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SCHUR–NEVANLINNA SEQUENCES OF RATIONAL FUNCTIONS

ADHEMAR BULTHEEL AND ANDREAS LASAROW

Department of Computer Science, Katholieke Universiteit Leuven, Celestijnenlaan 200A, 3001 Heverlee (Leuven), Belgium
(adhemar.bultheel@cs.kuleuven.be; andreas.lasarow@cs.kuleuven.be)

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Abstract We study certain sequences of rational functions with poles outside the unit circle. Such kinds of sequences are recursively constructed based on sequences of complex numbers with norm less than one. In fact, such sequences are closely related to the Schur–Nevanlinna algorithm for Schur functions on the one hand, and to orthogonal rational functions on the unit circle on the other. We shall see that rational functions belonging to a Schur–Nevanlinna sequence can be used to parametrize the set of all solutions of an interpolation problem of Nevanlinna–Pick type for Schur functions.

Keywords: Nevanlinna–Pick interpolation problem; Schur functions; rational functions; Christoffel–Darboux formulae; Schur–Nevanlinna algorithm; Schur parameters

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1. Introduction

A function g which maps the open unit disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ into the complex plane $\mathbb{C}$ is a *Schur function* or belongs to the *Schur class* $\mathcal{S}$ (in the open unit disc $\mathbb{D}$) if g is holomorphic in $\mathbb{D}$ and if its values $g(z)$ are bounded by 1 for $z \in \mathbb{D}$, i.e. g is a holomorphic function such that the kernel

$$S_g(z,\zeta) := \frac{1 - g(z)\overline{g(\zeta)}}{1 - z\bar{\zeta}}, \quad z, \zeta \in \mathbb{D},$$

is non-negative Hermitian. More explicitly this kernel condition means that for every choice of $m \in \mathbb{N} := \{1, 2, 3, \dots\}$ and of the m points $z_1, z_2, \dots, z_m \in \mathbb{D}$ the complex $(m \times m)$-matrix $(S_g(z_j, z_k))_{j,k=1}^{m}$ is non-negative Hermitian. The equivalence of these conditions follows from the considerations on the classical Nevanlinna–Pick interpolation problem [**18**, **20**].

In the present work, we study a multiple point Nevanlinna–Pick interpolation problem for Schur functions. A multiple point interpolation problem is a problem in which values not only for the function itself but also for its derivatives up to a certain order are prescribed. Here we consider the following problem.

(MNP) *Given $n \in \mathbb{N}$, mutually distinct points $z_1, z_2, \ldots, z_n \in \mathbb{D}$, $l_1, l_2, \ldots, l_n \in \mathbb{N}$, and $w_{js} \in \mathbb{C}$, $s = 0, 1, \ldots, l_j - 1$, $j = 1, 2, \ldots, n$, find necessary and sufficient conditions for the existence of a $g \in \mathcal{S}$ such that*

$$\frac{1}{s!} g^{(s)}(z_j) = w_{js}, \quad s = 0, 1, \ldots, l_j - 1,\ j = 1, 2, \ldots, n. \tag{1.1}$$

Moreover, describe the set of solutions $\mathcal{S}_\Delta$ of all $g \in \mathcal{S}$ fulfilling (1.1).

Note that problem (MNP) can be conceived as a generalization of the Schur coefficient problem [**1**, **23**, **24**] on the one hand and of the classical Nevanlinna–Pick problem on the other [**18**, **20**].

As is well known, it is frequently the case that a finite interpolation problem of Nevanlinna–Pick type can be reduced, in a suitable way, to the study of a truncated trigonometric moment problem. Moreover, there exist several approaches to the solution of such a problem and several generalizations of the problem, too (see, for example, [**2**, **3**, **6**, **7**, **10**–**12**, **16**, **25**, **26**]). In particular, it is well known that there is a $g \in \mathcal{S}$ satisfying (1.1) if and only if the generalized Schwarz–Pick matrix $\boldsymbol{P}_\Delta$, which can be computed from the data given in problem (MNP), is non-negative Hermitian. In § 2, we introduce $\boldsymbol{P}_\Delta$ briefly. Moreover, we recall some basic facts on the Schur–Nevanlinna algorithm for Schur functions which were introduced by Nevanlinna [**18**] as an extension of the classical algorithm of Schur [**23**, **24**]. In fact, we shall deduce that, starting from a $g \in \mathcal{S}_\Delta$, the feasibility of this algorithm for g is closely related to the case in which $\boldsymbol{P}_\Delta$ is even positive Hermitian. Furthermore, we present some basics on linear fractional transformations and on rational functions.

As the main result of this note, in § 6 we will see that, for the non-uniqueness case, i.e. if $\boldsymbol{P}_\Delta$ is a positive Hermitian matrix, the set of solutions $\mathcal{S}_\Delta$ can be characterized by a linear fractional transformation which is determined by certain rational functions, where $\mathcal{S}$ is the set of parameters. In fact, we find that a function g belongs to $\mathcal{S}_\Delta$ if and only if there exists a Schur function h fulfilling the equality

$$g(z) = \frac{\delta_m^{[m]}(z) + b_{\alpha_m}(z)\gamma_m(z)h(z)}{\gamma_m^{[m]}(z) + b_{\alpha_m}(z)\delta_m(z)h(z)}, \quad z \in \mathbb{D}, \tag{1.2}$$

where b_{α_m} is a Blaschke factor, the rational functions δ_m, γ_m are elements of a Schur–Nevanlinna sequence, and $\delta_m^{[m]}$, $\gamma_m^{[m]}$ denote their respective adjoint rational functions.

Some basic facts on Schur–Nevanlinna sequences of rational functions are explained in § 3. In fact, such sequences are connected on the one hand to the Schur–Nevanlinna algorithm for Schur functions and on the other hand to orthogonal rational functions on the unit circle. Similarly, as in the case of orthogonal functions, we will see that the validity of some Christoffel–Darboux formulae is an important property of Schur–Nevanlinna sequences of rational functions (see Theorem 4.2 and the inverse question discussed in § 5).

Note that in [**25**] the set of solutions $\mathcal{S}_\Delta$ is characterized by a linear fractional transformation determined by some polynomials (without calculating these functions precisely).

Note that in [**19**] a modified Schur–Nevanlinna algorithm is discussed with respect to the real line case and a multiple point interpolation problem for Nevanlinna functions. The essential new feature of the current paper is that the functions δ_m and γ_m that appear in (1.2) are closely related to the orthogonal rational functions on the unit circle which were introduced by Djrbashian [**8**] (see also [**4**] and papers cited therein). The explicit interplay between Schur–Nevanlinna sequences and orthogonal rational functions will be covered in a forthcoming paper.

2. Preliminaries

For problem (MNP), we assume that the following data are given: $n \in \mathbb{N}$, mutually different points $z_1, z_2, \dots, z_n \in \mathbb{D}$, numbers $l_1, l_2, \dots, l_n \in \mathbb{N}$ and $w_{js} \in \mathbb{C}$, $s = 0, 1, \dots, l_j - 1$, $j = 1, 2, \dots, n$. We denote this dataset by Δ, i.e.

$$\Delta := \{(z_j, l_j, (w_{js})_{s=0}^{l_j-1})_{j=1}^{n}\}, \tag{2.1}$$

and put

$$m := \sum_{j=1}^{n} l_j - 1. \tag{2.2}$$

For a given function $g \in \mathcal{S}$, we similarly define

$$\Delta_g := \left\{ \left(z_j, l_j, \left(\frac{1}{s!} g^{(s)}(z_j) \right)_{s=0}^{l_j-1} \right)_{j=1}^{n} \right\}.$$

In particular, $g \in \mathcal{S}_\Delta$ if and only if $\Delta_g = \Delta$.

Furthermore, the *generalized Schwarz–Pick matrix* (with respect to the data Δ) of size $(m+1) \times (m+1)$ is defined as

$$\boldsymbol{P}_\Delta := (\boldsymbol{P}_{jk})_{j,k=1}^{n},$$

where the complex $(l_j \times l_k)$-matrices

$$\boldsymbol{P}_{jk} := (p_{jk}^{s,t})_{\substack{s=0,1,\dots,l_j-1, \\ t=0,1,\dots,l_k-1}}, \quad j, k = 1, 2, \dots, n,$$

are determined by the entries

$$\begin{aligned} p_{jk}^{s,t} := \sum_{r=0}^{\min\{s,t\}} & \frac{(s+t-r)!}{(s-r)!r!(t-r)!} \frac{z_j^{t-r} \bar{z}_k^{s-r}}{(1 - z_j \bar{z}_k)^{s+t-r+1}} \\ & - \sum_{\ell=0}^{s} \sum_{h=0}^{t} \sum_{r=0}^{\min\{\ell,h\}} \frac{(h+\ell-r)!}{(\ell-r)!r!(h-r)!} \frac{z_j^{h-r} \bar{z}_k^{h-r}}{(1 - z_j \bar{z}_k)^{h+\ell-r+1}} w_{j,s-\ell} \bar{w}_{k,t-h}, \\ & \qquad s = 0, 1, \dots, l_j - 1,\ t = 0, 1, \dots, l_k - 1. \end{aligned}$$

In the following, 0 also stands for the zero matrix of appropriate size, and if $\boldsymbol{A}$, $\boldsymbol{B}$ are Hermitian matrices of the same size, then $\boldsymbol{A} \geqslant \boldsymbol{B}$ (respectively, $\boldsymbol{A} > \boldsymbol{B}$) means that $\boldsymbol{A} - \boldsymbol{B}$ is a non-negative (respectively, positive) Hermitian matrix.

Now we recall a well-known criterion for the solvability of problem (MNP) (see, for example, [**3**, **11**, **12**]).

Theorem 2.1. *For a given dataset Δ as in (2.1), problem (MNP) has a solution if and only if $\boldsymbol{P}_\Delta \geqslant 0$. The solution is unique if and only if $\boldsymbol{P}_\Delta \geqslant 0$ with* $\det \boldsymbol{P}_\Delta = 0$.

Since the main goal of this paper is to obtain the description of $\mathcal{S}_\Delta$ via (1.2) for the non-uniqueness case we will always assume in the further course

$$\boldsymbol{P}_\Delta > 0.$$

The next considerations are aimed at showing that $\boldsymbol{P}_\Delta > 0$ is closely related to the feasibility of the Schur–Nevanlinna algorithm at least $m+1$ times for a $g \in \mathcal{S}_\Delta$. The algorithm presented below goes back to Nevanlinna [**18**] and is based on the following version of Schwarz's lemma (see, for example, [**4**, Theorem 1.2.3] for a proof). Here b_z denotes the elementary Blaschke factor corresponding to $z \in \mathbb{D}$, i.e.

$$b_z(v) := \begin{cases} \dfrac{\bar{z}}{|z|} \dfrac{z - v}{1 - \bar{z}v} & \text{if } z \neq 0, \\ v & \text{if } z = 0. \end{cases} \tag{2.3}$$

Remark 2.2. If $g \in \mathcal{S}$ such that $g(z) = 0$, then also $h := g/b_z \in \mathcal{S}$.

Now we recall the *Schur–Nevanlinna algorithm*: given a function $g \in \mathcal{S}$ and some points $\alpha_0, \alpha_1, \alpha_2, \dots \in \mathbb{D}$, we set $g_0 := g$ and consider $s_0 := g_0(\alpha_0)$. If $s_0 \in \mathbb{D}$ we can define

$$g_1(z) := \frac{1}{b_{\alpha_0}(z)} \frac{g_0(z) - s_0}{1 - \bar{s}_0 g_0(z)}$$

and, if the function g_k, $k = 0, 1, 2, \dots$, is already defined and

$$s_k := g_k(\alpha_k) \tag{2.4}$$

belongs to $\mathbb{D}$, then

$$g_{k+1}(z) := \frac{1}{b_{\alpha_k}(z)} \frac{g_k(z) - s_k}{1 - \bar{s}_k g_k(z)}, \quad k = 0, 1, 2, \dots. \tag{2.5}$$

If $g \in \mathcal{S}$ and $\alpha_0, \alpha_1, \alpha_2, \dots \in \mathbb{D}$ such that the Schur–Nevanlinna algorithm can be carried out at least r times (that is, after obtaining g_r and s_r), then $(s_k)_{k=0}^r$ from (2.4) is called the sequence of *Schur parameters* associated with the pair $[g, (\alpha_k)_{k=0}^r]$.

The algorithm (cf. (2.5)) defines the Schur functions $g_0, g_1, g_2, \dots$. It breaks down after the kth step (that is, after obtaining g_k and s_k) if and only if $s_k \in \mathbb{T}$. In particular, the Schur–Nevanlinna algorithm breaks down after the kth step if and only if g is a Blaschke product of degree k (cf. [**18**]). Therefore, using some basic facts on generalized Schwarz–Pick matrices, one can conclude with the following statement (cf. [**14**, § 5], [**17**, Corollary 3.6]).

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Theorem 2.3. *If $g \in \mathcal{S}$, then the Schur–Nevanlinna algorithm can be carried out at least $m+1$ times for g (and any points $\alpha_0, \alpha_1, \dots, \alpha_{m+1} \in \mathbb{D}$) if and only if $\boldsymbol{P}_{\Delta_g} > 0$.*

In the following, we will also apply some well-known results on linear fractional transformations (see, for example, [**9**, **10**, **22**]). If

$$\boldsymbol{\Theta} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is a complex (2×2)-matrix and w is a complex number such that $cw + d \neq 0$, then we set

$$T_{\boldsymbol{\Theta}}(w) := \frac{aw + b}{cw + d}. \tag{2.6}$$

Note that the relation

$$T_{\boldsymbol{\Theta}_1}(T_{\boldsymbol{\Theta}_2}(w)) = T_{\boldsymbol{\Theta}_1\boldsymbol{\Theta}_2}(w) \tag{2.7}$$

is satisfied, and that in the case $\det \boldsymbol{\Theta} \neq 0$ the inverse mapping $T_{\boldsymbol{\Theta}}^{-1}$ is given by

$$T_{\boldsymbol{\Theta}}^{-1}(w) = T_{\boldsymbol{\Theta}^{-1}}(w) = \frac{dw - b}{-cw + a}. \tag{2.8}$$

Note that, in view of (2.6)–(2.8), the relation (2.5) can also be written as

$$g_k(z) = \frac{b_{\alpha_k}(z) g_{k+1}(z) + s_k}{\bar{s}_k b_{\alpha_k}(z) g_{k+1}(z) + 1} = T_{\widehat{\boldsymbol{\Xi}}_k(z)}(g_{k+1}(z))$$

with

$$\widehat{\boldsymbol{\Xi}}_k(z) := \begin{pmatrix} b_{\alpha_k}(z) & s_k \\ \bar{s}_k b_{\alpha_k}(z) & 1 \end{pmatrix} = \begin{pmatrix} 1 & s_k \\ \bar{s}_k & 1 \end{pmatrix} \begin{pmatrix} b_{\alpha_k}(z) & 0 \\ 0 & 1 \end{pmatrix} \tag{2.9}$$

and hence

$$\begin{aligned} g(z) \equiv g_0(z) &= T_{\widehat{\boldsymbol{\Xi}}_0(z)}(T_{\widehat{\boldsymbol{\Xi}}_1(z)}(\cdots (T_{\widehat{\boldsymbol{\Xi}}_k(z)}(g_{k+1}(z)) \cdots))) \\ &= T_{\widehat{\boldsymbol{\Xi}}_0(z)\widehat{\boldsymbol{\Xi}}_1(z)\cdots\widehat{\boldsymbol{\Xi}}_k(z)}(g_{k+1}(z)). \end{aligned} \tag{2.10}$$

In the next section, we shall treat those sequences of rational functions which are closely related to the Schur–Nevanlinna algorithm on the one hand and to the orthogonal rational functions on the unit circle introduced by Djrbashian [**8**] (see also [**4**]) on the other. Here, for fixed points $\alpha_0, \alpha_1, \alpha_2, \dots \in \mathbb{D}$ the notation $\mathfrak{H}_k$, $k = 0, 1, 2, \dots$, denotes the space of rational functions x that, for some complex polynomial p of degree not greater than k, admit the representation

$$x = \frac{p}{q_k},$$

where the complex polynomial q_k of degree not greater than $k+1$ is given by

$$q_k(v) = \prod_{j=0}^{k} (1 - \bar{\alpha}_j v).$$

As suggested in [**4**], the following transform of a rational function into another plays a key role. For $x \in \mathfrak{H}_k$, $k = 0, 1, 2, \dots$, by the *adjoint rational function* $x^{[k]}$ of x (adjoint with respect to $\alpha_0, \alpha_1, \dots, \alpha_k$) we mean the rational function which is uniquely determined via the formula

$$x^{[k]}(v) = \frac{1}{v} B_k(v) \overline{x\left(\frac{1}{\bar{v}}\right)}, \tag{2.11}$$

where B_k stands for the *Blaschke product* (of degree $k+1$) with respect to the points $\alpha_0, \alpha_1, \dots, \alpha_k$, i.e.

$$B_k(v) := \prod_{j=0}^{k} b_{\alpha_j}(v).$$

Some information on the calculation of $x^{[k]}$, $k = 0, 1, 2, \dots$, can be found in [**4**, § 2.2]. Note that the results on the adjoint rational function in [**4**] are explained relating to the special case $\alpha_0 = 0$. However, it is not hard to restate these with their proofs for the present situation. For instance, if $x \in \mathfrak{H}_k$, $k = 0, 1, 2, \dots$, then also $x^{[k]} \in \mathfrak{H}_k$ and $(x^{[k]})^{[k]} = x$ in that case.

3. Some basics on Schur–Nevanlinna sequences of rational functions

In this section, as a rational extension of the classical considerations of Schur [**23, 24**] and Nevanlinna [**18**] (see also [**13**, § 3] for an extension to the case of matrix-valued polynomials) we study some sequences of rational functions formed by given sequences of points and parameters belonging to $\mathbb{D}$.

If $\tau = 0$ or $\tau \in \mathbb{N}$ or $\tau = \infty$, if $\mathbb{I} := \{0, 1, 2, \dots, \tau\}$, and if $(\alpha_k)_{k\in\mathbb{I}}$ and $(\kappa_k)_{k\in\mathbb{I}}$ are sequences of points belonging to $\mathbb{D}$, then we define sequences of rational functions $(\gamma_k)_{k\in\mathbb{I}}$ and $(\delta_k)_{k\in\mathbb{I}}$ by the relations

$$\gamma_0(v) := \sqrt{\frac{1-|\alpha_0|^2}{1-|\kappa_0|^2}} \frac{1}{1-\bar{\alpha}_0 v}, \qquad \delta_0(v) := \bar{\kappa}_0 \sqrt{\frac{1-|\alpha_0|^2}{1-|\kappa_0|^2}} \frac{1}{1-\bar{\alpha}_0 v} \tag{3.1}$$

and, for $k \in \mathbb{I} \setminus \{0\}$, recursively:

$$\gamma_k(v) := \sqrt{\frac{1-|\alpha_k|^2}{(1-|\alpha_{k-1}|^2)(1-|\kappa_k|^2)}} \frac{1-\bar{\alpha}_{k-1} v}{1-\bar{\alpha}_k v} (b_{\alpha_{k-1}}(v)\gamma_{k-1}(v) + \bar{\kappa}_k \delta_{k-1}^{[k-1]}(v)),$$

$$\delta_k(v) := \sqrt{\frac{1-|\alpha_k|^2}{(1-|\alpha_{k-1}|^2)(1-|\kappa_k|^2)}} \frac{1-\bar{\alpha}_{k-1} v}{1-\bar{\alpha}_k v} (b_{\alpha_{k-1}}(v)\delta_{k-1}(v) + \bar{\kappa}_k \gamma_{k-1}^{[k-1]}(v)),$$

where $\delta_{k-1}^{[k-1]}$ and $\gamma_{k-1}^{[k-1]}$ stand for the adjoint rational functions of δ_{k-1} and γ_{k-1} (with respect to the points $\alpha_0, \alpha_1, \dots, \alpha_{k-1}$; see (2.11)). We call $[(\gamma_k)_{k\in\mathbb{I}}, (\delta_k)_{k\in\mathbb{I}}]$ the *Schur–Nevanlinna pair of rational functions corresponding to* $(\alpha_k, \kappa_k)_{k\in\mathbb{I}}$.

With the matrix function

$$\boldsymbol{\Theta}_k := \begin{pmatrix} b_{\alpha_k}\gamma_k & \delta_k^{[k]} \\ b_{\alpha_k}\delta_k & \gamma_k^{[k]} \end{pmatrix}, \quad k \in \mathbb{I}, \tag{3.2}$$

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the recurrence formulae above can be written, for $k \in \mathbb{I} \setminus \{0\}$, in matrix form as

$$\boldsymbol{\Theta}_k(v) = \sqrt{\frac{1-|\alpha_k|^2}{1-|\alpha_{k-1}|^2}}\frac{1-\bar{\alpha}_{k-1}v}{1-\bar{\alpha}_k v}\boldsymbol{\Theta}_{k-1}(v)\widehat{\boldsymbol{\Theta}}_k(v), \tag{3.3}$$

where

$$\widehat{\boldsymbol{\Theta}}_k(v) := \frac{1}{\sqrt{1-|\kappa_k|^2}}\begin{pmatrix} 1 & \kappa_k \\ \bar{\kappa}_k & 1 \end{pmatrix}\begin{pmatrix} b_{\alpha_k}(v) & 0 \\ 0 & \eta_k\bar{\eta}_{k-1} \end{pmatrix}, \quad k \in \mathbb{I} \setminus \{0\}, \tag{3.4}$$

and

$$\eta_k := \begin{cases} -1 & \text{if } \alpha_k = 0, \\ \dfrac{\bar{\alpha}_k}{|\alpha_k|} & \text{if } \alpha_k \neq 0, \end{cases} \qquad k \in \mathbb{I}. \tag{3.5}$$

Proposition 3.1. *Let $\tau \in \mathbb{N}$ or $\tau = \infty$, let $\mathbb{I} := \{0, 1, 2, \dots, \tau\}$, and let $(\alpha_k)_{k\in\mathbb{I}}$ and $(\kappa_k)_{k\in\mathbb{I}}$ be sequences of points belonging to $\mathbb{D}$. Further, let $(\gamma_k)_{k\in\mathbb{I}}$ and $(\delta_k)_{k\in\mathbb{I}}$ be sequences of rational functions such that γ_0, δ_0 are defined as in (3.1) and γ_k, δ_k belong to $\mathfrak{H}_k$ for $k \in \mathbb{I} \setminus \{0\}$. Then $[(\gamma_k)_{k\in\mathbb{I}}, (\delta_k)_{k\in\mathbb{I}}]$ is the Schur–Nevanlinna pair of rational functions corresponding to $(\alpha_k, \kappa_k)_{k\in\mathbb{I}}$ if and only if, for each $k \in \mathbb{I} \setminus \{0\}$, the following backward recurrence relations hold:*

$$\eta_k\bar{\eta}_{k-1}\gamma_k(v) - \bar{\kappa}_k\delta_k^{[k]}(v) = \frac{(1-\bar{\alpha}_k\alpha_{k-1})\sqrt{1-|\kappa_k|^2}}{\sqrt{(1-|\alpha_k|^2)(1-|\alpha_{k-1}|^2)}}(b_{\alpha_k}(v) - b_{\alpha_k}(\alpha_{k-1}))\gamma_{k-1}(v),$$

$$\eta_k\bar{\eta}_{k-1}\delta_k(v) - \bar{\kappa}_k\gamma_k^{[k]}(v) = \frac{(1-\bar{\alpha}_k\alpha_{k-1})\sqrt{1-|\kappa_k|^2}}{\sqrt{(1-|\alpha_k|^2)(1-|\alpha_{k-1}|^2)}}(b_{\alpha_k}(v) - b_{\alpha_k}(\alpha_{k-1}))\delta_{k-1}(v).$$

Proof. Let $k \in \mathbb{I} \setminus \{0\}$. Evidently, the identity

$$\begin{pmatrix} 1 & \kappa_k \\ \bar{\kappa}_k & 1 \end{pmatrix}\begin{pmatrix} 1 & -\kappa_k \\ -\bar{\kappa}_k & 1 \end{pmatrix} = \begin{pmatrix} 1-|\kappa_k|^2 & 0 \\ 0 & 1-|\kappa_k|^2 \end{pmatrix}$$

is satisfied. Therefore, (3.3) is equivalent to the relation

$$\begin{pmatrix} \gamma_k(v) & \bar{\eta}_k\eta_{k-1}\delta_k^{[k]}(v) \\ \delta_k(v) & \bar{\eta}_k\eta_{k-1}\gamma_k^{[k]}(v) \end{pmatrix}\begin{pmatrix} 1 & -\kappa_k \\ -\bar{\kappa}_k & 1 \end{pmatrix} = \sqrt{\frac{(1-|\alpha_k|^2)(1-|\kappa_k|^2)}{1-|\alpha_{k-1}|^2}}\frac{1-\bar{\alpha}_{k-1}v}{1-\bar{\alpha}_k v}\boldsymbol{\Theta}_{k-1}(v).$$

Hence, by considering the first column of $\boldsymbol{\Theta}_{k-1}(v)$ and using

$$\bar{\eta}_k\eta_{k-1}\frac{1-\bar{\alpha}_k\alpha_{k-1}}{1-|\alpha_k|^2}(b_{\alpha_k}(v) - b_{\alpha_k}(\alpha_{k-1})) = \frac{1-\bar{\alpha}_{k-1}v}{1-\bar{\alpha}_k v}b_{\alpha_{k-1}}(v), \tag{3.6}$$

one can finally conclude the assertion. □

Henceforth in this section $[(\gamma_k)_{k\in\mathbb{I}}, (\delta_k)_{k\in\mathbb{I}}]$ always denotes the Schur–Nevanlinna pair of rational functions corresponding to $(\alpha_k, \kappa_k)_{k\in\mathbb{I}}$, where $(\alpha_k)_{k\in\mathbb{I}}$ and $(\kappa_k)_{k\in\mathbb{I}}$ are some sequences of points belonging to $\mathbb{D}$.

By the combination of the forward recursions defining the Schur–Nevanlinna pair of rational functions $[(\gamma_k)_{k\in\mathbb{I}},(\delta_k)_{k\in\mathbb{I}}]$, the backward recursions stated in Proposition 3.1 and (3.6), one can see that the pair $[(\gamma_k)_{k\in\mathbb{I}},(\delta_k)_{k\in\mathbb{I}}]$ also satisfies the following three-term recurrence relations.

Corollary 3.2. *For each $k\in\mathbb{I}\setminus\{0,1\}$,*

$$\begin{aligned}\bar\kappa_{k-1}\gamma_k(v)&=\sqrt{\frac{1-|\alpha_k|^2}{(1-|\alpha_{k-1}|^2)(1-|\kappa_k|^2)}}\frac{1-\bar\alpha_{k-1}v}{1-\bar\alpha_k v}(\bar\kappa_{k-1}b_{\alpha_{k-1}}(v)+\bar\kappa_k\eta_{k-1}\bar\eta_{k-2})\gamma_{k-1}(v)\\&\quad-\bar\kappa_k\eta_{k-1}\bar\eta_{k-2}\sqrt{\frac{(1-|\alpha_k|^2)(1-|\kappa_{k-1}|^2)}{(1-|\alpha_{k-2}|^2)(1-|\kappa_k|^2)}}\frac{1-\bar\alpha_{k-2}v}{1-\bar\alpha_k v}b_{\alpha_{k-2}}(v)\gamma_{k-2}(v)\end{aligned}$$

and

$$\begin{aligned}\bar\kappa_{k-1}\delta_k(v)&=\sqrt{\frac{1-|\alpha_k|^2}{(1-|\alpha_{k-1}|^2)(1-|\kappa_k|^2)}}\frac{1-\bar\alpha_{k-1}v}{1-\bar\alpha_k v}(\bar\kappa_{k-1}b_{\alpha_{k-1}}(v)+\bar\kappa_k\eta_{k-1}\bar\eta_{k-2})\delta_{k-1}(v)\\&\quad-\bar\kappa_k\eta_{k-1}\bar\eta_{k-2}\sqrt{\frac{(1-|\alpha_k|^2)(1-|\kappa_{k-1}|^2)}{(1-|\alpha_{k-2}|^2)(1-|\kappa_k|^2)}}\frac{1-\bar\alpha_{k-2}v}{1-\bar\alpha_k v}b_{\alpha_{k-2}}(v)\delta_{k-2}(v),\end{aligned}$$

where $\gamma_0(v)$, $\delta_0(v)$ are given as in (3.1) and

$$\gamma_1(v)=\sqrt{\frac{1-|\alpha_1|^2}{(1-|\kappa_0|^2)(1-|\kappa_1|^2)}}\frac{1}{1-\bar\alpha_1 v}(b_{\alpha_0}(v)-\eta_0\bar\kappa_1\kappa_0),$$

$$\delta_1(v)=\sqrt{\frac{1-|\alpha_1|^2}{(1-|\kappa_0|^2)(1-|\kappa_1|^2)}}\frac{1}{1-\bar\alpha_1 v}(\bar\kappa_0 b_{\alpha_0}(v)-\eta_0\bar\kappa_1).$$

A key tool in the proof of the description (1.2) for the set $\mathcal{S}_\Delta$ is an application of some well-known results on Potapov's $\boldsymbol{J}$ theory (see, for example, [**21**,**22**]). Hereby, the special choice of the (2×2)-signature matrix

$$\boldsymbol{J}:=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

will be essential in the considerations below. Recall that a complex (2×2)-matrix $\boldsymbol{\Theta}(v)$ is said to be $\boldsymbol{J}$-*contractive* (respectively, $\boldsymbol{J}$-*unitary*), if

$$\boldsymbol{J}\geqslant\boldsymbol{\Theta}(v)^*\boldsymbol{J}\boldsymbol{\Theta}(v)\quad(\text{respectively, }\boldsymbol{J}=\boldsymbol{\Theta}(v)^*\boldsymbol{J}\boldsymbol{\Theta}(v)),$$

where $\boldsymbol{\Theta}(v)^*$ denotes the adjoint matrix of $\boldsymbol{\Theta}(v)$.

Theorem 3.3. *For each $k\in\mathbb{I}$,*

$$\boldsymbol{\Theta}_k(v)=\frac{\sqrt{1-|\alpha_k|^2}}{1-\bar\alpha_k v}\widehat{\boldsymbol{\Theta}}_0(v)\widehat{\boldsymbol{\Theta}}_1(v)\cdots\widehat{\boldsymbol{\Theta}}_k(v),\tag{3.7}$$

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where $\boldsymbol{\Theta}_k(v)$ *and* $\widehat{\boldsymbol{\Theta}}_\ell(v)$, $\ell = 1, 2, \dots, k$, *is given by (3.2) and (3.4) as well as*

$$\widehat{\boldsymbol{\Theta}}_0(v) := \frac{1}{\sqrt{1-|\kappa_0|^2}} \begin{pmatrix} 1 & \kappa_0 \\ \bar{\kappa}_0 & 1 \end{pmatrix} \begin{pmatrix} b_{\alpha_0}(v) & 0 \\ 0 & -\eta_0 \end{pmatrix}.$$

In particular, if $v \in \mathbb{D}$ (*respectively,* $v \in \mathbb{T}$), *then the matrix*

$$\frac{1-\bar{\alpha}_k v}{\sqrt{1-|\alpha_k|^2}} \boldsymbol{\Theta}_k(v)$$

is $\boldsymbol{J}$*-contractive* (*respectively,* $\boldsymbol{J}$*-unitary*).

Proof. Let $k \in \mathbb{I}$. The alleged representation (3.7) of $\boldsymbol{\Theta}_k(v)$ is an easy consequence of (3.3) and the choice of γ_0 and δ_0. It remains to prove that if $v \in \mathbb{D}$ (respectively, $v \in \mathbb{T}$), then the matrix

$$\frac{1-\bar{\alpha}_k v}{\sqrt{1-|\alpha_k|^2}} \boldsymbol{\Theta}_k(v)$$

is $\boldsymbol{J}$-contractive (respectively, $\boldsymbol{J}$-unitary). However, this follows immediately from (3.7) and the fact that, for all $\ell \in \{0, 1, \dots, k\}$,

$$\boldsymbol{J} = \frac{1}{1-|\kappa_\ell|^2} \begin{pmatrix} 1 & \kappa_\ell \\ \bar{\kappa}_\ell & 1 \end{pmatrix}^* \boldsymbol{J} \begin{pmatrix} 1 & \kappa_\ell \\ \bar{\kappa}_\ell & 1 \end{pmatrix} \tag{3.8}$$

and that if $v \in \mathbb{D}$ (respectively, $v \in \mathbb{T}$), then

$$\boldsymbol{J} \geqslant \begin{pmatrix} b_{\alpha_\ell}(v) & 0 \\ 0 & u \end{pmatrix}^* \boldsymbol{J} \begin{pmatrix} b_{\alpha_\ell}(v) & 0 \\ 0 & u \end{pmatrix} \quad \left(\text{respectively, } \boldsymbol{J} = \begin{pmatrix} b_{\alpha_\ell}(v) & 0 \\ 0 & u \end{pmatrix}^* \boldsymbol{J} \begin{pmatrix} b_{\alpha_\ell}(v) & 0 \\ 0 & u \end{pmatrix}\right) \tag{3.9}$$

for a $u \in \mathbb{T}$, i.e. the fact that $\widehat{\boldsymbol{\Theta}}_\ell(v)$ is $\boldsymbol{J}$-contractive (respectively, $\boldsymbol{J}$-unitary). □

By forming the determinants of both sides in (3.7), we obtain the following.

Corollary 3.4. *For each* $k \in \mathbb{I}$,

$$b_{\alpha_k}(v)(\gamma_k(v)\gamma_k^{[k]}(v) - \delta_k(v)\delta_k^{[k]}(v)) = -\eta_k \frac{1-|\alpha_k|^2}{(1-\bar{\alpha}_k v)^2} B_k(v).$$

Because $B_k(\alpha_\ell) = 0$, $\ell = 0, 1, \dots, k-1$ (cf. (2.3)), the equality in Corollary 3.4 implies in particular the following corollary.

Corollary 3.5. *For each* $k \in \mathbb{I} \setminus \{0\}$ *and each* $\ell \in \{0, 1, \dots, k-1\}$,

$$\gamma_k(\alpha_\ell)\gamma_k^{[k]}(\alpha_\ell) = \delta_k(\alpha_\ell)\delta_k^{[k]}(\alpha_\ell).$$

In view of some well-known results on $\boldsymbol{J}$-contractive matrices (see, for example, [**9**, Theorem 1.6.1]), Theorem 3.3 also yields the following.

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Corollary 3.6. *For each $k \in \mathbb{I}$ and each $v \in \mathbb{D} \cup \mathbb{T}$, the relations*

$$\gamma_k^{[k]}(v) \neq 0, \quad \frac{1}{|\gamma_k^{[k]}(v)|} \leqslant \frac{|1-\bar{\alpha}_k v|}{\sqrt{1-|\alpha_k|^2}}, \quad \left|\frac{\delta_k^{[k]}(v)}{\gamma_k^{[k]}(v)}\right| < 1 \quad \text{and} \quad \left|\frac{b_{\alpha_k}(v)\delta_k(v)}{\gamma_k^{[k]}(v)}\right| < 1$$

are satisfied.

Since Corollary 3.6 includes a localization of the zeros of $\gamma_k^{[k]}$, $k \in \mathbb{I} \setminus \{0\}$, the next corollary is an easy conclusion of Proposition 3.1 and (3.3) with $v = \alpha_{k-1}$.

Corollary 3.7. *For each $k \in \mathbb{I} \setminus \{0\}$, we have $\gamma_k^{[k]}(\alpha_{k-1}) \neq 0$,*

$$\bar{\kappa}_k = \eta_k \bar{\eta}_{k-1} \frac{\delta_k(\alpha_{k-1})}{\gamma_k^{[k]}(\alpha_{k-1})},$$

$$\sqrt{1-|\kappa_k|^2} = \eta_k \bar{\eta}_{k-1} \frac{\sqrt{(1-|\alpha_k|^2)(1-|\alpha_{k-1}|^2)}}{1-\bar{\alpha}_k \alpha_{k-1}} \frac{\gamma_{k-1}^{[k-1]}(\alpha_{k-1})}{\gamma_k^{[k]}(\alpha_{k-1})},$$

and, in particular, $\kappa_k = 0 \iff \delta_k(\alpha_{k-1}) = 0$.

Note that, if we put $\alpha_{-1} := 0$, $\eta_{-1} := -1$ and $\gamma_{-1}^{[-1]}(\alpha_{-1}) := 1$, then the relations in Corollary 3.7 hold also in the case when $k = 0$.

In spite of the symmetry of the recurrence relations, one cannot easily interchange the roles of γ_k and δ_k in Corollary 3.7. In fact, the condition $\delta_k^{[k]}(\alpha_{k-1}) \neq 0$ is not true in general.

Remark 3.8. For each $k \in \mathbb{I} \setminus \{0\}$, if $\delta_k^{[k]}(\alpha_{k-1}) \neq 0$, then Proposition 3.1 and (3.3) yield

$$\bar{\kappa}_k = \eta_k \bar{\eta}_{k-1} \frac{\gamma_k(\alpha_{k-1})}{\delta_k^{[k]}(\alpha_{k-1})},$$

$$\sqrt{1-|\kappa_k|^2} = \eta_k \bar{\eta}_{k-1} \frac{\sqrt{(1-|\alpha_k|^2)(1-|\alpha_{k-1}|^2)}}{1-\bar{\alpha}_k \alpha_{k-1}} \frac{\delta_{k-1}^{[k-1]}(\alpha_{k-1})}{\delta_k^{[k]}(\alpha_{k-1})},$$

and, in particular, $\kappa_k = 0 \iff \gamma_k(\alpha_{k-1}) = 0$. On the other hand, taking into account (3.3), $\gamma_k^{[k]}(\alpha_{k-1}) \neq 0$ and Corollary 3.5, one can see that

$$\delta_{k-1}^{[k-1]}(\alpha_{k-1}) = 0 \iff \delta_k^{[k]}(\alpha_{k-1}) = 0 \implies \gamma_k(\alpha_{k-1}) = 0, \quad k \in \mathbb{I} \setminus \{0\},$$

but $\kappa_k = 0$ also implies that $\gamma_k(\alpha_{k-1}) = 0$ and the condition $\gamma_k(\alpha_{k-1}) = 0$ conversely supplies $\delta_k^{[k]}(\alpha_{k-1}) = 0$ or $\kappa_k = 0$.

Remark 3.9. If $\kappa_0 = 0$, then obviously

$$\delta_0^{[0]}(\alpha_0) = 0$$

and, by an application of $\gamma_k^{[k]}(\alpha_0) \neq 0$ and Corollary 3.5, one can inductively derive from the recurrence relations of a Schur–Nevanlinna pair of rational functions (using, for example, (3.3)) that

$$\delta_k^{[k]}(\alpha_0) = 0, \quad \gamma_k(\alpha_0) = 0, \quad k \in \mathbb{I} \setminus \{0\}.$$

4. Christoffel–Darboux formulae

In this section we will show that, as for orthogonal rational functions (cf. [**4**, Theorem 3.1.3]), arbitrary Schur–Nevanlinna pairs of rational functions $[(\gamma_k)_{k\in\mathbb{I}},(\delta_k)_{k\in\mathbb{I}}]$ also satisfy some Christoffel–Darboux formulae. To prove these, we first note the following identities for $[(\gamma_k)_{k\in\mathbb{I}},(\delta_k)_{k\in\mathbb{I}}]$. Here and in the remainder of the section $[(\gamma_k)_{k\in\mathbb{I}},(\delta_k)_{k\in\mathbb{I}}]$ again denotes the Schur–Nevanlinna pair of rational functions corresponding to $(\alpha_k,\kappa_k)_{k\in\mathbb{I}}$, where $(\alpha_k)_{k\in\mathbb{I}}$ and $(\kappa_k)_{k\in\mathbb{I}}$ are some sequences of points belonging to $\mathbb{D}$.

Lemma 4.1. *For each $k\in\mathbb{I}\setminus\{0\}$, the following relations hold:*

$$\begin{aligned}(1-b_{\alpha_{k-1}}(v)\overline{b_{\alpha_{k-1}}(w)})(\delta_k^{[k]}(v)\overline{\delta_k^{[k]}(w)}-\gamma_k(v)\overline{\gamma_k(w)})\\ =(1-b_{\alpha_k}(v)\overline{b_{\alpha_k}(w)})(\delta_{k-1}^{[k-1]}(v)\overline{\delta_{k-1}^{[k-1]}(w)}-b_{\alpha_{k-1}}(v)\gamma_{k-1}(v)\overline{b_{\alpha_{k-1}}(w)\gamma_{k-1}(w)}),\end{aligned}$$

$$\begin{aligned}(1-b_{\alpha_{k-1}}(v)\overline{b_{\alpha_{k-1}}(w)})(\gamma_k^{[k]}(v)\overline{\gamma_k^{[k]}(w)}-\delta_k(v)\overline{\delta_k(w)})\\ =(1-b_{\alpha_k}(v)\overline{b_{\alpha_k}(w)})(\gamma_{k-1}^{[k-1]}(v)\overline{\gamma_{k-1}^{[k-1]}(w)}-b_{\alpha_{k-1}}(v)\delta_{k-1}(v)\overline{b_{\alpha_{k-1}}(w)\delta_{k-1}(w)}),\end{aligned}$$

and

$$\begin{aligned}(1-b_{\alpha_{k-1}}(v)\overline{b_{\alpha_{k-1}}(w)})(\delta_k^{[k]}(v)\overline{\gamma_k^{[k]}(w)}-\gamma_k(v)\overline{\delta_k(w)})\\ =(1-b_{\alpha_k}(v)\overline{b_{\alpha_k}(w)})(\delta_{k-1}^{[k-1]}(v)\overline{\gamma_{k-1}^{[k-1]}(w)}-b_{\alpha_{k-1}}(v)\gamma_{k-1}(v)\overline{b_{\alpha_{k-1}}(w)\delta_{k-1}(w)}).\end{aligned}$$

Proof. Note that, in view of (2.3), for $z\in\mathbb{D}$ and complex numbers v, w ($\neq 1/\bar{z}$), it follows that

$$1-b_z(v)\overline{b_z(w)}=\frac{(1-|z|^2)(1-v\bar{w})}{(1-v\bar{z})(1-z\bar{w})}. \tag{4.1}$$

Let $k\in\mathbb{I}\setminus\{0\}$. Considering the first row of the matrix function $\boldsymbol{\Theta}_k$, from (3.3), (3.8) and (3.9) one can see that

$$\begin{aligned}&\delta_k^{[k]}(v)\overline{\delta_k^{[k]}(w)}-\gamma_k(v)\overline{\gamma_k(w)}\\ &\quad=-\begin{pmatrix}\gamma_k(v) & \bar{\eta}_k\eta_{k-1}\delta_k^{[k]}(v)\end{pmatrix}\boldsymbol{J}\begin{pmatrix}\gamma_k(w) & \bar{\eta}_k\eta_{k-1}\delta_k^{[k]}(w)\end{pmatrix}^*\\ &\quad=-\Bigg(\sqrt{\frac{1-|\alpha_k|^2}{1-|\alpha_{k-1}|^2}}\frac{1-\bar{\alpha}_{k-1}v}{1-\bar{\alpha}_kv}\\ &\qquad\qquad\times\begin{pmatrix}b_{\alpha_{k-1}}(v)\gamma_{k-1}(v) & \delta_{k-1}^{[k-1]}(v)\end{pmatrix}\frac{1}{\sqrt{1-|\kappa_k|^2}}\begin{pmatrix}1 & \kappa_k\\ \bar{\kappa}_k & 1\end{pmatrix}\Bigg)\boldsymbol{J}\\ &\quad\times\Bigg(\sqrt{\frac{1-|\alpha_k|^2}{1-|\alpha_{k-1}|^2}}\frac{1-\bar{\alpha}_{k-1}w}{1-\bar{\alpha}_kw}\\ &\qquad\qquad\times\begin{pmatrix}b_{\alpha_{k-1}}(w)\gamma_{k-1}(w) & \delta_{k-1}^{[k-1]}(w)\end{pmatrix}\frac{1}{\sqrt{1-|\kappa_k|^2}}\begin{pmatrix}1 & \kappa_k\\ \bar{\kappa}_k & 1\end{pmatrix}\Bigg)^*\end{aligned}$$

$$= -\frac{1-|\alpha_k|^2}{1-|\alpha_{k-1}|^2}\frac{1-\bar{\alpha}_{k-1}v}{1-\bar{\alpha}_k v}\frac{1-\alpha_{k-1}\bar{w}}{1-\alpha_k\bar{w}}$$
$$\times \begin{pmatrix} b_{\alpha_{k-1}}(v)\gamma_{k-1}(v) & \delta_{k-1}^{[k-1]}(v)\end{pmatrix}\overline{\begin{pmatrix} b_{\alpha_{k-1}}(w)\gamma_{k-1}(w) \\ -\delta_{k-1}^{[k-1]}(w)\end{pmatrix}}$$
$$= \frac{1-|\alpha_k|^2}{1-|\alpha_{k-1}|^2}\frac{1-\bar{\alpha}_{k-1}v}{1-\bar{\alpha}_k v}\frac{1-\alpha_{k-1}\bar{w}}{1-\alpha_k\bar{w}}$$
$$\times \Big(\delta_{k-1}^{[k-1]}(v)\overline{\delta_{k-1}^{[k-1]}(w)} - b_{\alpha_{k-1}}(v)\gamma_{k-1}(v)\overline{b_{\alpha_{k-1}}(w)\gamma_{k-1}(w)}\Big).$$

and hence an application of (4.1) yields the first identity. Similarly, observing the second row (respectively, the first and the second row) of $\boldsymbol{\Theta}_k$, in view of (3.3), (3.8), (3.9) and (4.1), one can obtain the second (respectively, the third) identity. □

Theorem 4.2. *For $k \in \mathbb{I}$, the following Christoffel–Darboux formulae hold:*

$$(1 - b_{\alpha_k}(v)\overline{b_{\alpha_k}(w)})\sum_{\ell=0}^{k}\gamma_\ell(v)\overline{\gamma_\ell(w)}$$
$$= \delta_k^{[k]}(v)\overline{\delta_k^{[k]}(w)} - b_{\alpha_k}(v)\gamma_k(v)\overline{b_{\alpha_k}(w)\gamma_k(w)} + \frac{1-|\alpha_k|^2}{(1-\bar{\alpha}_k v)(1-\alpha_k\bar{w})}, \tag{4.2}$$
$$(1 - b_{\alpha_k}(v)\overline{b_{\alpha_k}(w)})\sum_{\ell=0}^{k}\delta_\ell(v)\overline{\delta_\ell(w)}$$
$$= \gamma_k^{[k]}(v)\overline{\gamma_k^{[k]}(w)} - b_{\alpha_k}(v)\delta_k(v)\overline{b_{\alpha_k}(w)\delta_k(w)} - \frac{1-|\alpha_k|^2}{(1-\bar{\alpha}_k v)(1-\alpha_k\bar{w})}, \tag{4.3}$$

and

$$(1 - b_{\alpha_k}(v)\overline{b_{\alpha_k}(w)})\sum_{\ell=0}^{k}\gamma_\ell(v)\overline{\delta_\ell(w)} = \delta_k^{[k]}(v)\overline{\gamma_k^{[k]}(w)} - b_{\alpha_k}(v)\gamma_k(v)\overline{b_{\alpha_k}(w)\delta_k(w)}. \tag{4.4}$$

Proof. According to the definition, we have

$$\gamma_0(v) = \sqrt{\frac{1-|\alpha_0|^2}{1-|\kappa_0|^2}}\frac{1}{1-\bar{\alpha}_0 v}, \qquad \delta_0(v) = \bar{\kappa}_0\sqrt{\frac{1-|\alpha_0|^2}{1-|\kappa_0|^2}}\frac{1}{1-\bar{\alpha}_0 v}.$$

Hence, in view of (4.1), using

$$(1-|\alpha_0|^2)(1-v\bar{w}) = 1-\bar{\alpha}_0 v - \alpha_0\bar{w} + |\alpha_0|^2 v\bar{w} - (|\alpha_0|^2 - \bar{\alpha}_0 v - \alpha_0\bar{w} + v\bar{w})$$
$$= (1-\bar{\alpha}_0 v)(1-\alpha_0\bar{w}) - (\alpha_0 - v)(\bar{\alpha}_0 - \bar{w}),$$

(2.11) and (2.3), we get

$$(1 - b_{\alpha_0}(v)\overline{b_{\alpha_0}(w)})\sum_{\ell=0}^{0}\gamma_\ell(v)\overline{\gamma_\ell(w)}$$
$$= \frac{(1-|\alpha_0|^2)(1-v\bar{w})}{(1-\bar{\alpha}_0 v)(1-\alpha_0\bar{w})}\frac{1-|\alpha_0|^2}{1-|\kappa_0|^2}\frac{1}{(1-\bar{\alpha}_0 v)(1-\alpha_0\bar{w})}$$

$$= \frac{(1-|\alpha_0|^2)(1-\bar{\alpha}_0 v)(1-\alpha_0\bar{w})}{(1-|\kappa_0|^2)(1-\bar{\alpha}_0 v)^2(1-\alpha_0\bar{w})^2} - \frac{(1-|\alpha_0|^2)(\alpha_0 - v)(\bar{\alpha}_0 - \bar{w})}{(1-|\kappa_0|^2)(1-\bar{\alpha}_0 v)^2(1-\alpha_0\bar{w})^2}$$
$$= \frac{(|\kappa_0|^2 - |\kappa_0|^2 + 1)(1-|\alpha_0|^2)}{(1-|\kappa_0|^2)(1-\bar{\alpha}_0 v)(1-\alpha_0\bar{w})} - \frac{(1-|\alpha_0|^2)(\alpha_0 - v)(\bar{\alpha}_0 - \bar{w})}{(1-|\kappa_0|^2)(1-\bar{\alpha}_0 v)^2(1-\alpha_0\bar{w})^2}$$
$$= \delta_0^{[0]}(v)\overline{\delta_0^{[0]}(w)} - b_{\alpha_0}(v)\gamma_0(v)\overline{b_{\alpha_0}(w)\gamma_0(w)} + \frac{1-|\alpha_0|^2}{(1-\bar{\alpha}_0 v)(1-\alpha_0\bar{w})}. \tag{4.5}$$

Thus, for the case when $k = 0$, the first identity is verified. Now we assume that, for $k \in \mathbb{I} \setminus \{0\}$, the formula

$$(1 - b_{\alpha_{k-1}}(v)\overline{b_{\alpha_{k-1}}(w)}) \sum_{\ell=0}^{k-1} \gamma_\ell(v)\overline{\gamma_\ell(w)}$$
$$= \delta_{k-1}^{[k-1]}(v)\overline{\delta_{k-1}^{[k-1]}(w)} - b_{\alpha_{k-1}}(v)\gamma_{k-1}(v)\overline{b_{\alpha_{k-1}}(w)\gamma_{k-1}(w)} + \frac{1-|\alpha_{k-1}|^2}{(1-\bar{\alpha}_{k-1}v)(1-\alpha_{k-1}\bar{w})}$$

has already been proved. Therefore, an application of the first equality in Lemma 4.1 and (4.1) implies that

$$(1 - b_{\alpha_k}(v)\overline{b_{\alpha_k}(w)}) \sum_{\ell=0}^{k} \gamma_\ell(v)\overline{\gamma_\ell(w)}$$
$$= \frac{1 - b_{\alpha_k}(v)\overline{b_{\alpha_k}(w)}}{1 - b_{\alpha_{k-1}}(v)\overline{b_{\alpha_{k-1}}(w)}} (1 - b_{\alpha_{k-1}}(v)\overline{b_{\alpha_{k-1}}(w)}) \sum_{\ell=0}^{k-1} \gamma_\ell(v)\overline{\gamma_\ell(w)}$$
$$+ \gamma_k(v)\overline{\gamma_k(w)} - b_{\alpha_k}(v)\gamma_k(v)\overline{b_{\alpha_k}(w)\gamma_k(w)}$$
$$= \frac{1 - b_{\alpha_k}(v)\overline{b_{\alpha_k}(w)}}{1 - b_{\alpha_{k-1}}(v)\overline{b_{\alpha_{k-1}}(w)}} (\delta_{k-1}^{[k-1]}(v)\overline{\delta_{k-1}^{[k-1]}(w)} - b_{\alpha_{k-1}}(v)\gamma_{k-1}(v)\overline{b_{\alpha_{k-1}}(w)\gamma_{k-1}(w)})$$
$$+ \frac{(1 - b_{\alpha_k}(v)\overline{b_{\alpha_k}(w)})(1-|\alpha_{k-1}|^2)}{(1 - b_{\alpha_{k-1}}(v)\overline{b_{\alpha_{k-1}}(w)})(1-\bar{\alpha}_{k-1}v)(1-\alpha_{k-1}\bar{w})}$$
$$+ \gamma_k(v)\overline{\gamma_k(w)} - b_{\alpha_k}(v)\gamma_k(v)\overline{b_{\alpha_k}(w)\gamma_k(w)}$$
$$= \delta_k^{[k]}(v)\overline{\delta_k^{[k]}(w)} - \gamma_k(v)\overline{\gamma_k(w)} + \frac{1-|\alpha_k|^2}{(1-\bar{\alpha}_k v)(1-\alpha_k\bar{w})}$$
$$+ \gamma_k(v)\overline{\gamma_k(w)} - b_{\alpha_k}(v)\gamma_k(v)\overline{b_{\alpha_k}(w)\gamma_k(w)}$$
$$= \delta_k^{[k]}(v)\overline{\delta_k^{[k]}(w)} - b_{\alpha_k}(v)\gamma_k(v)\overline{b_{\alpha_k}(w)\gamma_k(w)} + \frac{1-|\alpha_k|^2}{(1-\bar{\alpha}_k v)(1-\alpha_k\bar{w})}.$$

Consequently, for each $k \in \mathbb{I}$, the formula (4.2) is inductively shown. Similarly, (4.3) and (4.4) can be verified by using Lemma 4.1 and (4.1). $\square$

Obviously, the formulae in Theorem 4.2 can be restated as follows.

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Corollary 4.3. *For $k \in \mathbb{I} \setminus \{0\}$, the following Christoffel–Darboux formulae hold:*

$$(1 - b_{\alpha_k}(v)\overline{b_{\alpha_k}(w)}) \sum_{\ell=0}^{k-1} \gamma_\ell(v)\overline{\gamma_\ell(w)} = \delta_k^{[k]}(v)\overline{\delta_k^{[k]}(w)} - \gamma_k(v)\overline{\gamma_k(w)} + \frac{1 - |\alpha_k|^2}{(1 - \bar{\alpha}_k v)(1 - \alpha_k \bar{w})}, \tag{4.6}$$

$$(1 - b_{\alpha_k}(v)\overline{b_{\alpha_k}(w)}) \sum_{\ell=0}^{k-1} \delta_\ell(v)\overline{\delta_\ell(w)} = \gamma_k^{[k]}(v)\overline{\gamma_k^{[k]}(w)} - \delta_k(v)\overline{\delta_k(w)} - \frac{1 - |\alpha_k|^2}{(1 - \bar{\alpha}_k v)(1 - \alpha_k \bar{w})}, \tag{4.7}$$

$$(1 - b_{\alpha_k}(v)\overline{b_{\alpha_k}(w)}) \sum_{\ell=0}^{k-1} \gamma_\ell(v)\overline{\delta_\ell(w)} = \delta_k^{[k]}(v)\overline{\gamma_k^{[k]}(w)} - \gamma_k(v)\overline{\delta_k(w)}. \tag{4.8}$$

Using the same strategy as in the case of orthogonal rational functions (cf. [**4**, Corollary 3.1.4]), from the Christoffel–Darboux formulae (4.6) and (4.7) with $v = w$, one can conclude the following (cf. Corollary 3.6).

Corollary 4.4. *For $k \in \mathbb{I} \setminus \{0\}$, if $w \in \mathbb{C}$ with $|\kappa_0| > |b_{\alpha_0}(w)|$, then $\delta_k^{[k]}(w) \neq 0$ and*

$$\left| \frac{\gamma_k(w)}{\delta_k^{[k]}(w)} \right| < 1,$$

and for each $v \in \mathbb{D} \cup \mathbb{T}$ we have $\gamma_k^{[k]}(v) \neq 0$ and

$$\left| \frac{\delta_k(v)}{\gamma_k^{[k]}(v)} \right| < 1.$$

Remark 4.5. Let $k \in \mathbb{I} \setminus \{0\}$. Since $x^{[k]}(\alpha_k) = 0 \iff x \in \mathfrak{H}_{k-1}$ for each $x \in \mathfrak{H}_k$, Corollary 4.4 implies particularly that $\gamma_k \in \mathfrak{H}_k \setminus \mathfrak{H}_{k-1}$ and that if $|\kappa_0| > |b_{\alpha_0}(\alpha_k)|$, then also $\delta_k \in \mathfrak{H}_k \setminus \mathfrak{H}_{k-1}$. Furthermore, the case when $\delta_k \in \mathfrak{H}_{k-1}$ is possible in general (cf. Remarks 3.8 and 3.9) and Theorem 4.2 implies that

$$\delta_k \in \mathfrak{H}_{k-1} \iff \sum_{\ell=0}^{k} \gamma_\ell(v)\overline{\gamma_\ell(\alpha_k)} = \frac{1}{1 - \bar{\alpha}_k v} \iff \sum_{\ell=0}^{k} |\gamma_\ell(\alpha_k)|^2 = \frac{1}{1 - |\alpha_k|^2}$$
$$\iff \sum_{\ell=0}^{k} \gamma_\ell(v)\overline{\delta_\ell(\alpha_k)} = 0 \iff \sum_{\ell=0}^{k} \gamma_\ell(\alpha_k)\overline{\delta_\ell(\alpha_k)} = 0.$$

5. A characterization of Schur–Nevanlinna sequences

In the previous section (see, for example, Theorem 4.2), we explained that Schur–Nevanlinna sequences of rational functions fulfil some Christoffel–Darboux formulae. In this section we study an inverse problem. Roughly speaking, we shall see that (as in the case of orthogonal rational functions) the realization of Christoffel–Darboux formulae is in a way also a sufficient condition for rational functions to be a Schur–Nevanlinna pair.

Remark 5.1. Let $\tau \in \mathbb{N}$ or $\tau = \infty$, let $\mathbb{I} := \{0, 1, 2, \dots, \tau\}$ and let $(\alpha_k)_{k\in\mathbb{I}}$ be a sequence of points belonging to $\mathbb{D}$. Furthermore, let $k \in \mathbb{I} \setminus \{0\}$ and let $\gamma_\ell, \delta_\ell \in \mathfrak{H}_\ell$, $\ell = 0, 1, \dots, k$. Clearly, the following statements are equivalent:

(i) the first (respectively, second or third) identity of Theorem 4.2 is satisfied;

(ii) the first (respectively, second or third) identity of Corollary 4.3 is satisfied.

Lemma 5.2. *Let $\tau \in \mathbb{N}$ or $\tau = \infty$, let $\mathbb{I} := \{0, 1, 2, \dots, \tau\}$ and let $(\alpha_k)_{k\in\mathbb{I}}$ be a sequence of points belonging to $\mathbb{D}$. Furthermore, let $k \in \mathbb{I} \setminus \{0\}$, let $\gamma_k, \delta_k \in \mathfrak{H}_k$ and $\gamma_{k-1}, \delta_{k-1} \in \mathfrak{H}_{k-1}$. The following statements are equivalent:*

(i) *the first identity of Lemma 4.1 is satisfied;*

(ii) *the second identity of Lemma 4.1 is satisfied.*

Proof. If we fix the complex number w then, in view of (2.11) and by forming the adjoint with respect to the $k+2$ points $\alpha_0, \alpha_1, \dots, \alpha_k, \alpha_{k-1}$, the first identity of Lemma 4.1 is equal to

$$\begin{aligned}(b_{\alpha_{k-1}}(v) - b_{\alpha_{k-1}}(w))(\delta_k(v)\delta_k^{[k]}(w) - \gamma_k^{[k]}(v)\gamma_k(w))\\ = (b_{\alpha_k}(v) - b_{\alpha_k}(w))(b_{\alpha_{k-1}}(v)\delta_{k-1}(v)\delta_{k-1}^{[k-1]}(w) - \gamma_{k-1}^{[k-1]}(v)b_{\alpha_{k-1}}(w)\gamma_{k-1}(w)).\end{aligned}$$

Since, by fixing now the point v and taking the adjoint, this relation is equal to

$$\begin{aligned}(\overline{b_{\alpha_{k-1}}(v)}b_{\alpha_{k-1}}(w) - 1)(\overline{\delta_k(v)}\delta_k(w) - \overline{\gamma_k^{[k]}(v)}\gamma_k^{[k]}(w))\\ = (\overline{b_{\alpha_k}(v)}b_{\alpha_k}(w) - 1)(\overline{b_{\alpha_{k-1}}(v)\delta_{k-1}(v)}b_{\alpha_{k-1}}(w)\delta_{k-1}(w) - \gamma_{k-1}^{[k-1]}(v)\gamma_{k-1}^{[k-1]}(w)),\end{aligned}$$

we finally obtain the equivalence of (i) and (ii). □

Lemma 5.3. *Let $\tau \in \mathbb{N}$ or $\tau = \infty$, let $\mathbb{I} := \{0, 1, 2, \dots, \tau\}$ and let $(\alpha_k)_{k\in\mathbb{I}}$ be a sequence of points belonging to $\mathbb{D}$. Further, let $(\gamma_k)_{k\in\mathbb{I}}$ and $(\delta_k)_{k\in\mathbb{I}}$ be sequences of rational functions such that γ_0, δ_0 are given as in (3.1) for some $\kappa_0 \in \mathbb{D}$ and that γ_k, δ_k belong to $\mathfrak{H}_k$, $k \in \mathbb{I} \setminus \{0\}$. The following statements are equivalent:*

(i) *for each $k \in \mathbb{I} \setminus \{0\}$, the first (respectively, second or third) identity of Lemma 4.1 is satisfied;*

(ii) *for each $k \in \mathbb{I} \setminus \{0\}$, the first (respectively, second or third) identity of Theorem 4.2 is satisfied.*

Proof. By using the same arguments as in the proof of Theorem 4.2, one can inductively show that (i) implies (ii). It remains to verify that (ii) also implies (i). For each $k \in \mathbb{I} \setminus \{0\}$, it follows from (4.1), Remark 5.1, the first identity of Corollary 4.3, (4.5) and the first identity of Theorem 4.2 that

$$
\begin{aligned}
&(1 - b_{\alpha_{k-1}}(v)\overline{b_{\alpha_{k-1}}(w)})(\delta_k^{[k]}(v)\overline{\delta_k^{[k]}(w)} - \gamma_k(v)\overline{\gamma_k(w)}) \\
&\quad = (1 - b_{\alpha_{k-1}}(v)\overline{b_{\alpha_{k-1}}(w)})\left(\delta_k^{[k]}(v)\overline{\delta_k^{[k]}(w)} - \gamma_k(v)\overline{\gamma_k(w)} + \frac{1-|\alpha_k|^2}{(1-\bar{\alpha}_k v)(1-\alpha_k\bar{w})}\right) \\
&\qquad - \frac{(1-|\alpha_k|^2)(1-|\alpha_{k-1}|^2)(1-v\bar{w})}{(1-\bar{\alpha}_k v)(1-\alpha_k\bar{w})(1-\bar{\alpha}_{k-1}v)(1-\alpha_{k-1}\bar{w})} \\
&\quad = (1 - b_{\alpha_{k-1}}(v)\overline{b_{\alpha_{k-1}}(w)})(1 - b_{\alpha_k}(v)\overline{b_{\alpha_k}(w)})\sum_{\ell=0}^{k-1}\gamma_\ell(v)\overline{\gamma_\ell(w)} \\
&\qquad - \frac{(1-|\alpha_k|^2)(1-|\alpha_{k-1}|^2)(1-v\bar{w})}{(1-\bar{\alpha}_k v)(1-\alpha_k\bar{w})(1-\bar{\alpha}_{k-1}v)(1-\alpha_{k-1}\bar{w})} \\
&\quad = (1 - b_{\alpha_k}(v)\overline{b_{\alpha_k}(w)})(\delta_{k-1}^{[k-1]}(v)\overline{\delta_{k-1}^{[k-1]}(w)} - b_{\alpha_{k-1}}(v)\gamma_{k-1}(v)\overline{b_{\alpha_{k-1}}(w)\gamma_{k-1}(w)}).
\end{aligned}
$$

Consequently, with respect to the first kind of identity it is shown that (ii) yields (i). Similarly, one can prove by a straightforward calculation that the implication referring to the second (respectively, third) kind of identities in Theorem 4.2 and Lemma 4.1 are correct. □

Theorem 5.4. *Let $\tau = 0$ or $\tau \in \mathbb{N}$ or $\tau = \infty$, let $\mathbb{I} := \{0, 1, 2, \dots, \tau\}$ and let $(\alpha_k)_{k\in\mathbb{I}}$ be a sequence of points belonging to $\mathbb{D}$. Furthermore, for each $k \in \mathbb{I}$, let γ_k, δ_k be rational functions belonging to $\mathfrak{H}_k$ such that the following three conditions hold.*

(I) *The first or second identity, respectively, of Theorem 4.2 is achieved.*

(II) *The third identity of Theorem 4.2 is achieved.*

(III) *We have*

$$\arg[\gamma_k^{[k]}(\alpha_{k-1})] = \arg\left[\frac{\eta_k\bar{\eta}_{k-1}}{1-\bar{\alpha}_k\alpha_{k-1}}\gamma_{k-1}^{[k-1]}(\alpha_{k-1})\right],$$

where $\alpha_{-1} := 0$, $\eta_{-1} := -1$ and $\gamma_{-1}^{[-1]}(\alpha_{-1}) := 1$.

The relation $\gamma_k^{[k]}(\alpha_{k-1}) \neq 0$ holds and if we set

$$\kappa_k := \bar{\eta}_k\eta_{k-1}\frac{\overline{\delta_k(\alpha_{k-1})}}{\gamma_k^{[k]}(\alpha_{k-1})}, \quad k \in \mathbb{I},$$

then $(\kappa_k)_{k\in\mathbb{I}}$ is a sequence of points belonging to $\mathbb{D}$ and $[(\gamma_k)_{k\in\mathbb{I}}, (\delta_k)_{k\in\mathbb{I}}]$ is the Schur–Nevanlinna pair of rational functions corresponding to $(\alpha_k, \kappa_k)_{k\in\mathbb{I}}$.

Proof. First, we consider the case $k = 0$. From (I) and (II) we get

$$\gamma_0(v)\overline{\gamma_0(w)} = \delta_0^{[0]}(v)\overline{\delta_0^{[0]}(w)} + \frac{1-|\alpha_0|^2}{(1-\bar{\alpha}_0 v)(1-\alpha_0\bar{w})} \tag{5.1}$$

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or

$$\delta_0(v)\overline{\delta_0(w)} = \gamma_0^{[0]}(v)\overline{\gamma_0^{[0]}(w)} - \frac{1-|\alpha_0|^2}{(1-\bar{\alpha}_0 v)(1-\alpha_0\bar{w})}, \tag{5.2}$$

and

$$\gamma_0(v)\overline{\delta_0(w)} = \delta_0^{[0]}(v)\overline{\gamma_0^{[0]}(w)}. \tag{5.3}$$

Since (5.1) (respectively, (5.2)) implies that $|\gamma_0(v)|^2 \neq 0$ (respectively, $|\gamma_0^{[0]}(v)|^2 \neq 0$), in view of (2.11) it follows that

$$\gamma_0^{[0]}(0) \neq 0$$

and, hence, (5.3) and the definition of κ_0 yield

$$\delta_0^{[0]}(v) = -\eta_0\kappa_0\gamma_0(v).$$

Thus, from (5.1) (respectively, (5.2)), $\gamma_0 \in \mathfrak{H}_0$, (2.11) and (III), i.e. $-\bar{\eta}_0\gamma_0^{[0]}(0) \in [0,\infty)$, one may finally conclude that κ_0 belongs to $\mathbb{D}$ and that (3.1) is satisfied. In particular, for the case when $\tau = 0$, it is shown that $[(\gamma_k)_{k\in\mathbb{I}}, (\delta_k)_{k\in\mathbb{I}}]$ is the Schur–Nevanlinna pair of rational functions corresponding to $(\alpha_k, \kappa_k)_{k\in\mathbb{I}}$.

Now let $\tau \in \mathbb{N}$ or $\tau = \infty$ and let $k \in \mathbb{I} \setminus \{0\}$. By (I) and Lemmas 5.3 and 5.2 we obtain (cf. Corollary 4.4)

$$\gamma_k^{[k]}(v) \neq 0, \quad v \in \mathbb{D} \cup \mathbb{T},$$

and, by using $b_{\alpha_{k-1}}(\alpha_{k-1}) = 0$, we have

$$\begin{aligned}
&\gamma_k^{[k]}(v)\overline{\gamma_k^{[k]}(\alpha_{k-1})} - \delta_k(v)\overline{\delta_k(\alpha_{k-1})} \\
&\qquad = (1 - b_{\alpha_{k-1}}(v)\overline{b_{\alpha_{k-1}}(\alpha_{k-1})})(\gamma_k^{[k]}(v)\overline{\gamma_k^{[k]}(\alpha_{k-1})} - \delta_k(v)\overline{\delta_k(\alpha_{k-1})}) \\
&\qquad = (1 - b_{\alpha_k}(v)\overline{b_{\alpha_k}(\alpha_{k-1})})\gamma_{k-1}^{[k-1]}(v)\overline{\gamma_{k-1}^{[k-1]}(\alpha_{k-1})}.
\end{aligned} \tag{5.4}$$

In particular, we have

$$\gamma_k^{[k]}(\alpha_{k-1}) \neq 0.$$

Therefore, by the choice of κ_k, (4.1) and (III) it follows that $\kappa_k \in \mathbb{D}$ and

$$\begin{aligned}
\sqrt{1-|\kappa_k|^2} &= \sqrt{1 - \left|\frac{\delta_k(\alpha_{k-1})}{\gamma_k^{[k]}(\alpha_{k-1})}\right|^2} \\
&= \frac{\sqrt{(1-|\alpha_k|^2)(1-|\alpha_{k-1}|^2)}}{|1-\bar{\alpha}_k\alpha_{k-1}|}\left|\frac{\gamma_{k-1}^{[k-1]}(\alpha_{k-1})}{\gamma_k^{[k]}(\alpha_{k-1})}\right| \\
&= \eta_k\bar{\eta}_{k-1}\frac{\sqrt{(1-|\alpha_k|^2)(1-|\alpha_{k-1}|^2)}}{1-\bar{\alpha}_k\alpha_{k-1}}\frac{\gamma_{k-1}^{[k-1]}(\alpha_{k-1})}{\gamma_k^{[k]}(\alpha_{k-1})}.
\end{aligned} \tag{5.5}$$

In view of (2.11), the relation (5.4) implies that

$$\gamma_k(v)\gamma_k^{[k]}(\alpha_{k-1}) - \delta_k^{[k]}(v)\delta_k(\alpha_{k-1}) = (b_{\alpha_k}(v) - b_{\alpha_k}(\alpha_{k-1}))\gamma_{k-1}(v)\gamma_{k-1}^{[k-1]}(\alpha_{k-1}).$$

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Consequently, an application of (5.5) yields

$$\begin{aligned}\eta_k\bar{\eta}_{k-1}\gamma_k(v)-\bar{\kappa}_k\delta_k^{[k]}(v)&=\eta_k\bar{\eta}_{k-1}\gamma_k(v)-\eta_k\bar{\eta}_{k-1}\frac{\delta_k(\alpha_{k-1})}{\gamma_k^{[k]}(\alpha_{k-1})}\delta_k^{[k]}(v)\\&=\eta_k\bar{\eta}_{k-1}\frac{\gamma_{k-1}^{[k-1]}(\alpha_{k-1})}{\gamma_k^{[k]}(\alpha_{k-1})}(b_{\alpha_k}(v)-b_{\alpha_k}(\alpha_{k-1}))\gamma_{k-1}(v)\\&=\frac{(1-\bar{\alpha}_k\alpha_{k-1})\sqrt{1-|\kappa_k|^2}}{\sqrt{(1-|\alpha_k|^2)(1-|\alpha_{k-1}|^2)}}(b_{\alpha_k}(v)-b_{\alpha_k}(\alpha_{k-1}))\gamma_{k-1}(v).\end{aligned}\tag{5.6}$$

Furthermore, from (II), Lemma 5.3 and $b_{\alpha_{k-1}}(\alpha_{k-1})=0$ it follows that

$$\begin{aligned}&\delta_k^{[k]}(v)\overline{\gamma_k^{[k]}(\alpha_{k-1})}-\gamma_k(v)\overline{\delta_k(\alpha_{k-1})}\\&\quad=(1-b_{\alpha_{k-1}}(v)\overline{b_{\alpha_{k-1}}(\alpha_{k-1})})(\delta_k^{[k]}(v)\overline{\gamma_k^{[k]}(\alpha_{k-1})}-\gamma_k(v)\overline{\delta_k(\alpha_{k-1})})\\&\quad=(1-b_{\alpha_k}(v)\overline{b_{\alpha_k}(\alpha_{k-1})})\delta_{k-1}^{[k-1]}(v)\overline{\gamma_{k-1}^{[k-1]}(\alpha_{k-1})}.\end{aligned}$$

By forming the adjoint rational functions, we get

$$\delta_k(v)\gamma_k^{[k]}(\alpha_{k-1})-\gamma_k^{[k]}(v)\delta_k(\alpha_{k-1})=(b_{\alpha_k}(v)-b_{\alpha_k}(\alpha_{k-1}))\delta_{k-1}(v)\gamma_{k-1}^{[k-1]}(\alpha_{k-1}).$$

Accordingly, the equality (5.5) provides

$$\begin{aligned}\eta_k\bar{\eta}_{k-1}\delta_k(v)-\bar{\kappa}_k\gamma_k^{[k]}(v)&=\eta_k\bar{\eta}_{k-1}\delta_k(v)-\eta_k\bar{\eta}_{k-1}\frac{\delta_k(\alpha_{k-1})}{\gamma_k^{[k]}(\alpha_{k-1})}\gamma_k^{[k]}(v)\\&=\eta_k\bar{\eta}_{k-1}\frac{\gamma_{k-1}^{[k-1]}(\alpha_{k-1})}{\gamma_k^{[k]}(\alpha_{k-1})}(b_{\alpha_k}(v)-b_{\alpha_k}(\alpha_{k-1}))\delta_{k-1}(v)\\&=\frac{(1-\bar{\alpha}_k\alpha_{k-1})\sqrt{1-|\kappa_k|^2}}{\sqrt{(1-|\alpha_k|^2)(1-|\alpha_{k-1}|^2)}}(b_{\alpha_k}(v)-b_{\alpha_k}(\alpha_{k-1}))\delta_{k-1}(v).\end{aligned}\tag{5.7}$$

Finally, by virtue of (5.6), (5.7) and Proposition 3.1 we can conclude that $[(\gamma_k)_{k\in\mathbb{I}},(\delta_k)_{k\in\mathbb{I}}]$ is the Schur–Nevanlinna pair of rational functions corresponding to $(\alpha_k,\kappa_k)_{k\in\mathbb{I}}$. □

6. Solution of problem (MNP) in the non-uniqueness case

In this section, we shall show that the set $\mathcal{S}_\Delta$ of solutions of problem (MNP) can be parametrized by the linear fractional transformation (1.2), where γ_m and δ_m are some elements of a Schur–Nevanlinna pair of rational functions with m defined as in (2.2), if a dataset Δ (as in (2.1)) is given such that $\boldsymbol{P}_\Delta>0$.

With the points $z_1,z_2,\dots,z_n$ in Δ we form here a sequence $(\alpha_k)_{k=0}^m$ in which z_j appears according to its multiplicity l_j. For instance, we can choose $\alpha_k:=\beta_k$ with

$$\beta_k:=z_j\quad\text{if}\ \sum_{r=1}^{j-1}l_r\leqslant k<\sum_{r=1}^{j}l_r,\ j=1,2,\dots,n.$$

But in fact, in the following it is not essential that equal points are successors, i.e. for an arbitrary bijective mapping p of $\{0,1,\dots,m\}$ onto itself we can put

$$\alpha_k := \beta_{p(k)}, \quad k = 0,1,\dots,m. \tag{6.1}$$

In the following $[(\gamma_k)_{k\in\mathbb{I}},(\delta_k)_{k\in\mathbb{I}}]$ stands for the Schur–Nevanlinna pair of rational functions corresponding to $(\alpha_k,\kappa_k)_{k\in\mathbb{I}}$, where $\mathbb{I} := \{0,1,\dots,m\}$ and $(\kappa_k)_{k\in\mathbb{I}}$ is a certain sequence of points belonging to $\mathbb{D}$.

Note that, Corollary 3.6 implies that, for all $h \in \mathcal{S}$,

$$\gamma_m^{[m]}(z) + b_{\alpha_m}(z)\delta_m(z)h(z) \neq 0, \quad z \in \mathbb{D}, \tag{6.2}$$

and, moreover, that the function

$$g_0(z) := \frac{\delta_m^{[m]}(z)}{\gamma_m^{[m]}(z)}, \quad z \in \mathbb{D}, \tag{6.3}$$

belongs to $\mathcal{S}$.

Lemma 6.1. *For each $h \in \mathcal{S}$, the function g, where*

$$g(z) := \frac{\delta_m^{[m]}(z) + b_{\alpha_m}(z)\gamma_m(z)h(z)}{\gamma_m^{[m]}(z) + b_{\alpha_m}(z)\delta_m(z)h(z)}, \quad z \in \mathbb{D}, \tag{6.4}$$

belongs to $\mathcal{S}$ and $\Delta_g = \Delta_{g_0}$.

Proof. Let $z \in \mathbb{D}$. In view of (2.6), with

$$\boldsymbol{\Theta}(z) := \frac{1-\bar{\alpha}_m z}{\sqrt{1-|\alpha_m|^2}} \begin{pmatrix} b_{\alpha_m}(z)\gamma_m(z) & \delta_m^{[m]}(z) \\ b_{\alpha_m}(z)\delta_m(z) & \gamma_m^{[m]}(z) \end{pmatrix},$$

the relation (6.4) can be written as

$$g(z) = T_{\boldsymbol{\Theta}(z)}(h(z)).$$

From Theorem 3.3 we see that the matrix $\boldsymbol{\Theta}(z)$ is $\boldsymbol{J}$-contractive. Therefore, $\boldsymbol{\Theta}$ is a $\boldsymbol{J}$-contractive holomorphic matrix function and, as a well-known result on linear fractional transformations (cf. [**9**, Theorem 1.6.1]), $T_{\boldsymbol{\Theta}}$ maps the class $\mathcal{S}$ into the class $\mathcal{S}$. Hence, it follows that, according to (6.4), we have $g \in \mathcal{S}$ if $h \in \mathcal{S}$.

Now let the function g_0 be defined as in (6.3). Consequently, by virtue of (6.2), (6.4) and Corollary 3.4 we obtain

$$\begin{aligned} g(z) - g_0(z) &= \frac{\delta_m^{[m]}(z) + b_{\alpha_m}(z)\gamma_m(z)h(z)}{\gamma_m^{[m]}(z) + b_{\alpha_m}(z)\delta_m(z)h(z)} - \frac{\delta_m^{[m]}(z)}{\gamma_m^{[m]}(z)} \\ &= \frac{b_{\alpha_m}(z)h(z)(\gamma_m(z)\gamma_m^{[m]}(z) - \delta_m(z)\delta_m^{[m]}(z))}{(\gamma_m^{[m]}(z) + b_{\alpha_m}(z)\delta_m(z)h(z))\gamma_m^{[m]}(z)} \\ &= \frac{-\eta_m(1-|\alpha_m|^2)h(z)}{(1-\bar{\alpha}_m z)^2(\gamma_m^{[m]}(z) + b_{\alpha_m}(z)\delta_m(z)h(z))\gamma_m^{[m]}(z)} B_m(z). \end{aligned} \tag{6.5}$$

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Since the Blaschke product B_m has a zero of order l_j at the point z_j, $j = 1, 2, \dots, n$, one can finally conclude that

$$g^{(s)}(z_j) = g_0^{(s)}(z_j), \quad s = 0, 1, \dots, l_j - 1, \ j = 1, 2, \dots, n,$$

i.e. $\Delta_g = \Delta_{g_0}$. □

Lemma 6.2. *If $[(\tilde{\gamma}_k)_{k\in\mathbb{I}}, (\tilde{\delta}_k)_{k\in\mathbb{I}}]$ is a Schur–Nevanlinna pair of rational functions corresponding to $(\alpha_k, \tilde{\kappa}_k)_{k\in\mathbb{I}}$ with some $\tilde{\kappa}_k \in \mathbb{D}$, $k \in \mathbb{I}$, such that $\Delta_{\tilde{g}_0} = \Delta_{g_0}$, where g_0 is defined as in (6.3) and the function $\tilde{g}_0$ similarly by*

$$\tilde{g}_0(z) := \frac{\tilde{\delta}_m^{[m]}(z)}{\tilde{\gamma}_m^{[m]}(z)}, \quad z \in \mathbb{D}, \tag{6.6}$$

then for each $k \in \mathbb{I}$ the equality $\tilde{\kappa}_k = \kappa_k$ holds, i.e. $[(\tilde{\gamma}_k)_{k\in\mathbb{I}}, (\tilde{\delta}_k)_{k\in\mathbb{I}}]$ is the Schur–Nevanlinna pair of rational functions corresponding to $(\alpha_k, \kappa_k)_{k\in\mathbb{I}}$.

Proof. Since (3.1) and (2.11) imply that

$$\tilde{\kappa}_0 = \frac{\tilde{\delta}_0^{[0]}(z)}{\tilde{\gamma}_0^{[0]}(z)}, \qquad \kappa_0 = \frac{\delta_0^{[0]}(z)}{\gamma_0^{[0]}(z)}, \tag{6.7}$$

in the case $m = 0$ it evidently follows that $\tilde{\kappa}_0 = \kappa_0$, i.e. that $[(\tilde{\gamma}_k)_{k\in\mathbb{I}}, (\tilde{\delta}_k)_{k\in\mathbb{I}}]$ is the Schur–Nevanlinna pair of rational functions corresponding to $(\alpha_k, \kappa_k)_{k\in\mathbb{I}}$.

Now let $m > 0$. In view of the recursions defining a Schur–Nevanlinna pair of rational functions, (2.11) and $\Delta_{\tilde{g}_0} = \Delta_{g_0}$, we find that the values of the functions

$$\frac{\tilde{\delta}_{m-1}^{[m-1]}(z) + b_{\alpha_{m-1}}(z)\tilde{\gamma}_{m-1}(z)\tilde{\kappa}_m}{\tilde{\gamma}_{m-1}^{[m-1]}(z) + b_{\alpha_{m-1}}(z)\tilde{\delta}_{m-1}(z)\tilde{\kappa}_m}, \quad \frac{\delta_{m-1}^{[m-1]}(z) + b_{\alpha_{m-1}}(z)\gamma_{m-1}(z)\kappa_m}{\gamma_{m-1}^{[m-1]}(z) + b_{\alpha_{m-1}}(z)\delta_{m-1}(z)\kappa_m}, \quad z \in \mathbb{D},$$

and their derivatives up to the order $l_j - 1$ at the points z_j, $j = 1, 2, \dots, n$, coincide. Because of Lemma 6.1, a successive continuation of this procedure yields that, for each $k \in \mathbb{I} \setminus \{0\}$, the values of the functions

$$\frac{\tilde{\delta}_{k-1}^{[k-1]}(z) + b_{\alpha_{k-1}}(z)\tilde{\gamma}_{k-1}(z)\tilde{\kappa}_k}{\tilde{\gamma}_{k-1}^{[k-1]}(z) + b_{\alpha_{k-1}}(z)\tilde{\delta}_{k-1}(z)\tilde{\kappa}_k}, \quad \frac{\delta_{k-1}^{[k-1]}(z) + b_{\alpha_{k-1}}(z)\gamma_{k-1}(z)\kappa_k}{\gamma_{k-1}^{[k-1]}(z) + b_{\alpha_{k-1}}(z)\delta_{k-1}(z)\kappa_k}, \quad z \in \mathbb{D}, \tag{6.8}$$

and their derivatives up to the order $r_j - 1$ at the points z_j contained in the sequence $(\alpha_\ell)_{\ell=0}^k$ (where r_j stands for the multiplicity of z_j in the sequence $(\alpha_\ell)_{\ell=0}^k$) coincide and, in particular, that

$$\frac{\tilde{\delta}_0^{[0]}(\alpha_0)}{\tilde{\gamma}_0^{[0]}(\alpha_0)} = \frac{\delta_0^{[0]}(\alpha_0)}{\gamma_0^{[0]}(\alpha_0)}. \tag{6.9}$$

In the following, by induction on k, we verify that $\tilde{\kappa}_k = \kappa_k$, $k \in \mathbb{I}$. For $k = 0$, the equalities (6.7) and (6.9) supply immediately

$$\tilde{\kappa}_0 = \kappa_0.$$

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Now let $k \in \mathbb{I} \setminus \{0\}$ and we assume that $\tilde{\kappa}_\ell = \kappa_\ell$, $\ell = 0, 1, \dots, k-1$. In view of (6.8) and the induction assumption, we find that the values of the functions

$$\frac{\delta_{k-1}^{[k-1]}(z) + b_{\alpha_{k-1}}(z)\gamma_{k-1}(z)\tilde{\kappa}_k}{\gamma_{k-1}^{[k-1]}(z) + b_{\alpha_{k-1}}(z)\delta_{k-1}(z)\tilde{\kappa}_k}, \quad \frac{\delta_{k-1}^{[k-1]}(z) + b_{\alpha_{k-1}}(z)\gamma_{k-1}(z)\kappa_k}{\gamma_{k-1}^{[k-1]}(z) + b_{\alpha_{k-1}}(z)\delta_{k-1}(z)\kappa_k}, \quad z \in \mathbb{D},$$

and their derivatives up to the order $r_j - 1$ at the points z_j contained in the sequence $(\alpha_\ell)_{\ell=0}^k$ (where r_j stands again for the multiplicity of z_j in the sequence $(\alpha_\ell)_{\ell=0}^k$) coincide on the one hand, and on the other hand Corollary 3.4 provides

$$\begin{aligned}
&\frac{\delta_{k-1}^{[k-1]}(z) + b_{\alpha_{k-1}}(z)\gamma_{k-1}(z)\tilde{\kappa}_k}{\gamma_{k-1}^{[k-1]}(z) + b_{\alpha_{k-1}}(z)\delta_{k-1}(z)\tilde{\kappa}_k} - \frac{\delta_{k-1}^{[k-1]}(z) + b_{\alpha_{k-1}}(z)\gamma_{k-1}(z)\kappa_k}{\gamma_{k-1}^{[k-1]}(z) + b_{\alpha_{k-1}}(z)\delta_{k-1}(z)\kappa_k} \\
&\quad = \frac{(\tilde{\kappa}_k - \kappa_k) b_{\alpha_{k-1}}(z)(\gamma_{k-1}(z)\gamma_{k-1}^{[k-1]}(z) - \delta_{k-1}(z)\delta_{k-1}^{[k-1]}(z))}{(\gamma_{k-1}^{[k-1]}(z) + b_{\alpha_{k-1}}(z)\delta_{k-1}(z)\tilde{\kappa}_k)(\gamma_{k-1}^{[k-1]}(z) + b_{\alpha_{k-1}}(z)\delta_{k-1}(z)\kappa_k)} \\
&\quad = \frac{-(\tilde{\kappa}_k - \kappa_k)\eta_{k-1}(1 - |\alpha_{k-1}|^2) B_{k-1}(z)}{(1 - \bar{\alpha}_{k-1}z)^2(\gamma_{k-1}^{[k-1]}(z) + b_{\alpha_{k-1}}(z)\delta_{k-1}(z)\tilde{\kappa}_k)(\gamma_{k-1}^{[k-1]}(z) + b_{\alpha_{k-1}}(z)\delta_{k-1}(z)\kappa_k)}.
\end{aligned}$$

Since $\eta_{k-1}(1 - |\alpha_{k-1}|^2) \neq 0$ and since the Blaschke product B_{k-1} has only k zeros (at the points $\alpha_0, \alpha_1, \dots, \alpha_{k-1}$), one can finally conclude that $\tilde{\kappa}_k = \kappa_k$. Thus, for each $k \in \mathbb{I}$ the identity $\tilde{\kappa}_k = \kappa_k$ is satisfied, i.e. $[(\tilde{\gamma}_k)_{k\in\mathbb{I}}, (\tilde{\delta}_k)_{k\in\mathbb{I}}]$ is the Schur–Nevanlinna pair of rational functions corresponding to $(\alpha_k, \kappa_k)_{k\in\mathbb{I}}$. □

Now we are able to prove the main result of this paper, i.e. that in the non-uniqueness case the set $\mathcal{S}_\Delta$ of all solutions of problem (MNP) is given by a linear fractional transformation of the form stated in (1.2).

Theorem 6.3. *Let Δ be a dataset as in (2.1) whereby $\boldsymbol{P}_\Delta > 0$. Further, let $g_\bullet \in \mathcal{S}_\Delta$, let $(\alpha_k)_{k=0}^m$ be given as in (6.1) and let $(s_k)_{k=0}^m$ be the sequence of Schur parameters associated with $[g_\bullet, (\alpha_k)_{k=0}^m]$. If $g \in \mathcal{S}$, then the following statements are equivalent:*

(i) *$g \in \mathcal{S}_\Delta$;*

(ii) *$(s_k)_{k=0}^m$ is the sequence of Schur parameters associated with $[g, (\alpha_k)_{k=0}^m]$.*

Moreover, if we put $\mathbb{I} := \{0, 1, \dots, m\}$, $\kappa_0 := s_0$, and $\kappa_k := -s_k\eta_{k-1}$, $k = 1, 2, \dots, m$, and if $[(\gamma_k)_{k\in\mathbb{I}}, (\delta_k)_{k\in\mathbb{I}}]$ stands for the Schur–Nevanlinna pair of rational functions corresponding to $(\alpha_k, \kappa_k)_{k\in\mathbb{I}}$, then the relation

$$g(z) = \frac{\delta_m^{[m]}(z) + b_{\alpha_m}(z)\gamma_m(z)h(z)}{\gamma_m^{[m]}(z) + b_{\alpha_m}(z)\delta_m(z)h(z)}, \quad z \in \mathbb{D},$$

establishes a bijective correspondence between the set $\mathcal{S}_\Delta$ of all solutions g of problem (MNP) and the class $\mathcal{S}$ of all Schur functions h.

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Proof. Let $z \in \mathbb{D}$. Note that, in view of $\boldsymbol{P}_\Delta > 0$ and Theorem 2.3, for any solution of problem (MNP), the Schur–Nevanlinna algorithm can be carried out (at least) $m+1$ times. Consequently, we can always suppose in the following a given Schur function g for which the Schur–Nevanlinna algorithm can be carried out (at least) $m+1$ times. In particular (cf. (2.4) and (2.10)), we find Schur parameters $\tilde{s}_0, \tilde{s}_1, \dots, \tilde{s}_m \in \mathbb{D}$ associated with the pair $[g, (\alpha_k)_{k=0}^m]$ and a unique Schur function h_{m+1} such that the relation

$$g(z) = T_{\boldsymbol{\Xi}(z)}(h_{m+1}(z)) \tag{6.10}$$

is satisfied, where

$$\boldsymbol{\Xi}(z) := \prod_{k=0}^{\stackrel{\curvearrowright}{m}} \widetilde{\boldsymbol{\Xi}}_k(z) \quad (:= \widetilde{\boldsymbol{\Xi}}_0(z)\widetilde{\boldsymbol{\Xi}}_1(z)\cdots\widetilde{\boldsymbol{\Xi}}_m(z))$$

and

$$\widetilde{\boldsymbol{\Xi}}_k(z) := \begin{pmatrix} b_{\alpha_k}(z) & \tilde{s}_k \\ \overline{\tilde{s}_k} b_{\alpha_k}(z) & 1 \end{pmatrix}, \quad k \in \mathbb{I}.$$

By virtue of (2.9) and (3.5), with $\eta_{-1} := -1$, one can also write

$$\boldsymbol{\Xi}(z) = \left(\prod_{k=0}^{\stackrel{\curvearrowright}{m}} \begin{pmatrix} 1 & -\tilde{s}_k\eta_{k-1} \\ -\overline{\tilde{s}_k\eta_{k-1}} & 1 \end{pmatrix} \begin{pmatrix} b_{\alpha_k}(z) & 0 \\ 0 & \eta_k\bar{\eta}_{k-1} \end{pmatrix} \right) \begin{pmatrix} 1 & 0 \\ 0 & -\bar{\eta}_m \end{pmatrix}. \tag{6.11}$$

According to § 3, we now define $[(\tilde{\gamma}_k)_{k\in\mathbb{I}}, (\tilde{\delta}_k)_{k\in\mathbb{I}}]$ as the Schur–Nevanlinna pair of rational functions corresponding to $(\alpha_k, \tilde{\kappa}_k)_{k\in\mathbb{I}}$ with $\tilde{\kappa}_k := -\tilde{s}_k\eta_{k-1}$, $k \in \mathbb{I}$. Thus, setting

$$\widetilde{\boldsymbol{\Theta}}_k(v) := \frac{1}{\sqrt{1-|\tilde{\kappa}_k|^2}} \begin{pmatrix} 1 & \tilde{\kappa}_k \\ \overline{\tilde{\kappa}_k} & 1 \end{pmatrix} \begin{pmatrix} b_{\alpha_k}(v) & 0 \\ 0 & \eta_k\bar{\eta}_{k-1} \end{pmatrix}, \quad k \in \mathbb{I},$$

Theorem 3.3 yields the identity

$$\begin{pmatrix} b_{\alpha_m}(v)\tilde{\gamma}_m(v) & \tilde{\delta}_m^{[m]}(v) \\ b_{\alpha_m}(v)\tilde{\delta}_m(v) & \tilde{\gamma}_m^{[m]}(v) \end{pmatrix} = \frac{\sqrt{1-|\alpha_m|^2}}{1-\bar{\alpha}_m v} \widetilde{\boldsymbol{\Theta}}_0(v)\widetilde{\boldsymbol{\Theta}}_1(v)\cdots\widetilde{\boldsymbol{\Theta}}_m(v)$$

on the one hand, and from (6.11) it follows that

$$\boldsymbol{\Xi}(z) = \left(\prod_{k=0}^{\stackrel{\curvearrowright}{m}} \sqrt{1-|\tilde{\kappa}_k|^2}\, \widetilde{\boldsymbol{\Theta}}_k(z) \right) \begin{pmatrix} 1 & 0 \\ 0 & -\bar{\eta}_m \end{pmatrix}$$

on the other. Hence, by (6.10), (6.2), (2.6) and (2.7), we see that

$$g(z) = \frac{\tilde{\delta}_m^{[m]}(z) + b_{\alpha_m}(z)\tilde{\gamma}_m(z)(-\eta_m h_{m+1}(z))}{\tilde{\gamma}_m^{[m]}(z) + b_{\alpha_m}(z)\tilde{\delta}_m(z)(-\eta_m h_{m+1}(z))}. \tag{6.12}$$

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Moreover, via the construction of the rational functions $\tilde{\gamma}_m$ and $\tilde{\delta}_m$, Lemma 6.1 implies that the Schur function $\tilde{g}_0$ given as in (6.6) satisfies $\Delta_g = \Delta_{\tilde{g}_0}$. In particular, since $g_\bullet \in \mathcal{S}_\Delta$, the considerations above tell us that $g_\bullet$ admits the following representation,

$$g_\bullet(z) = \frac{\delta_m^{[m]}(z) + b_{\alpha_m}(z)\gamma_m(z)(-\eta_m h_\bullet(z))}{\gamma_m^{[m]}(z) + b_{\alpha_m}(z)\delta_m(z)(-\eta_m h_\bullet(z))}$$

for a unique $h_\bullet \in \mathcal{S}$, and that the Schur function g_0 given as in (6.3) satisfies the identity $\Delta_{g_\bullet} = \Delta_{g_0}$. Consequently, if $g \in \mathcal{S}_\Delta$ then

$$\Delta_{\tilde{g}_0} = \Delta_g = \Delta_{g_\bullet} = \Delta_{g_0}$$

and, hence, for each $k \in \mathbb{I}$, Lemma 6.2 yields the identity $\tilde{\kappa}_k = \kappa_k$ (i.e. $\tilde{s}_k = s_k$). Therefore, (i) implicates (ii) and, in addition, (6.12) leads to (1.2). Conversely, if g admits the representation (1.2) for some $h \in \mathcal{S}$, then from Lemma 6.1 one can get

$$\Delta_g = \Delta_{g_0} = \Delta_{g_\bullet},$$

i.e. $g \in \mathcal{S}_\Delta$. Finally, if (ii) is satisfied, then (6.12) implies that g admits the representation (1.2) with $h(z) := -\eta_m h_{m+1}(z)$, $z \in \mathbb{D}$. □

Observe that the equivalence of (i) and (ii) in Theorem 6.3 is closely related to Schur's result that, for each $l \in \mathbb{N}$, there is a one-to-one correspondence between the first l Taylor coefficients of a Schur function at the point $z = 0$ and the first l corresponding Schur parameters. Clearly, on applying appropriate conformal mappings of the open unit disc $\mathbb{D}$ onto itself, one can obtain a similar result with respect to arbitrarily points $z_1, z_2, \dots, z_n \in \mathbb{D}$. Nevertheless, it seems to be really hard and unwieldy to derive directly from this classical result the equivalence of (i) and (ii), since the underlying sequence $(\alpha_k)_{k=0}^m$ has only to satisfy (6.1) and hence the points $\alpha_0, \alpha_1, \dots, \alpha_m$ are not strictly in the order $z_1, z_1, \dots, z_1, z_2, z_2, \dots, z_2, z_n, z_n, \dots, z_n$.

If in (1.2) (i.e. in the description of $\mathcal{S}_\Delta$ by the linear fractional transformation according to Theorem 6.3) the point $z \in \mathbb{D}$ is fixed, then the set

$$\mathfrak{K}_\Delta(z) := \{g(z) : g \in \mathcal{S}_\Delta\} \tag{6.13}$$

is a closed disc in the unit disc $\mathbb{D}$, the boundary of which is sometimes called a *Weyl circle*. Using some well-known properties of linear fractional transformations (cf. [**25**, Proposition 2]), it can easily be shown that the centre c_z and the radius r_z of this Weyl circle are given by

$$c_z = \frac{\delta_m^{[m]}(z)\overline{\gamma_m^{[m]}(z)} - b_{\alpha_m}(z)\gamma_m(z)\overline{b_{\alpha_m}(z)\delta_m(z)}}{|\gamma_m^{[m]}(z)|^2 - |b_{\alpha_m}(z)\delta_m(z)|^2},$$

$$r_z = \frac{|b_{\alpha_m}(z)||\gamma_m(z)\gamma_m^{[m]}(z) - \delta_m(z)\delta_m^{[m]}(z)|}{|\gamma_m^{[m]}(z)|^2 - |b_{\alpha_m}(z)\delta_m(z)|^2}.$$

Consequently, in view of Theorem 4.2, (4.1) and Corollary 3.4 the parameters of this Weyl circle can also be computed by the formulae

$$c_z = \frac{(1-|z|^2)\sum_{\ell=0}^{m} \gamma_\ell(z)\overline{\delta_\ell(z)}}{(1-|z|^2)\sum_{\ell=0}^{m} |\delta_\ell(z)|^2 + 1}, \qquad r_z = \frac{|B_m(z)|}{(1-|z|^2)\sum_{\ell=0}^{m} |\delta_\ell(z)|^2 + 1},$$

where B_m denotes the Blaschke product (of degree $m+1$) with respect to the points $\alpha_0, \alpha_1, \dots, \alpha_m$ given via (6.1). Furthermore, (3.1) and Corollary 4.4 imply that

$$|\gamma_m^{[m]}(z)|^2 - |b_{\alpha_m}(z)\delta_m(z)|^2 > (1 - |b_{\alpha_m}(z)|^2)|\gamma_m^{[m]}(z)|^2 > 0$$

as well as Corollary 3.6 and (4.1) yield

$$\frac{1}{(1-|z|^2)|\gamma_m^{[m]}(z)|^2} \leqslant \frac{1}{1-|b_{\alpha_m}(z)|^2}.$$

To summarize, we have proved the following corollary.

Corollary 6.4. *Under the assumptions of Theorem 6.3, if $z \in \mathbb{D}$ is fixed, then the set $\mathfrak{K}_\Delta(z)$ in (6.13) can be described by*

$$\mathfrak{K}_\Delta(z) = \{w : |w - c_z| \leqslant r_z\},$$

where the parameters c_z and r_z are given by the relations above. In particular, if a point $z \in \mathbb{D} \setminus \{z_1, z_2, \dots, z_n\}$ is fixed, then for each $g \in \mathcal{S}_\Delta$ the following estimate holds:

$$|g(z) - c_z| < \frac{|B_m(z)|}{(1-|z|^2)|\gamma_m^{[m]}(z)|^2} \leqslant \frac{|B_m(z)|}{1-|b_{\alpha_m}(z)|^2}.$$

Following the geometrical considerations, one can also see that the Weyl circle with centre c_z and radius r_z can be described as an Apollonius circle [**5**, **15**].

Corollary 6.5. *Under the assumptions of Theorem 6.3, if $z \in \mathbb{D}$ is fixed, then the set $\mathfrak{K}_\Delta(z)$ in (6.13) can be described by*

$$\mathfrak{K}_\Delta(z) = \left\{v : \left|\frac{v - a_{1,z}}{v - a_{2,z}}\right| \leqslant |b_{\alpha_m}(z)| d_z\right\},$$

where

$$a_{1,z} := \frac{\delta_m^{[m]}(z)}{\gamma_m^{[m]}(z)}, \qquad a_{2,z} := \frac{\gamma_m(z)}{\delta_m(z)}, \qquad d_z := \left|\frac{\delta_m(z)}{\gamma_m^{[m]}(z)}\right|.$$

Remark 6.6. Clearly, $\mathfrak{K}_\Delta(z_j)$, $j = 1, 2, \dots, n$, contains only the value w_{j0}. But following the idea of [**25**, § 6], if we consider instead the set

$$\mathfrak{K}'_\Delta(z_j) := \left\{\frac{1}{l_j!} g^{(l_j)}(z_j) : g \in \mathcal{S}_\Delta\right\}, \quad j = 1, 2, \dots, n,$$

and if, in (6.1), we choose a sequence $(\alpha_k)_{k\in\mathbb{I}}$ so that $\alpha_m = z_j$, then by a straightforward calculation from (6.5) it follows that

$$\mathfrak{K}'_\Delta(z_j) = \{w : |w - c'_{z_j}| \leqslant r'_{z_j}\},$$

where the Schur function g_0 is defined as in (6.3) and

$$c'_{z_j} = \frac{1}{l_j!} g_0^{(l_j)}(z_j), \quad r'_{z_j} = \frac{\prod_{k=1,\, k\neq j}^{n} |b_{z_k}(z_j)|^{l_k}}{|\gamma_m^{[m]}(z_j)|^2 (1-|z_j|^2)^{l_j+1}} \leqslant \frac{1}{(1-|z_j|^2)^{l_j}} \prod_{k=1, k\neq j}^{n} |b_{z_k}(z_j)|^{l_k}.$$

Finally, we point out that the rational functions γ_m and δ_m, which occur in the linear fractional transformation of Theorem 6.3, can be constructed from the interpolation data Δ, but indirectly. One needs to determine the corresponding Schur parameters first, which is not easy to do in general. A work around is the following. Since the definition of Schur–Nevanlinna sequences of rational functions according to § 3 is done with a view to orthogonal rational functions on the unit circle and their recurrence relations presented in [**4**, Chapter 4], one can also use the theory of orthogonal rational functions to compute the corresponding Schur parameters or the functions γ_m and δ_m. This will be explained in detail in a forthcoming work.

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