ON QUASI-LINEAR PARABOLIC EQUATIONS

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To Professor Kiyoshi Noshiro on the occasion of his 60th birthday.

§1 Introduction

In this paper we consider the following quasi-linear parabolic equations

$$Lu = u_t - \operatorname{div} A(x, t, u, u_x) + B(x, t, u, u_x) = 0\ , \tag{1.1}$$

where A is a given vector function of the variables x, t, u, u_x, and B is a given scalar function of the some variables. We assume that they are difined in the rectangle

$$\begin{aligned} R &= \{(x, t) \in E^{n+1} \mid x = (x_1, \ldots\ldots, x_n) \in E^n, |x_i| < 2r, 0 < t < 2r^2\} \\ &= Q_{2r} \times (0, 2r^2),\ \text{where } Q_{2r} = \{x \mid |x_i| < 2r\}\ . \end{aligned}$$

Moreover we assume that

$$\begin{cases} |A(x, t, u, p)| \leqslant M|p| + c(x, t)|u| + e(x, t) \\ |B(x, t, u, p)| \leqslant b(x, t)|p| + d(x, t)|u| + f(x, t) \\ pA(x, t, u, p) \geqslant \lambda |p|^2 - d(x, t)|u|^2 - g(x, t) \end{cases} \tag{1.2}$$

for any real vector $p = (p_1, \ldots\ldots, p_n)$. Here M and λ are positive constants, and $b, c, d, e,\ f$ and g are non-negative functions of the variables $x,\ t$ such that

$$\begin{cases} b, c, e \in L^\infty[0, 2r^2; L^{n+\varepsilon}(Q_{2r})],\ \ d, f, g \in L^\infty[0, 2r^2; L^{\frac{n+\varepsilon}{2}}(Q_{2r})] \\ \text{for arbitrary } \varepsilon > 0 \text{ and} \\ \max\limits_{0<t<2r^2} \|d\|_{n+\varepsilon}(t) + \max\limits_{0<t<2r^2} \|c\|_{n+\varepsilon}(t) + \max\limits_{0<t<2r^2} \|e\|_{n+\varepsilon}(t) + \max\limits_{0<t<2r^2} \|d\|_{\frac{n+\varepsilon}{2}}(t) \\ + \max\limits_{0<t<2r^2} \|f\|_{\frac{n+\varepsilon}{2}}(t) + \max\limits_{0<t<2r^2} \|g\|_{\frac{n+\varepsilon}{2}}(t) < M, \end{cases} \tag{1.3}$$

where $\|w\|_p(t) = \left(\int_{Q_{2r}} |w|^p dx\right)^{1/p}$

We denote by $L^q[0, 2r^2; L^p(Q_{2r})]$ the space of function $\varphi(x, t)$ with the following properties:

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(i) φ is defined and measurable in $R=Q_{2r}\times(0,2r^2)$,

(ii) for almost all $t\in(0,2r^2)$, $\varphi(x,t)\in L^p(Q_{2r})$,

(iii) $\|\varphi\|_{L^p(Q_{2r})}(t)\in L^q(0,2r^2)$.

The function u is said to be a weak solution of (1.1) in R if u with u_x, u_t is square integrable and if u satisfies the following equality

$$(1.4)\qquad \iint_R [u_t\phi+A(x,t,u,u_x)\phi_x+B(x,t,u,u_x)\phi]dxdt=0$$

for any $\phi(x,t)\in H^{1,2}[0,2r^2;L^2(Q_{2r})]\cap L^2[0,2r^2;H_0^{1,2}(Q_{2r})]$.

Let $R'=\{(x,t)|\ |x_i|<2\rho,\ 0<t<2\rho^2\}$,

$R^+=\{(x,t)|\ |x_i|<k\rho,\ h^+\rho^2<t<2\rho^2\}$,

$R^-=\{(x,t)|\ |x_i|<k\rho,\ h_1^-\rho^2<t<h_2^-\rho^2\}$

where ρ, h_1^-, h_2^-, h^+ and k are arbitrary numbers such that $0<\rho\leqslant r$, $0<h_1^-<h_2^-<\frac{2}{3}$, $\frac{4}{3}<h^+<2$ and $0<k<\sqrt{\frac{2}{3}}$.

Then we can prove the following.

THEOREM 1. *If u is a non-negative weak solution of (1.1) in R, then*

$$(1.5)\qquad \max_{R^-} u\leqslant\gamma\min_{R^+}\{u+l(\rho)\},$$

where $l(\rho)=\rho^{\frac{\varepsilon}{n+\varepsilon}}\left\{\max_t\|e\|_{n+\varepsilon}(t)+\max_t\|f\|_{\frac{n+\varepsilon}{2}}(t)+\left(\max_t\|g\|_{\frac{n+\varepsilon}{2}}(t)\right)^{\frac{1}{2}}+1\right\}$ *and*

$\gamma>1$ is a constant depending only on n, ε, λ, M, k, h_1^-, h_2^-, h^+, and γ.

Remark. Moser [3] proved the Harnack inequality

$$(1.6)\qquad \max_{R^-} u\leqslant\gamma\min_{R^+} u$$

for every positive solution u of the uniformely parabolic equations

$$\mathrm{L}u=u_t-\sum_{i,j=1}^n(a_{i,j}(x,t)u_{x_i})_{x_j}=0$$

with measurable coefficients.

Theorem 1 does not imply the inequality (1.6). However, we can get (1.6) by the same argument as in the proof of Theorem 1.

THEOREM 2. *Every weak solution of (1.1) in R is bouoded in subdomain of R.*

We shall give the proof of Theorem 1 in §2 and prove Theorem 2 in §3. In §4 we shall deal with the removable singularities for solutions of parabolic equations (1.1). (cf. [1]). These results are extensions of the results of Serrin [5], who considered the equation

$$-div A(x, u, u_x)+B(x, u, u_x)=0 \quad \text{in } \Omega\subset E^n ,$$

of elliptic type, where

$$|A(x, u, p)|< a|p|^{\alpha-1}+c|u|^{\alpha-1}+e,$$
$$|B(x, u, p)|<b|p|^{\alpha-1}+d|u|^{\alpha-1}+f,$$
$$p\cdot A>|p|^{\alpha}-d|u|^{\alpha}-g$$

for $x\in\Omega$. Here $\alpha>1$ is a fixed exponent, a is a positive constant and if $1<\alpha<n$, then

$$(1.7) \qquad c, e\in L_{n/(\alpha-1)},\ b\in L_{n/(1-\varepsilon)},\ d.\ f, g\in L_{n/(\alpha-\varepsilon)}.$$

Our conditions (1.3) with respect to the coefficients b, d, f, g correspond to the conditions (1.7) in the case $\alpha=2$.

We state two lemmas which will be often used in this paper.

LEMMA *1.* (*Sobolev's Lemma*) (*cf.* [4])

If $u\in H_0^{1,2}(\Omega)$, *then*

$$\left(\int_\Omega |u|^{2*}dx\right)^{\frac{1}{2}*}\leqslant K\sum_{i=1}^n\left(\int_\Omega u_{x_i}^2 dx\right)^{\frac{1}{2}},$$

where K is a positive constant depending only on n, and $\dfrac{1}{2^*}=\dfrac{1}{2}-\dfrac{1}{n}$.

LEMMA 2. *If $f(x,t)$ belongs to $L^\infty[0, 2r^2;\ L^q(Q_{2r})]$ and if* $\max\limits_{0<t<2r^2}\left(\int|f|^q dx\right)^{1/q}\leqslant M$ *for $q>p$, then $f(x,t)$ can be written in the form $f(x,t)=f'(xt)+f''(x,t)$, where* $\max\limits_t\left(\int_{Q_{2r}}|f''|^p dx\right)^{1/p}\leqslant\eta$ *and* $\sup\limits_R |f'|\leqslant K(\eta)$ *for any $\eta>0$ and for a positivwe function $K(\eta)$ of η. Moreover $K(\eta)$ may be taken as the value* $c(M)\eta^{-\frac{p}{q-p}}$, *where $c(M)$ is a constant depending on M, p and q.* (*cf.* [6])

Proof. We put

$$f'(x,t)=\begin{cases} k, & \text{if } k\leqslant f,\\ f(x,t), & \text{if } |f|<k,\\ -k, & \text{if } f\leqslant -k,\end{cases}$$

and $f''=f-f'$. Then

$$\left(\int_{Q_{2r}}|f''|^p dx\right)^{1/p}\leqslant 2\left(\int_{A_k(t)}|f|^p dx\right)^{1/p}\leqslant 2\left(\int_{Q_{2r}}|f|^q dx\right)^{1/q}\left[\text{meas }(A_k(t))\right]^{1/p-1/q}$$

where $A_k(t)=\{x\in Q_{2r}\,|\,|f|>k\}$,

Since meas $(A_k(t))\leqslant k^{-q}\int_{Q_{2r}}|f|^q dx$, we have

$$\left(\int_{Q_{2r}}|f''(|^p dx\right)^{1/p}\leqslant 2M^{\frac{q}{p}}k^{1-\frac{q}{p}}.$$

The right hand side of this inequality does not depend on t. Therefore $\max\limits_{0<t<2r^2}\left(\int_{Q_{2r}}|f''|^p dx\right)^{1/p}\leqslant 2M^{\frac{q}{p}}k^{1-\frac{q}{p}}$. From this we can easily verify the assertion of Lemma.

§2 Harnack's inequality.

In this section we give the proof of Theorem 1, i.e. (1.5), which Kurihara [2] recently has proved under the following conditions

$$b, c, e\in L^\infty[0, 2r^2; L^{2n}(Q_{2r})],\quad d, f, g\in L^\infty[0, 2r^2; L^n(Q_{2r})]. \tag{2. 1}$$

If we prove lemmas corresponding to Lemmas 9 and 10 of Kurihara [2], then the proof of Theorem 1 can be completed, since the remaining part of the proof follows by the same method as Kurihara's.

First, we introduce some notation;

$$\begin{cases} R_{\rho\tau}=\{(x, t)\,|\,|x_i|<\rho, -\tau<t<0\}, \\ H_{\rho\tau}(w)=\rho^{-n}\tau^{-1}\iint_{R_{\rho\tau}} w^2(x, t)dxdt, \\ D_{\rho\tau}(w)=\rho^{-n+2}\tau^{-1}\iint_{R_{\rho\tau}}|w_x|^2 dxdt, \\ M_{\rho\tau}(w)=\rho^{-n}\max\limits_{-\tau<t<0}\int_{Q_\rho} w^2 dx. \end{cases} \tag{2. 2}$$

LEMMA 2. 1. *Assume that u is a non-negative weak solution of (1.1). Let $v=(u+l(\rho))^{q/2}$ and $R_{\rho'\tau'}\subset R_{\rho'\tau'}$,*

(i) *If $q>1$, then*

$$D_{\rho'\tau'}(v)\leqslant c\left(\frac{q}{q-1}\right)^{\frac{2n}{\varepsilon}+2}q^{\frac{2n}{\varepsilon}+2}\left\{\frac{\rho'^2}{(\rho-\rho')^2}+\frac{\rho'^2}{\tau-\tau'}+\rho'^2+\frac{\rho'^2}{\rho}+\frac{\rho'^2}{\rho^2}\right\}\times \tag{2. 3}$$

$$\times\left(\frac{\rho}{\rho'}\right)^n\left(\frac{\tau}{\tau'}\right)H_{\rho\tau}(v),$$

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$$(2.4)\qquad M_{\rho'\tau'}(v) \leqslant c\Big(\frac{q}{q-1}\Big)^{\frac{2n}{\varepsilon}+1} q^{\frac{2n}{\varepsilon}+2}\Big\{\frac{\tau'}{(\rho-\rho')^2}+\frac{\tau'}{\tau-\tau'}+\tau'+\frac{\tau'}{\rho}+\frac{\tau'}{\rho^2}\Big\}\times$$

$$\times\Big(\frac{\rho}{\rho'}\Big)^n\Big(\frac{\tau}{\tau'}\Big)H_{\rho\tau}(v)$$

(ii) *If* $q<0$, *then*

$$(2.5)\qquad D_{\rho'\tau'}(v) \leqslant c\Big(1+|q|^{\frac{2n}{\varepsilon}+2}\Big)\Big\{\frac{\rho'^2}{(\rho-\rho')^2}+\frac{\rho'^2}{\tau-\tau'}+\rho'^2+\frac{\rho'^2}{\rho}+\frac{\rho'^2}{\rho^2}\Big\}\times$$

$$\times\Big(\frac{\rho}{\rho'}\Big)^n\Big(\frac{\tau}{\tau'}\Big)H_{\rho\tau}(v),$$

$$(2.6)\qquad M_{\rho'\tau'}(v) \leqslant c\Big(1+|q|^{\frac{2n}{\varepsilon}+2}\Big)\Big(\frac{q-1}{q}\Big)\Big\{\frac{\tau'}{(\rho-\rho')^2}+\frac{\tau'}{\tau-\tau'}+\tau'+\frac{\tau'}{\rho}$$

$$+\frac{\tau'}{\rho^2}\Big\}\Big(\frac{\rho}{\rho'}\Big)^n\Big(\frac{\tau}{\tau'}\Big)H_{\rho\tau}(v).$$

Here c is a constant depending only on n, ε, λ and M.

Proof. Now we suppose that $q>1$. We put

$$\bar{u}=u+l(\rho),\ \bar{c}=c+l(\rho)^{-1}e \quad\text{and}\quad \bar{d}=d+l(\rho)^{-1}f+l(\rho)^{-2}g.$$

Then from (1.2) we have

$$(2.7)\qquad \begin{cases}|A(x,t,u,p)|\leqslant M|p|+\bar{c}\bar{u},\\ |B(x,t,u,p)|\leqslant b|p|+\bar{d}\bar{u},\\ pA(x,t,u,p)\geqslant \lambda|p|^2-\bar{d}\bar{u}^2\end{cases}$$

and it is clear that $\bar{c}\in L^\infty[0,2r^2;L^{n+\varepsilon}(Q_{2r})]$ and $\bar{d}\in L^\infty[0,2r^2;L^{\frac{n+\varepsilon}{2}}(Q_{2r})]$.

We put $\phi(x,t)=q\bar{u}^{q-1}\psi(x,t)^2$, where $\psi\geqslant 0$ has compact support in Q_ρ. From (2.7) we see

$$\phi u_t+\phi_x A(x,t,u,u_x)+\phi B(x,t,u,u_x)\geqslant (v^2\psi^2)_t-2v^2\psi\psi_t+4\lambda\frac{q-1}{q}v_x^2\psi^2-q^2\bar{d}v^2\psi^2$$
$$-4Mv|v_x|\psi|\psi_x|-2q\bar{c}v^2\psi|\psi_x|-2bv|v_x|\psi^2.$$

Thus we obtain

$$(2.8)\qquad \iint_{R_{\rho\tau}}\Big\{(v^2\psi^2)_t+4\lambda\frac{q-1}{q}v_x^2\psi^2\Big\}dxdt$$
$$\leqslant\iint_{R_{\rho\tau}}2v^2\psi|\psi_t|+4Mv|v_x|\psi|\psi_x|+2bv|v_x|\psi^2+2q\bar{c}v^2|\psi_x|\psi+q^2\bar{d}v^2\psi^2\Big\}dxdt.$$

By using Lemmas 1 and 2, we estimate each terms of (2.8).
First, we see

$$\iint_{R_{\rho\tau}} 4Mv|v_x||\psi||\psi_x|dxdt \leqslant 4M\|v_x\psi\|\cdot\|v\psi_x\| \tag{2.9}$$

$$\leqslant \frac{q-1}{q}\eta\|v_x\psi\|^2 + \frac{q}{q-1}\frac{4M^2}{\eta}\|v\psi_x\|^2$$

for any $\eta>0$. Here $\|f\|=\left(\iint_{R_{\rho\tau}} f^2dxdt\right)^{1/2}$,

Next, we have

$$\int_{Q_\rho} 2b\psi^2v|v_x|dx = 2\int_{Q_\rho}|b'+b''|\Psi^2v|v_x|dx$$

$$\leqslant 2\sup|b'|\cdot\|v_x\psi\|_2(t)\cdot\|v\psi\|_2(t) + 2\|b''\|_n(t)\cdot\|v_x\psi\|_2(t)\cdot\|v\psi\|_2{}^*(t)$$

$$\leqslant 2B_{\eta'}\|v_x\psi\|_2(t)\cdot\|v\psi\|_2(t) + 2\eta'\|v_x\psi\|_2(t)\cdot\|v\psi\|_2{}^*(t)$$

$$\leqslant \frac{q-1}{q}\eta\|v_x\psi\|_2^2(t) + \frac{q}{q-1}\frac{B_{\eta'}^2}{\eta}\|v\psi\|_2^2(t) + 2\eta' K\|v_x\psi\|_2(t)\cdot\|(v\psi)_x\|_2(t)$$

$$\leqslant \frac{q-1}{q}\eta\|v_x\psi\|_2^2(t) + \frac{q}{q-1}\frac{B_{\eta'}^2}{\eta}\|v\psi\|_2^2(t) + 4\eta' K\|v_x\psi\|_2^2(t) + 2\eta' K\|v\psi_x\|_2^2(t)$$

for an arbitrary positive constant η. We put $B_{\eta'}=\left(\frac{q}{q-1}\right)^{\frac{n}{\varepsilon}}c(\eta)$, where $\eta'=\frac{q-1}{q}$ and $c(\eta)$ depends only on ε, n, M and η. Then we have

$$\iint_{R_{\rho\tau}} 2b\psi^2v|v_x|dxdt < \left(\frac{q-1}{q}\right)\left(1+4K\right)\eta\|v_x\psi\|^2 \tag{2.10}$$

$$+c(\eta)\left(\frac{q}{q-1}\right)^{\frac{2n}{\varepsilon}+1}\|v\psi\|^2 + 2\left(\frac{q-1}{q}\right)K\eta\|v\psi_x\|^2.$$

Now we put

$$\iint_{R_{\rho\tau}} 2q\bar{c}v^2\psi|\psi_x|dxdt = \iint_{R_{\rho\tau}} 2qcv^2\psi|\psi_x|dxdt + \iint_{R_{\rho\tau}} 2ql(\rho)^{-1}ev^2\psi|\psi_x|dxdt$$

$$=C+E.$$

Similarly as above, we see

$$\int 2qcv^2\psi|\psi_x|dx \leqslant 2qC_{\eta'}\|v\psi\|_2(t)\cdot\|v\psi_x\|_2(t) + 2q\eta'\|v\psi_x\|_2(t)\cdot\|v\psi\|_2{}^*(t)$$

$$\leqslant 2qC_{\eta'}\|v\psi\|_2^2(t) + 2qC_{\eta'}\|v\psi_x\|_2^2(t) + 2q\eta' K\|v_x\psi\|_2^2(t) + 4q\eta' K\|v\psi_x\|_2^2(t).$$

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Here we put $\eta'=\frac{q-1}{q}\eta q^{-1}$. Then by Lemma 2 we may put $C_{\eta'}=\left(\frac{q}{q-1}\right)^{\frac{n}{\varepsilon}}q^{\frac{n}{\varepsilon}}c(\eta)$. Thus we obtain

$$C \leqslant 2\left(\frac{q-1}{q}\right)\eta K\,||\,v_x\psi\,||^2+c(\eta)\left(\frac{q}{q-1}\right)^{\frac{n}{\varepsilon}}q^{\frac{n}{\varepsilon}+1}||\,v\psi\,||^2$$

$$+\left[c(\eta)\left(\frac{q}{q-1}\right)^{\frac{n}{\varepsilon}}q^{\frac{n}{\varepsilon}+1}+4\left(\frac{q-1}{q}\right)K\eta\right]||\,v\psi\,||^2$$

Similarly we have

$$E \leqslant 2\left(\frac{q-1}{q}\right)\eta K\,||\,v_x\psi\,||^2+c(\eta)\left(\frac{q}{q-1}\right)^{\frac{2n}{\varepsilon}}q^{\frac{2n}{\varepsilon}+1}l(\rho)^{-2\frac{n+\varepsilon}{\varepsilon}}||\,v\psi\,||^2$$

$$+\left[q+4\,\frac{q-1}{q}\,\eta K\right]||\,v\psi_x\,||^2 .$$

Therefore we obtain

$$2q\iint_{R_{\rho\tau}}\bar{c}v^2\psi\,|\,\psi_x\,|\,dxdt \leqslant 4\left(\frac{q-1}{q}\right)\eta K\,|\,|\,v_x\,\psi\,||^2 \tag{2.11}$$

$$+c(\eta)\left(\frac{q}{q-1}\right)^{\frac{2n}{\varepsilon}}q^{\frac{2n}{\varepsilon}+1}\left(1+l(\rho)^{-2\frac{n+\varepsilon}{\varepsilon}}\right)||\,v\psi\,||^2$$

$$+c(\eta)\left(\frac{q}{q-1}\right)^{\frac{n}{\varepsilon}}q^{\frac{n}{\varepsilon}+1}||\,v\psi_x\,||^2 .$$

Here we have used the fact that $0<\frac{q-1}{q}<1<\frac{q}{q-1}$ for $q>1$.

Finally we consider

$$\iint_{R_{\rho\tau}}q^2\bar{d}\psi^2v^2dxdt=\iint_{R_{\rho\tau}}q^2d\psi^2v^2dxdt+\iint_{R_{\rho\tau}}q^2l(\rho)^{-1}f\psi^2v^2dxdt$$

$$+\iint_{R_{\rho\tau}}q^2l(\rho)^{-2}g\psi^2v^2dxdt=D+F+G .$$

We observe

$$\int q^2d\psi^2v^2dx \leqslant q^2D_{\eta'}\,||\,v\psi\,||_2^2(t)+2q^2\eta'K^2\,||\,v_x\psi\,||_2^2(t)+2q^2\eta'K^2\,||\,v\psi_x\,||_2^2(t),$$

where $\eta'=\frac{q-1}{q}\eta\frac{1}{q^2}$ and $D_{\eta'}=\left(\frac{q}{q-1}\right)^{\frac{n}{\varepsilon}}q^{\frac{2n}{\varepsilon}}c(\eta)$.

So we have

$$D \leqslant 2\Big(\frac{q-1}{q}\Big)K^2\eta\,||\,v_x\psi\,||^2 + c(\eta)\Big(\frac{q}{q-1}\Big)^{\frac{n}{\varepsilon}} q^{\frac{2n}{\varepsilon}+2}\,||\,v\psi\,||^2 + 2\Big(\frac{q-1}{q}\Big)K^2\eta\,||\,v\psi_x\,||^2.$$

Similarly, we have

$$F \leqslant 2\Big(\frac{q-1}{q}\Big)K^2\eta\,||\,v_x\psi\,||^2 + c(\eta)\Big(\frac{q}{q-1}\Big)^{\frac{n}{\varepsilon}} q^{\frac{2n}{\varepsilon}+2}\,l(\rho)^{-\frac{n+\varepsilon}{\varepsilon}}\,||\,\psi v\,||^2$$
$$+2\Big(\frac{q-1}{q}\Big)K^2\eta\,||\,v\psi_x\,||^2$$

$$G \leqslant 2\Big(\frac{q-1}{q}\Big)K^2\eta\,||\,v_x\psi\,||^2 + c(\eta)\Big(\frac{q}{q-1}\Big)^{\frac{n}{\varepsilon}} q^{\frac{2n}{\varepsilon}+2}\,l(\rho)^{-2\left(\frac{n+\varepsilon}{\varepsilon}\right)}\,||\,v\psi\,||^2$$
$$+2\Big(\frac{q-1}{q}\Big)K^2\eta\,||\,v\psi_x\,||^2.$$

Therefore, using the fact that $0 < \frac{q-1}{q} < 1 < \frac{q}{q-1}$ for $q > 1$, we obtain

(2.12) $$\iint_{R_{\rho\tau}} q^2\bar{d}v^2\psi^2\,dx\,dt \leqq 6\Big(\frac{q-1}{q}\Big)K^2\eta\,||\,v_x\psi^2\,|| + c(\eta)\Big(\frac{q}{q-1}\Big)^{\frac{n}{\varepsilon}} q^{\frac{2n}{\varepsilon}+2}\Big(1+l(\rho)^{-\frac{n+\varepsilon}{\varepsilon}}$$
$$+l(\rho)^{-2\left(\frac{n+\varepsilon}{\varepsilon}\right)}\Big)||\,v\psi\,||^2 + 6\Big(\frac{q-1}{q}\Big)K^2\eta\,||\,v\psi_x\,||^2.$$

It follows from (2.9)$\sim$(2.12) that

$$\iint_{R_{\rho\tau}}(v^2\psi^2)_t\,dx\,dt + \Big(\frac{q-1}{q}\Big)\Big\{4\lambda-(2+8K+6K^2)\eta\Big\}\iint_{R_{\rho\tau}} v_x^2\psi^2\,dx\,dt$$
$$\leqslant c(\eta)\Big\{\Big(\frac{q}{q-1}\Big)^{\frac{2n}{\varepsilon}+1} q^{\frac{2n}{\varepsilon}+2}\Big(1+l(\rho)^{-\frac{n+\varepsilon}{\varepsilon}} + l(\rho)^{-\frac{2(n+\varepsilon)}{\varepsilon}}\Big)\iint_{R_{\rho\tau}} v^2\psi^2\,dx\,dt$$
$$+\Big(\frac{q}{q-1}\Big)^{\frac{2n}{\varepsilon}+1} q^{\frac{2n}{\varepsilon}+2}\iint_{R_{\rho\tau}} v^2\psi^2\,dx\,dt + \iint_{R_{\rho\tau}} v^2\psi\,|\,\psi_t\,|\,dx\,dt\Big\}.$$

Denoting that

$$l(\rho)^{-\frac{n+\varepsilon}{\varepsilon}} = \Big[\rho^{\frac{\varepsilon}{n+\varepsilon}}\Big\{\max\,||\,e\,||_{n+\varepsilon}(t) + \max\,||\,f\,||_{\frac{n+\varepsilon}{2}}(t)$$
$$+(\max\,||\,g\,||_{\frac{n+\varepsilon}{2}}(t))^{1/2}+1\Big\}\Big]^{-\frac{n+\varepsilon}{\varepsilon}} \leqslant \rho^{-1}$$

and putting $\eta = \dfrac{3\lambda}{2+8K+6K^2}$, we have

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$$(2.13)\qquad \iint_{R_{\rho\tau}}(v^2\psi^2)_t dxdt+\lambda\Big(\frac{q-1}{q}\Big)\iint_{R_{\rho\tau}}v_x^2\psi^2 dxdt$$
$$\leqslant c_1\Big(\frac{q}{q-1}\Big)^{\frac{2n}{\varepsilon}+1}q^{\frac{2n}{\varepsilon}+2}\iint_{R_{\rho\tau}}v^2[\psi|\psi_t|+\psi_x^2+\psi^2(1+\rho^{-1}+\rho^{-2})]dxdt\,.$$

We define $\psi(x,t)=\psi_1(|x|)\cdot\psi_2(t)$, where

$$\psi_1(x)=\begin{cases}0 & \text{if } |x|\geqslant\rho\,,\\ \dfrac{\rho-x}{\rho-\rho'} & \text{if } \rho'<|x|<\rho,\\ 1 & \text{if } |x|<\rho'\,,\end{cases}\quad\text{and}\quad \psi_2(t)=\begin{cases}0 & \text{if } t\leqslant-\tau\,,\\ \dfrac{\tau-t}{\tau-\tau'} & \text{if } -\tau<t<-\tau'\,,\\ 1 & \text{if } -\tau'\leqslant t<0\end{cases}$$

Then we have

$$\psi|\psi_t|+\psi_x^2+\psi^2(1+\rho^{-1}+\rho^{-2})\leqslant c\Big\{\frac{1}{(\rho-\rho')^2}+\frac{1}{\tau-\tau'}+1+\frac{1}{\rho}+\frac{1}{\rho^2}\Big\}\,.$$

Therefore we obtain

$$(2.14)\qquad \iint_{R_{\rho'\tau'}}(v^2)_t\,dxdt+\lambda\Big(\frac{q-1}{q}\Big)\iint_{R_{\rho'\tau'}}v_x^2 dxdt$$
$$\leqslant c_2\Big(\frac{q}{q-1}\Big)^{\frac{2n}{\varepsilon}+1}q^{\frac{2n}{\varepsilon}+2}\Big\{\frac{1}{(\rho-\rho')^2}+\frac{1}{\tau-\tau'}+1+\frac{1}{\rho}+\frac{1}{\rho^2}\Big\}\iint_{R_{\rho\tau}}v^2 dxdt\,.$$

Using this, we can get (2.3) and (2.4) in the quite similar manner to Kurihara's [2].

We also obtain (2.5) and (2.6) in the similar manner.

LEMMA 2.2. *Let u be a non-negative weak solution of (1.1) and let $v=\bar{u}(x,-t)^{q/2}=(u(x,-t)+l(\rho))^{q/2}$ for $0<q<1$. Then for $R_{\rho'\tau'}\subset R_{\rho\tau}$*

$$(2.15)\qquad D_{\rho'\tau'}(v)\leqslant c\Big[\Big(\frac{q}{q-1}\Big)^{\frac{2n}{\varepsilon}+2}+1\Big]\Big\{\frac{\rho'^2}{(\rho-\rho')^2}+\frac{\rho'^2}{\tau-\tau'}+\rho'^2+\frac{\rho'^2}{\rho}+\frac{\rho'^2}{\rho^2}\Big\}\times$$
$$\times\Big(\frac{\rho}{\rho'}\Big)^n\Big(\frac{\tau}{\tau'}\Big)H_{\rho\tau}(v)\,,$$

$$(2.16)\qquad M_{\rho'\tau'}(v)\leqslant c\Big[\Big(\frac{q}{1-q}\Big)^{\frac{2n}{\varepsilon}+1}+\Big(\frac{1-q}{q}\Big)\Big]\Big\{\frac{\tau'}{(\rho-\rho')^2}+\frac{\tau'}{\tau-\tau'}+\tau'+\frac{\tau'}{\rho}$$
$$+\frac{\tau'}{\rho^2}\Big\}\Big(\frac{\rho}{\rho'}\Big)^n\Big(\frac{\tau}{\tau'}\Big)H_{\rho\tau}(v)\,.$$

This lemma is obtained by the same calculations as in the proof of Lemma 2. 1. Hence we omit the proof here.

As stated in the beginning of this section, the above two lemmas imply Theorem 1.

§3. Boundedness of weak solutions.

To prove Theorem 2, we put

$$F(\bar{u})=\begin{cases}\bar{u}^q & \text{if } \bar{u}<l \\ ql^{q-1}\bar{u}-(q-1)l^q & \text{if } \bar{u}>l\end{cases}$$

and

$$G(u)=\text{sign } u\{F(\bar{u})F'(\bar{u})-q\}$$

for $l>1$ and $q>1$. Here $\bar{u}=|u|+1$.

At first, we prove the following lemma,

LEMMA 3. 1. *Let u be a weak solution of (1. 1) in $R_{2\rho,2\rho^2}$. Then, for $q>1$,*

$$D_{\rho'\tau'}(v)\leqslant cq^{\frac{n}{\varepsilon}+2}\left\{\frac{\rho'^2}{(\rho-\rho')^2}+\frac{\rho'^2}{\tau-\tau'}+\rho'^2\right\}\left(\frac{\rho}{\rho'}\right)^n\left(\frac{\tau}{\tau'}\right)H_{\rho\tau}(v)\,, \tag{3. 1}$$

$$M_{\rho'\tau'}(v)\leqslant cq^{\frac{n}{\varepsilon}+2}\left\{\frac{\tau'}{(\rho-\rho')^2}+\frac{\tau'}{\tau-\tau'}+\tau'\right\}\left(\frac{\rho}{\rho'}\right)^n\left(\frac{\tau}{\tau'}\right)H_{\rho\tau}(v)\,, \tag{3. 2}$$

where $F=v$ and c is a constant depending on M, ε, n and λ.

Proof. We put $\bar{c}=c+e$ and $\bar{d}=d+f+g$. Then from (1. 2), we see

$$\begin{cases}|A(x,t,u,p)|\leqslant M|p|+\bar{c}|\bar{u}|, \\ |B(x,t,u,p)|\leqslant b|p|+\bar{d}|\bar{u}|, \\ pA(x,t,u,p)\geqslant \lambda|p|^2-\bar{d}|\bar{u}|^2.\end{cases} \tag{3. 3}$$

We take $\phi=\psi^2 G$ as the test function in (1. 4), where ψ is a non-negative, piecewise differentiable function with support in $\rho=\{x|\ |x_i|<\rho(<r)\}$ and $\psi(x,t)=0$ for $t\leqslant-\tau$. Since u is a weak solution of (1. 1), we have

$$\iint_{R_{\rho\tau}}[\psi G u_t+(\psi^2 G)_x A+\psi^2 GB]dxdt=0\,,$$

where

$$\begin{aligned}\psi^2 Gu_t+(\psi^2G)_xA+\psi^2GB&=\psi^2\text{ sign } u\{FF'-q\}u_t+2\psi\psi_x\text{ sign } u\{FF'-q\}A \\ &\quad+\psi^2u_xG'A+\psi^2\text{ sign } u\{FF'-q\}B \\ &\geqslant \psi^2vv_t+\lambda\psi^2v_x^2-2M|v_x\psi|\cdot|v\psi_x|-b|v_x\psi|\cdot|v\psi|-2q\bar{c}|v\psi|\cdot|v\psi_x|-2q^2\bar{d}\psi^2v^2\,.\end{aligned}$$

Here we used the fact that

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$$G'(u)=\begin{cases}(F')^2\cdot\dfrac{2q-1}{q} & \text{if } |u|<l-1\\ (F')^2 & \text{if } |u|>l-1\end{cases}$$

and $\bar{u}F^r < q\cdot F$.

Hence we have

$$(3.4)\qquad \iint_{R_{\rho,\tau}}\left[\frac{1}{2}(\phi^2v^2)_t+\lambda\phi^2v_x^2\right]dxdt \leqslant \iint_{R_{\rho\tau}}[2M|\phi v_x||\phi_x v|+b|\phi v_x|\cdot|\phi v| +2q\bar{c}|\phi v|\,|\phi_x v|+2q^2\bar{d}\phi^2v^2+v^2|\phi\phi_t|]dxdt.$$

We estimate each terms of (3. 4). First we get

$$(3.5)\qquad \iint_{R_{\rho\tau}}2M|\phi v_x||\phi_x v|dxdt \leqslant 2M\|\phi v_x\|\cdot\|\phi_x v\| \leqslant \eta\|\phi v_x\|^2+\frac{4M^2}{\eta}\|\phi_x v\|^2.$$

Further we see

$$\int b|\phi v_x|\cdot|\phi v|dx \leqslant B_\eta\cdot\|\phi v_x\|_2(t)\|\phi v\|_2(t)+\eta\|\phi v_x\|_2(t)\cdot\|\phi v\|_2{}^*(t)$$

$$\leqslant \eta\|\phi v_x\|_2^2(t)+\frac{B^2\eta}{\eta}\|\phi v\|_2^2(t)+2K\eta\|\phi v_x\|_2^2(t)+K\eta\|\phi_x v\|_2^2(t).$$

Thus we obtain

$$(3.6)\qquad \iint_{R_{\rho\tau}}b|\phi v_x|\cdot|\phi v|dxdt \leqslant (1+2K)\eta\|\phi v_x\|^2+\frac{B^2\eta}{\eta}\|\phi v\|^2+\eta K\|\phi_x v\|^2.$$

Similarly

$$\int 2q\bar{c}|\phi v|\cdot|\phi_x v|dx \leqslant 2qC_{\eta'}\|\phi v\|_2(t)\|\phi_x v\|_2(t)+2q\eta'\|\phi v\|_2{}^*(t)\|\phi_x v\|_2(t)$$

where $\eta'=\dfrac{1}{q}\eta$ and $C_{\eta'}=c(\eta)q^{\frac{n}{\varepsilon}}$.

Thus we obtain

$$(3.7)\qquad \iint_{R_{\rho\tau}}2q\bar{c}|\phi v|\cdot|\phi_x v|dxdt \leqslant 2K\eta\|v_x\phi\|^2+c(\eta)q^{\frac{n}{\varepsilon}+1}\|\phi v\|^2 +[c(\eta)q^{\frac{n}{\varepsilon}+1}+4K\eta]\|\phi_x v\|^2.$$

Similarly

$$(3.8)\qquad \iint_{R_{\rho\tau}}2q^2\bar{d}\phi^2v^2dxdt \leqslant 4K^2\eta\|\phi v_x\|^2+c(\eta)q^{\frac{n}{\varepsilon}+2}\|\phi v\|^2+4K^2\eta\|\phi_x v\|^2.$$

It follows immediately from (3. 5)$\sim$(3. 8) that

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$$\iint_{R_{\rho\tau}}\Big[\frac{1}{2}(v^2\psi^2)_t+\big\{\lambda-(4K^2+4K+2)\eta\big\}v_x^2\psi^2\Big]dxdt \tag{3. 9}$$
$$\leqslant c_1 q^{\frac{n}{\varepsilon}+2}\iint_{R_{\rho\tau}}(\psi^2+\psi_x^2+|\psi\psi_t|)v^2dxdt\,.$$

Putting $\eta=\dfrac{\lambda}{2(4K^2+4K+2)}$, we see from (3. 9)

$$\iint_{R_{\rho\tau}}[(v^2\psi^2)_t+\lambda\psi^2v_x^2]dxdt \leqslant 2c_1q^{\frac{n}{\varepsilon}+2}\iint_{R_{\rho\tau}}(\psi^2+\psi_x^2+|\psi\psi_t|)v^2dxdt\,. \tag{3. 10}$$

From this we obtain (3. 1) and (3. 2). We thus obtain Lemma 3. 1.

If we let l tend to ∞, then v tends to $\bar{u}^q$. Therefore by letting $l\to\infty$ in (3. 1) and (3. 2), and taking ψ as in the proof of Theorem 1, we get

$$D_{\rho'\tau'}(\bar{u}^{q/2}) \leqslant cq^{\frac{n}{\varepsilon}+2}\Big\{\frac{\rho'^2}{(\rho-\rho')^2}+\frac{\rho'^2}{\tau-\tau'}+\rho'^2\Big\}\Big(\frac{\rho}{\rho'}\Big)^n\Big(\frac{\tau}{\tau'}\Big)H_{\rho\tau}(\bar{u}^{q/2})\,, \tag{3. 1)$'$}$$

$$M_{\rho'\tau'}(\bar{u}^{q/2}) \leqslant cq^{\frac{n}{\varepsilon}+2}\Big\{\frac{\tau'}{(\rho-\rho')^2}+\frac{\tau'}{\tau-\tau'}+\tau'\Big\}\Big(\frac{\rho}{\rho'}\Big)^n\Big(\frac{\tau}{\tau'}\Big)H_{\rho\tau}(\bar{u}^{q/2})\,. \tag{3. 2)$'$}$$

It follows from (3. 1)$'$ and (3. 2)$'$ that we have

$$\max_{R_{k'\rho,\,h'\rho^2}} u \leqslant \Big(\frac{1}{|R_{k\rho,\,h\rho^2}|}\iint_{R_{k\rho,\,h\rho^2}}\bar{u}^q dxdt\Big)^{1/q} \tag{3. 11}$$

for $q \geqslant q' > 1$, $0<k'<k\leqslant 2$ and $0<h'<h\leqslant 2$. Here $\gamma\geqslant 1$ is a constant depending only on n, ε, λ, M, k, k', h', h, q' and r. (cf. [2]). Hence we have Theorem 2.

COROLLARY. Let u be a weak solution of (1. 1) in R. Then u is Hölder continuous in any compact subset of R (cf. [3]).

§4. Removable singularities.

First we introduce some notations and definition. Let $U(Q)$ be the class of functions $\psi=\psi(x,t)$ such that $\psi\in C^1(E^n\times E^1)$, $\psi\equiv 1$ in a neighborhood of Q, $\psi\equiv 0$ outside some fixed sphere in E^n and $0\leqslant\psi\leqslant 1$. Here Q is a compact set in the $(n+1)$-dimensional (x,t)-space $E^n\times E^1$.

We say that Q is an (α,β)-null set if

$$\inf_{\psi\in U(Q)}\Big[\int_I\Big\{\int_{E^n}(|\psi_x|^2+\psi_t^-)^{\alpha/2}dx\Big\}^{\beta/\alpha}dt\Big]^{1/\beta}=0 \tag{4. 1}$$

where $\alpha\geqslant 2$, $\beta\geqslant 2$, and I is a bounded open interval in E^1 such that $Q\subset E^n\times I$ and $\psi_t^-=\max(0,-\psi_t)$ (cf. [1]).

We can prove the following.

THEOREM 3. *Let Q be an (α,β)-null set for some $2\leqslant\alpha\leqslant n$ and $\beta\geqslant 2$. Let u be a continuous weak solution of* (1. 1) *in $R_{2r,2r^2}-Q$ such that u_t is square integrable and*

(4. 2) $u\in L^b[-2r^2, 0; L^a(Q_{2r})]$ *with* $a=\dfrac{\alpha}{\alpha-2}(1+\theta)$ *and* $b=\dfrac{\beta}{\beta-2}(1+\theta)$ *for some*

θ $(0<\theta<1)$. *We take $a=\infty$ when $\alpha=2$, and $b=\infty$ when $\beta=2$. Then u can be defined over Q so that the resulting function satisfies* (1. 4) *throughout* $R_{2r,2r^2}$ *and u is Hölder continuous in any compacrt subset of* $R_{2r,2r^2}$.

Proof. It suffices to show that u can be made a continuous solution in the neighborhood of any point in $R_{2r,2r^2}$. Let $P'(x',y')$ be in $R_{2r,2r^2}$ and let $R_\rho(P')=\{P(x,t)|\,|||P-P'|||<2\rho\}$ be such that $R_\rho(P')$ is in $R_{2r,2r^2}$,

where $|||P|||=\begin{cases}\max\{x_1,\ldots\ldots,x_n,\sqrt{-2t}\} & \text{if } t\leqslant 0,\\ \infty & \text{if } t>0\end{cases}$

and $P=(x,t)=(x_1,\ldots\ldots,x_n;t)$.

By a suitable parallel transformation of variables it may supposed that $P'=(0,0)$ and $R'_\rho(P)=\{(x,t)|\,|x_i|<2\rho, -2\rho^2<t<0\}=R_{2\rho,2\rho^2}$. As in the proof of Theorem 2, we put $\bar{u}=|u|+1$ for $(x,t)\in R_{2r,2r^2}-Q$.

We now introduce an appropriate test function $\phi(x,t)$. Let $\bar{\psi}$ and ψ be non-negative smooth functions, ψ having compact support in $Q_\rho=\{x|\,|x_i|<\rho\}$ with respect to x and $\psi(x,t)=0$ for $t\leqslant-\rho^2$, and $\bar{\psi}$ vanishing in some neighborhood of Q. Let

$$\phi(x,t)=(\psi\bar{\psi})^2\text{ sign } u\left\{\bar{u}^{2q-1}-1\right\}\times q$$

for $q\geqslant q_0>\dfrac{1}{2}$.

By the same manner as in the proof of Lemma 2. 1 or 3. 1, it follows that

$$\int v^2(\psi\bar{\psi})^2dx+\lambda\left(\frac{2q-1}{q}\right)\iint_{R_{2\rho,2\rho^2}-Q}v_x^2(\psi\bar{\psi})^2dxdt \tag{4. 3}$$
$$\leqslant c_1q^{\frac{2n}{\varepsilon}+2}\iint_{R_{2\rho,2\rho^2}-Q}[(\psi\bar{\psi})^2+(\psi\bar{\psi})_x^2]v^2dxdt+\iint_{R_{2\rho,2\rho^2}-Q}(\psi\bar{\psi})(\psi\bar{\psi})_tv^2dxdt.$$

Here $v=\bar{u}^q\left(q\geqslant q_0>\dfrac{1}{2}\right)$.

Now, we use the following lemma (cf. [1]).

LEMMA 4. 1. *If Q is an (α,β)-null set for some $2\leqslant\alpha\leqslant n$ and $\beta\geqslant 2$, then Q is of measure zero and there exists a sequence $\eta^\nu\in U(Q)$ such that $\eta^\nu\to 0$ almost everywhere in $E^n\times I$ as $\nu\to\infty$.*

We replace $\bar\phi$ in (4. 1) by the elements $\bar\phi^\nu=1-\eta^\nu$. Then $\bar\phi^\nu=0$ in the neighborhood of Q and $\bar\phi^\nu\to 1$ almost everytwhere as $\nu\to\infty$. Since

$$(\psi\bar\phi^\nu)(\psi\bar\phi^\nu)_t=(\psi\psi_t)(\bar\phi^\nu)^2+\psi^2(\bar\phi^\nu\bar\phi_t)\leqslant|\psi\psi_t|(\bar\phi^\nu)^2+\psi^2(\eta^\nu)_t^-$$

and since

$$(\psi\bar\phi^\nu)_x^2=(\psi_x\bar\phi^\nu+\psi(-\eta_x^\nu))^2\leqslant 2\psi_x^2(\bar\phi^\nu)^2+2\psi^2(\eta_x^\nu)^2,$$

it follows from (4. 3) that

$$\int(\psi\bar\phi^\nu)^2v^2dx+\lambda\Big(\frac{2q-1}{q}\Big)\iint_{R_{2\rho,2\rho^2}-Q}(\psi\bar\phi^\nu)^2v_x^2dxdt \tag{4. 4}$$
$$\leqslant c_2q^{\frac{2n}{\varepsilon}+2}\iint_{R_{2\rho,2\rho^2}-Q}[(\psi\bar\phi^\nu)^2+\psi_x^2(\bar\phi^\nu)^2+|\psi\psi_t|(\bar\phi^\nu)^2+\psi^2\{(\eta_x^\nu)^2+(\eta^\nu)_t^-]v^2dxdt\ .$$

Now consider

$$\iint\{(\eta_x^\nu)^2+(\eta^\nu)_t^-\}v^2dxdt\leqslant\int\Big[\Big(\int\{(\eta_x^\nu)^2+(\eta^\nu)_t^-\}^{\alpha/2}dx\Big)^{2/\alpha}\Big(\int\bar u^{\frac{2q\alpha}{\alpha-2}}dx\Big)^{\frac{\alpha-2}{\alpha}}\Big]dt$$
$$\leqslant\Big[\int\Big(\int\{(\eta_x^\nu)^2+(\eta^\nu)_t^-\}^{\alpha/2}dx\Big)^{\beta/\alpha}dt\Big]^{2/\beta}\Big[\int\Big(\int u^{\frac{2q\alpha}{\alpha-2}}dx\Big)^{\frac{\alpha-2}{\alpha}\cdot\frac{\beta}{\beta-2}}dt\Big]^{\frac{\beta-2}{\beta}}\ .$$

If we put $q=q_0=\dfrac{\theta+1}{2}$, (4. 2) implies

$$\iint\{(\eta_x^\nu)^2+(\eta^\nu)_t^-\}v^2dxdt\leqslant c\Big[\int\Big(\int\{(\eta_x^\nu)^2+(\eta^\nu)_t^-\}^{\alpha/2}dx\Big)^{\beta/\alpha}dt\Big]^{2/\beta}.$$

Letting $\nu\to\infty$, we obtain from the dominated convergence theorem that

$$\int\psi^2v^2dx+\lambda\Big(\frac{2q_0-1}{q_0}\Big)\iint_{R_{2\rho,2\rho^2}-Q}\psi^2v_x^2dxdt\leqslant c_2q_0^{\frac{2n}{\varepsilon}+2}\times \tag{4. 5}$$
$$\times\iint_{R_{2\rho,2\rho^2}-Q}\{\psi^2+\psi_x^2+|\psi\psi_t|\}v^2dxdt,$$

where $v=\bar u^{q_0}$, and c_2 does not depend on q. Therefore, if ψ is chosen as in Theorem 1, we have

$$H'_{\rho,\rho^2}(\bar u^{kp_0/2})\leqslant\gamma_1H'_{2\rho,2\rho^2}(\bar u^{p_0/2})^k \tag{4. 6}$$

where $p_0=2q_0>1$, $k=1+\dfrac{2}{n}$ for $n>2$ and $k=\dfrac{5}{3}$ for $n=1,2$ and $H'_{\rho,\rho^2}(u)$

$=(\rho)^{-n}(\rho^2)^{-1}\iint_{R_{\rho,\rho^2-Q}} u^2\, dxdt$. Here γ_1 depends only on $p_0, M, \lambda, n, \varepsilon, \alpha, \beta$ and ρ.

Now, to proceed the arguments, we define for $q \geqslant p_0 > 1$,

$$F(\bar{u})=\begin{cases} \bar{u}^q, & \text{if } 1 \leqslant \bar{u} \leqslant l\,, \\ p_0^{-1}[ql^{q-p_0}\bar{u}^{p_0}+(p_0-q)l^q], & \text{if } l \leqslant \bar{u}\,, \end{cases}$$

and

$$G(u)=\operatorname{sign}\{F(\bar{u})F'(\bar{u})-q\} \qquad -\infty < u < +\infty.$$

Then it is clear that F is a continuouly differentiable function of $\bar{u}$, and G is a piecewise smooth function of u with corners at $u=\pm(l-1)$. Moreover, these functions have the properties:

$$F \leqslant (q/p_0)l^{q-p_0}\bar{u}^{p_0}, \quad \bar{u}F' \leqslant q\,F$$

and

$$G'(u)=q^{-1}(2q-1)(F')^2\,.$$

We may now substitute $\phi(x, t)=(\psi\bar{\psi})^2G(u)$ into (1. 4). Then we find by the same argument as in Theorem 1 or Theorem 2 that

$$(4.\ 7) \qquad \int(\psi\bar{\psi})^2v^2dx+\lambda\Big(\frac{2q-1}{q}\Big)\iint_{R_{2\rho,2\rho^2-Q}}(\psi\bar{\psi})^2v_x^2dxdt$$

$$\leqslant c_3q^{\frac{2n}{\varepsilon}+2}\iint_{R_{2\rho,2\rho^2-Q}}[(\psi\bar{\psi})^2+\psi_x^2\bar{\psi}^2+|\psi\psi_t|\bar{\psi}^2+\bar{\psi}^2\{\bar{\psi}_x^2+\bar{\psi}_t^-\}]v^2dxdt\,,$$

where $v=v(x, t)=F(\bar{u})$.

Since $v \leqslant \text{Const.}\ \bar{u}^{p_0}$, it is clear as in the earlier part of the proof that

$$\iint_{R_{2\rho,2\rho^2-Q}}\{\bar{\psi}_x^2+\bar{\psi}_t^-\}v^2dxdt \leqslant \text{Const.}\Big[\int\Big(\int\{\bar{\psi}_x^2+\bar{\psi}_t^-\}^{\alpha/2}dx\Big)^{\beta/\alpha}dt\Big]^{2/\beta}$$

Replacing $\bar{\psi}$ by $\bar{\psi}^\nu=1-\eta^\nu$ and letting $\nu\to\infty$, we obtain

$$(4.\ 8) \qquad \int\psi^2v^2dx+\lambda\Big(\frac{2q-1}{q}\Big)\iint_{R_{2\rho^2,\rho^2-Q}}v_x^2\psi^2dxdt$$

$$\leqslant c_3q^{\frac{2n}{\varepsilon}+2}\iint_{R_{2\rho,2\rho^2-Q}}\{\psi^2+\psi_x^2+|\psi\psi_t|\}v^2dxdt\,.$$

Let $l\to\infty$. Then $v\to\bar{u}^q$. If we choose ψ as in the proof of Theorem 1 or Theorem 2, we see

$$(4.\ 9) \qquad H'_{\rho,\rho^2}(\bar{u}^{kq}) \leqslant \gamma_2 H'_{2\rho,2\rho^2}(\bar{u}^q)^k \qquad (q \geqslant p_0)\,.$$

Now iterate the inequality (4. 6), (4. 9) starting with $q=q_0$. This yields the conclusion

(4. 10) $$\max_{R_{\rho,\rho^2}-Q} \bar{u} \leqslant \gamma \left\{ \iint_{R_{2\rho,2\rho^2}-Q} \bar{u}^{p_0}\, dx dt \right\}^{1/p_0}$$

On the other hand,

$$\left\{ \iint \bar{u}^{p_0}\, dx dt \right\}^{1/p_0} \leqslant \left[\int \left\{ \int \bar{u}^a dx \right\}^{b/a} dt \right]^{1/b},$$

so that the right side of (4. 10) is finite. Thus we have shown that u is uniformly bounded on the set R_{ρ,ρ^2-Q}.

Next we show that u can be extended to a continuous solution of (1. 1) throughout R_{ρ,ρ^2}. Choosing ψ such that $\psi \equiv 1$ in R_{ρ,ρ^2}, we have from (4. 5)

$$\iint_{R_{\rho,\rho^2}-Q} \bar{u}^{p_0-2} |u_x|^2\, dx dt \leqslant c_5 \iint_{R_{2\rho,2\rho^2}-Q} \bar{u}^{p_0} dx dt \leqslant \text{Const.}$$

Since $p_0 < 2$ and $\bar{u}$ is bounded in $R_{\rho,\rho^2} - Q$, this proves that u_x is in $L_2(R_{\rho,\rho^2} - Q)$.

We shall show that if u is put to be equal to zero on Q, the resulting function is strongly differentiable in R_{ρ,ρ^2}.

For any smooth function $\phi(x, t)$ with compact support in $Q_\rho - Q$ we have

$$\iint_{R_{\rho,\rho^2}} u\phi_x dx dt = -\iint_{R_{\rho,\rho^2}} u_x \phi\, dx dt .$$

Putting $\phi = \psi\bar{\psi}$ where ψ has compact support in Q_ρ, we get

$$\iint_{R_{\rho,\rho^2}-Q} u(\psi\bar{\psi}_x + \bar{\psi}\psi_x) dx dt = -\iint_{R_{\rho,\rho^2}-Q} \psi\bar{\psi} u_x dx .$$

Thus, replacing $\bar{\psi}$ by $\bar{\psi}^\nu = 1 - \eta^\nu$ where η^ν is given in Lemma 4. 1 and letting $\nu \to \infty$, we have from the dominated convergence theorem

(4. 11) $$\iint u\psi_x dx dt = -\iint \psi u_x dx dt ,$$

the integrals being evaluated over the set $R_{\rho,\rho^2} - Q$. If we put $u_x = 0$ on Q, the relation (4. 11) becomes valid over all of R_{ρ,ρ^2}. Thus the assertion is proved. Finally if ϕ has compact support in Q_ρ and if $\phi = 0$ on Q, then

$$\iint \{u_t \phi + A\phi_x + B\phi\} dx dt = 0 .$$

Again setting $\phi = \psi\bar{\psi}^\nu$, we easily obtain, in the limit as $\nu \to \infty$,

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$$\iint \{u_t\psi + A\psi_x + B\psi\} dx dt = 0 ,$$

which is valid whenever ψ has compact support in R_{ρ, ρ^2}. It follows that u, defined over Q as above, is a weak solution of (1. 1) in R_{ρ, ρ^2}. By Corollary in the end of §3, we can redefine u on a set of measure zero so that it is Hölder continuous in R_{ρ, ρ^2}. The redefinition cannot effect the values of u on $R_{\rho, \rho^2} - Q$, where it is already continouous. Since measure of Q is zero, the resulting function u is a (Hölder) continuous solution of (1. 1) in R_{ρ, ρ^2}, that is, in a non-empty neighborhood of the point P. This completes the proof.

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Added in proofs: During the proofs of this paper, Professor Serrin informed me that he and Aronson obtained more precise results than mine (cf. Notices of Amer. Math. soc., 13 (1966), p. 381) and that Ivanov also gave the same results as mine.

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