REAL HYPERSURFACES OF A COMPLEX PROJECTIVE SPACE

M. KIMURA

We study real hypersurfaces $M$ of a complex projective space and show that a condition on the derivative of the Ricci Tensor of $M$ implies $M$ is locally homogeneous with two or three principal curvatures.

0. Introduction.

Let $P^n(\mathbb{C})$ be an $n$-dimensional complex projective space with Fubini-Study metric of constant holomorphic sectional curvature 4. We consider a real hypersurface $M$ of $P^n(\mathbb{C})$. Let $(\phi, \xi, \eta, g)$ be an almost contact metric structure induced from the complex structure on $P^n(\mathbb{C})$ (§1). If the Ricci transformation of $M$ satisfies

\[ S\xi = a\xi + b\eta(X)\xi, \]

where $a$ and $b$ are constant, we call $M$ a pseudo-Einstein hypersurface [3]. Pseudo-Einstein real hypersurfaces in $P^n(\mathbb{C})$ are completely classified by Kon [3] (see [4]). This result shows that if the Ricci tensor of $M$ has a nice form, then $M$ is determined (see [5]). In this paper, we consider the following problem: If the derivative of the Ricci tensor of $M$ has a nice form, what can we say about $M$?

Received 6 August 1985. The author would like to express his thanks to Professor K. Ogiue and Professor N. Ejiri for valuable suggestions.
We have the following

**THEOREM 1.** Let $M$ be a real hypersurface of $P^n(\mathbb{C})$. If the Ricci transformation $S$ of $M$ satisfies

$$(0.2) \quad (\nabla^S_X Y) = c\{g(AX, Y)\xi + \eta(Y)\phi AX\},$$

where $c$ is a non-zero constant, and $A$ denotes the shape operator ($\S1$). Then $M$ is locally congruent to a homogeneous hypersurface with two or three distinct principal curvatures.

We note that pseudo-Einstein hypersurfaces satisfy (0.2). Moreover,

**THEOREM 2.** There are no real hypersurfaces with parallel Ricci tensor on which $\xi$ is principal.

1. Preliminaries.

Let $M$ be a real hypersurface of $P^n(\mathbb{C})$. In a neighbourhood of each point, we choose a unit normal vector field $N$ in $P^n(\mathbb{C})$. The Riemannian connections $\nabla$ in $P^n(\mathbb{C})$ and $\nabla$ in $M$ are related by the following formulas for arbitrary vector fields $X$ and $Y$ on $M$:

$$(1.1) \quad \nabla^N_X Y = \nabla_X Y + g(AX, Y)N,$$

$$(1.2) \quad \nabla^N_X N = -AX,$$

where $g$ denotes the Riemannian metric on $M$ induced from the Fubini-Study metric $\bar{g}$ on $P^n(\mathbb{C})$ and $A$ is the shape operator of $M$ in $P^n(\mathbb{C})$.

An eigenvector $X$ of the shape operator $A$ is called a principal curvature vector. Also an eigenvalue $\lambda$ of $A$ is called a principal curvature.

It is known that $M$ has an almost contact metric structure induced from the complex structure $J$ on $P^n(\mathbb{C})$, (see [6]), that is, we define a tensor field $\phi$ of type $(1,1)$, a vector field $\xi$ and a 1-form $\eta$ on $M$ by

$$g(\phi X, Y) = \bar{g}(JX, Y) \quad \text{and} \quad g(\xi, X) = \eta(X) = \bar{g}(JX, N).$$

Then we have

$$(1.3) \quad \phi^2 X = -X + \eta(X)\xi, \quad g(\xi, \xi) = 1, \quad \phi\xi = 0.$$
From (1.1), we easily have

\[ (\nabla_X^\phi)Y = n(Y)AX - g(AX, Y)\xi , \]

\[ (\nabla_X^\phi)Y = \phi AX . \]

Let \( \overline{R} \) and \( R \) be the curvature tensors of \( F^N(\mathcal{C}) \) and \( M \) respectively. Since the curvature tensor \( \overline{R} \) has a nice form, we have the following Gauss and Codazzi equations.

\[ g(R(X, Y)Z, W) = g(Y, Z)g(X, W) - g(X, Z)g(Y, W) \]
\[ + g(\phi Y, Z)g(\phi X, W) - g(\phi X, Z)g(\phi Y, Z) - 2g(\phi X, Y)g(\phi Z, W) + g(AY, Z)g(AX, W) - g(AX, Z)g(AY, W) \]

and

\[ g(\nabla_X^\phi)Y - g(\nabla_Y^\phi)X = n(X)\phi Y - n(Y)\phi X - 2g(\phi X, Y)\xi . \]

Using (1.3), (1.5), (1.6) and (1.7), we get

\[ SX = (2n+1)X - 3n(X)\xi + hAX - A^2X , \]
\[ (\nabla_X^S)Y = -3\{g(\phi AX, Y)\xi + n(Y)\phi AX\} + (Xh)AY + (h - A)(\nabla_X^A)Y - (\nabla_X^A)AY , \]

where \( h = \text{trace } A \) and \( S \) denotes the Ricci tensor on \( M \).

2. Proof of Theorems.

First, we determine the hypersurface \( M \) satisfying (0.2). Using (1.9), we see that (0.2) is equivalent to

\[ (\sigma + 3)\{n(W)g(\phi AX, Y) + n(Y)g(\phi AX, W)\} - (Xh)g(AY, W) \]
\[ + g((A - h)(\nabla_X^A)Y + (\nabla_X^A)AY, W) = 0 . \]

Contraction with respect to \( Y \) and \( W \), together with (1.3), yields

\[ -(Xh)h + \text{trace}(\nabla_X^A)(2A - h) = 0 . \]

It follows that \( h^2 - \text{trace } A^2 \) is constant. Next, using (1.7), we see that (2.1) becomes
(2.3) $$(c+3)(\eta(W)g(\phi AX, Y) + \eta(Y)g(\phi AX, W)) - (Xh)g(AY, W)$$
$$+ g((\nabla_X A)X + \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi, (A - h)W)$$
$$+ g((\nabla_{A\xi} A)X + \eta(X)\phi AY - \eta(AY)\phi X - 2g(\phi X, AY)\xi, W) = 0.$$ 

Contraction with respect to $X$ and $W$ yields

(2.4) $$ (c+3)g(\phi A\xi, Y) - (AY)h + \text{trace}(A - h)(\nabla_X A) + \eta((A - h)\phi Y)$$
$$+ 2g(\phi Y, (A - h)\xi) + \text{trace}(\nabla_{A\xi} A) + 3\eta(\phi AY) = 0. $$

since $\phi$ and $\phi A$ are skew-symmetric. (1.3) and commutativity of contraction and covariant differentiation imply

(2.5) $$-c\eta(A\xi, \phi Y) + \text{trace}(\nabla_X A)A - h \text{trace}(\nabla_X A) = 0.$$

and

(2.6) $$-c\eta(A\xi, \phi Y) + \frac{1}{2}Y(\text{trace} A^2 - h^2) = 0.$$

so that $g(A\xi, \phi Y) = 0$. Consequently, $\xi$ is principal. Let $A\xi = \mu\xi$.

Then Lemma 2.4 of [5] implies $\mu$ is locally constant. If we replace $Y$ by $\xi$, (2.1) becomes

(2.7) $$(c+3)\phi A\xi - (Xh)A\xi + (A - h + \mu)(\nabla_X A)\xi = 0.$$

by (1.3). From (1.5), we have $$(\nabla_X A)\xi = \nabla_X (A\xi) - A\nabla_X \xi = (\mu - A)\phi AX. $$

Then (2.7) gives

(2.8) $$((A - h + \mu)(\mu - A) + (c + 3))\phi AX - \mu(Xh)\xi = 0.$$

Since $A\phi(T_x M)$ is orthogonal to $\xi$, both first term and second term are zero, so that $\mu(Xh) = 0$.

Let $X$ be a principal vector with principal curvature $\lambda$, which is orthogonal to $\xi$. Then Lemma 2.2 of [5] implies that $\phi X$ is a principal vector with principal curvature $$(\lambda + 2)/(2\lambda - \mu),$$ and $2\lambda - \mu \neq 0$. Hence (2.8) gives

(2.9) $$\lambda\{(\frac{\lambda\mu + 2}{2\lambda - \mu} - h + \mu)(\mu - \frac{\lambda\mu + 2}{2\lambda - \mu} + c + 3) = 0. $$

If $\mu = 0$, then $\lambda \neq 0$, $A\phi X = \lambda^{-1}\phi X$ and $(\lambda h - 1) + \lambda^2(c + 3) = 0$.

Let $A_0$ be the restriction of $A$ to the orthogonal complement
Hypersurfaces of a Projective Space

$\xi_1 \ (= \phi T_x M)$ of $\xi$. Then $A_0$ has at most two distinct eigenvalues, so that $M$ has at most three distinct principal curvatures. From the proof of Theorem 4 of [1], $M$ is a homogeneous hypersurface with 2 or 3 distinct principal curvatures.

If $\mu \neq 0$, then $h$ is constant. From (2.9), $A_0$ has at most three distinct constant eigenvalues so that $M$ has at most four distinct constant principal curvatures. Since $\xi$ is principal, Theorem 1 and Theorem 4 in [2] implies that $M$ is a homogeneous hypersurface with 2 or 3 distinct principal curvatures. Thus Theorem 1 is proved.

The same argument implies that if $M$ has parallel Ricci tensor and $\xi$ is principal, then $M$ is a homogeneous hypersurface with 2 or 3 distinct principal curvatures. But there is no homogeneous hypersurface with parallel Ricci tensor. Hence Theorem 2 is proved.

References


Department of Mathematics,
Tokyo Metropolitan University,
Setagayaku, Tokyo 158,
Japan.