} \omega - z}{uv(2u - v)}.
\]

**Step 4.** Now we calculate the function \( \phi_2 \), which represents the rotated image of \( \phi_1 \) by an angle of 90 degrees clockwise, that is
\[
\phi_2(x, y) = \phi_1(y, x),
\]
by taking the Fourier transform of this equation
\[
\hat{\phi}_2(u, v) = \hat{\phi}_1(v, u).
\]
We find
\[
\hat{\phi}_2(u, v) = \frac{z \omega + z^2 - \omega^{-1} - z^{-1}}{uv(2u + v)} + \frac{z \omega - \omega^{-1}}{uv(u + 2v)}
\]
\[
+ \frac{-z^2 - z \omega^{-1} + z^{-1} + \omega}{uv(2u - v)} + \frac{\omega - z \omega^{-1}}{uv(u - 2v)},
\]
see Figure 9(b).
Step 5. We may define \( \psi \) as in (3.5), and by its Fourier transform \( \hat{\psi} \):

\[
\hat{\psi}(u, v) = \frac{P(z, \omega)}{uv(u + 2v)} + \frac{Q(z, \omega)}{uv(u - 2v)} + \frac{R(z, \omega)}{uv(2u - v)} + \frac{S(z, \omega)}{uv(2u + v)}. \tag{3.8}
\]

Here \( P, Q, R \) and \( S \) are Laurent polynomials with real coefficients, defined by

\[
P(z, \omega) = a_0 + a_1 z + a_2 z \omega + a_3 \omega + a_4 z^{-1} + a_5 z^{-1} \omega^{-1} + a_6 z^{-1} \omega^{-1} + \ldots,
\]

and similarly for \( Q, R \) and \( S \), see Figure 10.

The function \( \psi \) is symmetric as in

\[
\psi(-x, y) = -\psi(x, y), \quad \psi(x, -y) = -\psi(x, y), \quad \psi(x, y) = -\psi(y, x),
\]

by taking the Fourier transform

\[
\hat{\psi}(-u, v) = -\hat{\psi}(u, v), \quad \hat{\psi}(u, -v) = -\hat{\psi}(u, v), \quad \hat{\psi}(u, v) = -\hat{\psi}(v, u).
\]

By studying the above, and also the continuity conditions for \( \hat{\psi}(u, v) \) in (3.8), and by taking \( a_2 = 1 \), \( \hat{\psi}(u, v) \) has the following form:

\[
\hat{\psi}(u, v) = \left( \frac{z^{-1} + z^{-1} \omega - z - z \omega^{-1}}{uv(2u - v)} \right) + \left( \frac{z^{-1} + z^{-1} \omega^{-1} - z - z \omega}{uv(2u + v)} \right) + \left( \frac{z \omega + \omega - \omega^{-1} - z^{-1} \omega^{-1}}{uv(u + 2v)} \right) + \left( \frac{\omega^{-1} + \omega^{-1} - \omega - z \omega^{-1}}{uv(u - 2v)} \right). \tag{3.9}
\]

Step 6. Let

\[
\Phi(x, y) = \psi_1(x, y) + \psi_1(x - 1, y) - \psi_2(x, y) - \psi_2(x, y - 1).
\]

By taking the Fourier transform

\[
\hat{\Phi}(u, v) = \hat{\psi}_1(u, v) + e^{-iu} \hat{\psi}_1(u, v) - \hat{\psi}_2(u, v) - e^{-iv} \hat{\psi}_2(u, v),
\]

where \( z = e^{-iu} \) and \( \omega = e^{-iv} \), then

\[
\hat{\Phi}(u, v) = \hat{\psi}_1(u, v) + z \hat{\psi}_1(u, v) - \hat{\psi}_2(u, v) - \omega \hat{\psi}_2(u, v),
\]
by substituting the values of $\hat{\psi}_1$ and $\hat{\psi}_2$, we find that

$$\hat{\Phi}(u, v) = \frac{z^{-1} \omega - z^2 - z \omega^2 + \omega^{-1}}{uv(2u + v)} + \frac{\omega^2 - z^{-1} + z^2 \omega - z \omega^{-1}}{uv(u + 2v)}$$

$$+ \frac{z^{-1} \omega - \omega^{-1} + z \omega^2 - z^2}{uv(u - 2v)} + \frac{z \omega^{-1} - z^{-1} + z^2 \omega - \omega^2}{uv(2u - v)},$$

(3.10)

see Figure 11.

We now show that $\Phi$ can be written in the form

$$\Phi(x, y) = aX + bY + cZ,$$

where

$$X(x, y) = \phi_1(x, y - 1) + \phi_1(x - 1, y - 1) + \phi_1(x, y + 1) + \phi_1(x - 1, y + 1)$$
$$- \phi_2(x + 1, y) - \phi_2(x + 1, y - 1) - \phi_2(x - 1, y) - \phi_2(x - 1, y - 1),$$

$$Y(x, y) = \phi_1(x, y) + \phi_1(x - 1, y) - \phi_2(x, y) - \phi_2(x, y - 1),$$

$$Z(x, y) = \psi(x, y) - \psi(x - 1, y) - \psi(x, y - 1) + \psi(x - 1, y - 1).$$

By taking the Fourier transform

$$\hat{\Phi} = a\hat{X} + b\hat{Y} + c\hat{Z},$$

(3.11)

by substituting the values of $\hat{X}$, $\hat{Y}$ and $\hat{Z}$, we find that

$$\hat{\Phi}(u, v) = \frac{(-a - c)(\omega^2 - z^{-1} + z^2 \omega - z \omega^{-1}) + (a - c)(z^{-1} \omega^{-1} - \omega - z^2 \omega^2 + z)}{uv(u + 2v)}$$

$$+ \frac{b(\omega^{-1} - z \omega - z^2 \omega + 1)}{uv(u + 2v)}$$

$$+ \frac{(-a - c)(z \omega^{-1} - z^{-1} + z^2 \omega - \omega^2) + (a - c)(z^{-1} \omega^2 - z \omega + 1 - z^2 \omega^{-1})}{uv(2u - v)}$$

$$+ \frac{b(z^{-1} \omega - z + \omega - z^2)}{uv(2u - v)}.$$
By substituting the values of Step 7. Let \( \phi_3 \) be defined as
\[
\phi_3 = \psi_1 + \frac{3c}{2} \psi - \frac{c}{2} \psi(\cdot - (0, 1)),
\]
we had \( c = -\frac{1}{2} \), so
\[
\phi_3(x, y) = \psi_1(x, y) - \frac{3}{4} \psi(x, y) + \frac{1}{4} \psi(x, y - 1).
\]
By taking the Fourier transform we have
\[
\hat{\phi}_3(u, v) = \hat{\psi}_1(u, v) - \frac{3}{4} \hat{\psi}(u, v) + \frac{1}{4} \omega \hat{\psi}(u, v).
\]
By substituting the values of \( \hat{\psi}_1(u, v) \) and \( \hat{\psi}(u, v) \), we get
\[
\hat{\phi}_3(u, v) = \frac{z \omega + 5 \omega^2 - 5 z^{-1} - \omega^{-1} - 3 \omega + 3 z^{-1} \omega^{-1} - 1 + z \omega^2}{4uv(u + 2v)} + \frac{2z^{-1} \omega - 2z - 3z^{-1} + 3z \omega^{-1} - z \omega + z^{-1} \omega^2}{4uv(2u - v)} + \frac{6z \omega - 6z^{-1} - 3z \omega^{-1} - 1 + z \omega^2}{4uv(2u + v)} + \frac{3 \omega^2 + 7z \omega - 7 \omega^{-1} - 3z - 3z \omega^{-1} + 3 \omega + 1 - z^{-1} \omega^2}{4uv(u - 2v)}.
\]
We may write \( \hat{\phi}_3 \) in the following matrix form
\[
(\hat{\phi}_3(u, v)) = M_3(z, \omega) \times \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix} \times \frac{1}{uv(u + v)(u - v)(u + 2v)(2u + v)},
\]
where
\[
\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix} = \begin{pmatrix} (u - v)(u + 2v)(2u + v) \\ u(u - v)(2u + v) \\ (u + v)(u + 2v)(2u + v) \\ v(u - v)(u + 2v) \end{pmatrix},
\]
and
\[
M_3(z, \omega) = \begin{pmatrix} P_3(z, \omega) & Q_3(z, \omega) & R_3(z, \omega) & S_3(z, \omega) \end{pmatrix},
\]
where

\[ P_3(z, \omega) = (1 - z^2 \omega^{-1})(2z^{-1}\omega - 3z^{-1} + z^{-1}\omega^2), \]
\[ Q_3(z, \omega) = (1 - z^2 \omega)(z^{-1}\omega - 3z^{-1}\omega^{-1} - 6z^{-1}), \]
\[ R_3(z, \omega) = (1 - zw^2)(-5z^{-1} + 3z^{-1}\omega^{-1} - 1 - \omega^{-1}), \]
\[ S_3(z, \omega) = (1 - z^{-1}\omega^2)(-7\omega^{-1} + 1 - 3z\omega^{-1} - 3z). \]

Now we calculate \( \hat{\phi}_4(u, v) \) such that

\[ \phi_4(x, y) = \phi_3(y, x). \]

By taking the Fourier transform we have

\[ \hat{\phi}_4(u, v) = \hat{\phi}_3(v, u). \]

By substituting the value of \( \hat{\phi}_3 \) we get

\[
\begin{align*}
\hat{\phi}_4(u, v) &= \frac{z\omega + 5z^2 - 5\omega^{-1} - z^{-1} - 3z + 3\omega^{-1}z^{-1} - 1 + z^2\omega}{4uv(2u + v)} \\
&\quad + \frac{-2z\omega^{-1} + 2\omega + 3\omega^{-1} - 3z^{-1}\omega + z\omega - z^2\omega^{-1}}{4uv(u - 2v)} \\
&\quad + \frac{6z\omega - 6\omega^{-1} - 3z^{-1}\omega^{-1} + 3\omega + z\omega^{-1} - z^2\omega}{4uv(u + 2v)} \\
&\quad + \frac{-3z^2 - 7z\omega^{-1} + 7z^{-1} + 3\omega + 3z^{-1}\omega - 3z - 1 + z^2\omega^{-1}}{4uv(2u - v)}.
\end{align*}
\]

(3.16)

We may write \( \hat{\phi}_4 \) in the following matrix form

\[
\begin{pmatrix}
\hat{\phi}_4(u, v)
\end{pmatrix} = M_4(z, \omega) \times \begin{pmatrix}
\lambda_1 \\
\lambda_2 \\
\lambda_3 \\
\lambda_4
\end{pmatrix} \times \frac{1}{uv(u + v)(u - v)(u + 2v)(2u + v)},
\]

where

\[
\begin{pmatrix}
\lambda_1 \\
\lambda_2 \\
\lambda_3 \\
\lambda_4
\end{pmatrix} = \begin{pmatrix}
(u - v)(u + 2v)(2u + v) \\
u(u - v)(2u + v) \\
(u + v)(u + 2v)(2u + v) \\
v(u - v)(u + 2v)
\end{pmatrix},
\]

and

\[ M_4(z, \omega) = \begin{pmatrix}
P_4(z, \omega) & Q_4(z, \omega) & R_4(z, \omega) & S_4(z, \omega)
\end{pmatrix}, \]

where

\[ P_4(z, \omega) = (1 - z^2\omega^{-1})(7z^{-1} - 1 + 3\omega + 3z^{-1}\omega), \]
\[ Q_4(z, \omega) = (1 - z^2\omega)(-5\omega^{-1} + 3z^{-1}\omega^{-1} - 1 - z^{-1}), \]
\[ R_4(z, \omega) = (1 - zw^2)(-6\omega^{-1} - 3z^{-1}\omega^{-1} + z\omega^{-1}), \]
\[ S_4(z, \omega) = (1 - z^{-1}\omega^2)(-z^2\omega^{-1} - 2z\omega^{-1} + 3\omega^{-1}). \]

We can confirm that

\[
\begin{align*}
\phi_3 + \phi_3(\cdot - (1, 0)) - \phi_4 - \phi_4(\cdot - (0, 1)) \\
= \Phi + 3c\psi - c\psi(\cdot - (1, 1)) + c\psi(\cdot - (1, 0)) + c\psi(\cdot - (0, 1)) \\
= aX + bY + 4c\psi.
\end{align*}
\]
So $\psi$ is in the span of shifts of $\phi_1, \phi_2, \phi_3$ and $\phi_4$. This suggests that $\phi = (\phi_1, \phi_2, \phi_3, \phi_4)$ is a multi-box spline.

### 3.3. Properties

Now we show that $\phi = (\phi_1, \phi_2, \phi_3, \phi_4)$ is a local generator, and study its properties.

We may write $\phi$ in the following form

$$
\begin{pmatrix}
\hat{\phi}_1(u, v) \\
\hat{\phi}_2(u, v) \\
\hat{\phi}_3(u, v) \\
\hat{\phi}_4(u, v)
\end{pmatrix} = M(z, \omega) \times \begin{pmatrix}
\lambda_1 \\
\lambda_2 \\
\lambda_3 \\
\lambda_4
\end{pmatrix} \times \frac{1}{uv(u + v)(u - v)(u + 2v)(2u + v)},
$$

where

$$
\begin{pmatrix}
\lambda_1 \\
\lambda_2 \\
\lambda_3 \\
\lambda_4
\end{pmatrix} = \begin{pmatrix}
(u - v)(u + 2v)(2u + v) \\
(u - v)(2u + v) \\
(u + v)(u + 2v)(2u + v) \\
v(u - v)(u + 2v)
\end{pmatrix},
$$

and

$$
M(z, \omega) = \begin{pmatrix}
P_1(z, \omega) & Q_1(z, \omega) & R_1(z, \omega) & S_1(z, \omega) \\
P_2(z, \omega) & Q_2(z, \omega) & R_2(z, \omega) & S_2(z, \omega) \\
P_3(z, \omega) & Q_3(z, \omega) & R_3(z, \omega) & S_3(z, \omega) \\
P_4(z, \omega) & Q_4(z, \omega) & R_4(z, \omega) & S_4(z, \omega)
\end{pmatrix},
$$

where $\hat{\phi}_1$, $\hat{\phi}_2$, $\hat{\phi}_3$ and $\hat{\phi}_4$ are defined as follows.

For $\phi_1$,

$$
\begin{aligned}
P_1(z, \omega) &= 4\omega z^{-1}(1 - z^2 \omega^{-1}), \\
Q_1(z, \omega) &= -4z^{-1}(1 - z^2 \omega), \\
R_1(z, \omega) &= 0, \\
S_1(z, \omega) &= 0.
\end{aligned}
$$

For $\phi_2$,

$$
\begin{aligned}
P_2(z, \omega) &= 0, \\
Q_2(z, \omega) &= 0, \\
R_2(z, \omega) &= -4\omega^{-1}(1 - z \omega^2), \\
S_2(z, \omega) &= -4z\omega^{-1}(1 - z^{-1} \omega^2).
\end{aligned}
$$

For $\phi_3$,

$$
\begin{aligned}
P_3(z, \omega) &= (1 - z^2 \omega^{-1})(2z^{-1} \omega - 3z^{-1} + z^{-1} \omega^2), \\
Q_3(z, \omega) &= (1 - z^2 \omega)(z^{-1} \omega - 3z^{-1} \omega^{-1} - 6z^{-1}), \\
R_3(z, \omega) &= (1 - z \omega^2)(-7z^{-1} + 3z^{-1} \omega^{-1} - 1 - \omega^{-1}), \\
S_3(z, \omega) &= (1 - z^{-1} \omega^2)(-7\omega^{-1} + 1 - 3z \omega^{-1} - 3z).
\end{aligned}
$$

Finally for $\phi_4$,

$$
\begin{aligned}
P_4(z, \omega) &= (1 - z^2 \omega^{-1})(7z^{-1} - 1 + 3\omega + 3z^{-1} \omega), \\
Q_4(z, \omega) &= (1 - z^2 \omega)(-5\omega^{-1} + 3z^{-1} \omega^{-1} - 1 - z^{-1}), \\
R_4(z, \omega) &= (1 - z \omega^2)(-6\omega^{-1} - 3z^{-1} \omega^{-1} + z \omega^{-1}), \\
S_4(z, \omega) &= (1 - z^{-1} \omega^2)(-z^2 \omega^{-1} - 2z \omega^{-1} + 3\omega^{-1}).
\end{aligned}
$$
After constructing $M(z, \omega)$, we find the determinant of $M(z, \omega)$ using Maple, as follows:

$$
\det M(z, \omega) = -2^{10} z^{-2} \omega^{-2} (1 - z)(1 - \omega) \\
\times (1 - z^2 \omega^{-1})(1 - z^2 \omega)(1 - z \omega^2)(1 - z^{-1} \omega^2).
$$

By Theorem 4, the function $\phi$ is a multi-box spline if the determinant of $M(z, \omega)$ is in the following form

$$
\det M(\tilde{z}) = cz^k \prod_{j=0}^{n+r} (1 - \tilde{z}^v_j), \quad \tilde{z} \in (\mathbb{C} \setminus \{0\})^2,
$$

for some $k \in \mathbb{Z}^2$, $c \in \mathbb{R}$, $c \neq 0$. By comparing this determinant with the above, we find that $k = (-2, -2)$, $c = -2^{10}$. So $\phi$ is a multi-box spline.

Note that only three lines in the mesh $M = M(v_0, \ldots, v_5)$ intersect other than in $\mathbb{Z}^2$. So the conditions of Theorem 3 are not satisfied.

Since only three lines in $M$ intersect other than in $\mathbb{Z}^2$, then $\phi$ is stable as in Theorem 5.

The equation (1.2), which defines the symmetry of any function $f_j$, shows that $\phi_1$ and $\phi_2$ are symmetric, however $\phi_3$ and $\phi_4$ are not symmetric, therefore the multi-box spline $\phi = (\phi_1, \phi_2, \phi_3, \phi_4)$ is not symmetric.

**Remark 1.** We can see from this case that the multi-box spline is not necessarily symmetric. Our experience suggests that it is not possible to construct a symmetric multi-box spline for this space which, if true, would disprove the conjecture made by Goodman in [5] that any space $S_n = S_n(v_0, \ldots, v_{n+r})$ has a symmetric local generator.

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**References**


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