A DECOMPOSITION THEOREM FOR MATRICES

MARTIN H. PEARL

According to a classical theorem originally proved by L. Autonne (**1**; **3**) in 1915, every $m \times n$ matrix of rank r with entries from the complex field can be decomposed as

$$A = U_1 D U_2,$$

where U_1 and U_2 are unitary matrices of order m and n respectively and D is an $m \times n$ matrix having the form

$$D = \begin{bmatrix} \Delta & 0 \\ 0 & 0 \end{bmatrix}, \tag{1}$$

where Δ is a non-singular diagonal matrix whose rank is r. If $r = m$, then the row of zero matrices of (1) does not actually appear. If $r = n$, then the column of zero matrices of (1) does not appear. The main purpose of this paper is to give a necessary and sufficient condition under which both U_1 and U_2 may be chosen to be real orthogonal matrices. The result is contained in

THEOREM 1. *Let A be a rectangular matrix. Then A can be expressed as*

$$A = O_1 D O_2 \tag{2}$$

where O_1 and O_2 are real orthogonal matrices and D has the form of (1) *if and only if AA^* and A^*A are both real. Here X^*, X^T, and $\bar{X}$ denote the conjugate transpose, the transpose, and the conjugate of X, respectively.*

The necessity of the condition is immediate, for if such a decomposition exists, then

$$AA^* = O_1 D\bar{D} O_1^T, \qquad A^*A = O_2^T \bar{D} D O_2$$

are both real.

For any real orthogonal matrices Q_1 and Q_2, and any real number θ, Theorem 1 is true for A if and only if it is true for

$$\tilde{A} = e^{i\theta} Q_1 A Q_2. \tag{3}$$

We shall say that two matrices, A and $\tilde{A}$, related as in equation (3), are *orthogonally equivalent*. Before demonstrating the sufficiency of the condition of Theorem 1, we shall perform a sequence of orthogonal equivalences, beginning with A and ending with a matrix (also called A) which has a simpler form. Let

$$A = B + iC,$$

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where B and C are real, $i = \sqrt{(-1)}$. Then a direct computation shows that

LEMMA 1. *AA^* is real if and only if*

$$BC^T = CB^T, \tag{4}$$

in which case

$$AA^* = BB^T + CC^T. \tag{5}$$

Similarly, A^*A is real if and only if

$$B^TC = C^TB, \tag{4'}$$

in which case

$$A^*A = B^TB + C^TC. \tag{5'}$$

Henceforth we shall assume that both AA^* and A^*A are real and, consequently, that equations (4), (5), (4′), and (5′) hold.

LEMMA 2. *The matrices BB^T, CC^T, and $BC^T = CB^T$ are real and symmetric and commute in pairs. Similarly, the matrices B^TB, C^TC, and $B^TC = C^TB$ are real and symmetric and commute in pairs.*

Proof. It is sufficient to prove the first assertion. That all of the matrices are real and symmetric is obvious. In addition, by repeated use of equations (4) and (4′), we have

$$\begin{aligned} BB^TCC^T &= BC^TBC^T = CB^TCB^T = CC^TBB^T, \\ BB^TBC^T &= BB^TCB^T = BC^TBB^T, \\ CC^TBC^T &= CB^TCC^T = BC^TCC^T. \end{aligned} \tag{6}$$

COROLLARY 1. *There exists a real orthogonal matrix Q such that*

$$QBB^TQ^T = D_1, \qquad QCC^TQ^T = D_2, \qquad QBC^TQ^T = D_3$$

where D_1, D_2, and D_3 are diagonal matrices of order m (**2**, p. 56). *Moreover, D_1 and D_2 are non-negative, and, according to equation* (6),

$$D_1 D_2 = (D_3)^2. \tag{7}$$

We now perform an orthogonal equivalence, using Q, and call the resulting matrix A. That is, first we set $QA = \tilde{A} = \tilde{B} + i\tilde{C}$ (and consequently $\tilde{B} = QB$, $\tilde{C} = QC$). Then we drop the tildes and obtain a new matrix $A = B + iC$ which satisfies equations (4), (5), (4′), and (5′) and for which we have, in addition,

$$BB^T = D_1, \qquad CC^T = D_2, \qquad BC^T = D_3. \tag{8}$$

Let us denote the rank of the matrix X by $\mathrm{r}(X)$. It is well known that for any matrix X (**4**, p. 147),

$$\mathrm{r}(X) = \mathrm{r}(X^*) = \mathrm{r}(XX^*) = \mathrm{r}(X^*X).$$

COROLLARY 2. *For any matrix $A = B + iC$ for which AA^* and A^*A are both real,*

$$\mathrm{r}(A) \geqslant \max[\mathrm{r}(B), \mathrm{r}(C)].$$

Proof. Note that the orthogonal equivalence just performed above does not change the rank of A, B, or C. Hence we may assume (8) without loss of generality. Then it follows from the fact that D_1 and D_2 are non-negative that

$$\text{(9)}\quad \mathrm{r}(A) = \mathrm{r}(AA^*) = \mathrm{r}(BB^T + CC^T) = \mathrm{r}(D_1 + D_2) \geqslant \max[\mathrm{r}(D_1), \mathrm{r}(D_2)] = \max[\mathrm{r}(B), \mathrm{r}(C)]$$

since $\mathrm{r}(D_1) = \mathrm{r}(B)$ and $\mathrm{r}(D_2) = \mathrm{r}(C)$.

LEMMA 3. *There exists a real number θ for which the real part of $\tilde{A} = e^{i\theta}A$ has the same rank as $\tilde{A}$.*

Proof. Set $\tilde{A} = \tilde{B} + i\tilde{C}$. Then a computation shows that

$$\tilde{B} = \cos\theta B - \sin\theta C, \qquad \tilde{C} = \sin\theta B + \cos\theta C.$$

Consequently, by equations (4) and (8), we have

$$\text{(10)}\quad (\cos\theta B - \sin\theta C)(\sin\theta B^T - \cos\theta C^T) = \sin\theta\cos\theta\,(BB^T + CC^T) - BC^T = \tfrac{1}{2}\sin 2\theta(D_1 + D_2) - D_3.$$

By (7), D_3 has zero on the diagonal in any position in which either D_1 or D_2 has a zero. Since D_1 and D_2 are non-negative, it follows that for $2\theta \neq k\pi$, $k = 0, \pm 1, \pm 2, \ldots,$

$$\text{(11)}\qquad \mathrm{r}[\tfrac{1}{2}\sin 2\theta(D_1 + D_2) - D_3] \leqslant \mathrm{r}(D_1 + D_2).$$

Moreover, it is clear that for a proper choice of θ (which we now select and henceforth use) we can obtain equality in (11). Then, from equations (9) and (10), we have

$$\mathrm{r}(\tilde{B}) \geqslant \mathrm{r}[\tfrac{1}{2}\sin 2\theta(D_1 + D_2) - D_3] = \mathrm{r}(D_1 + D_2) = \mathrm{r}(A) = \mathrm{r}(\tilde{A}).$$

On the other hand, by Corollary 2, $\mathrm{r}(\tilde{B}) \leqslant \mathrm{r}(\tilde{A})$ and consequently $\mathrm{r}(\tilde{B}) = \mathrm{r}(\tilde{A})$.

We define the diagonal matrices $\tilde{D}_1$, $\tilde{D}_2$, and $\tilde{D}_3$ as follows:

$$\tilde{D}_1 = \tilde{B}\tilde{B}^T = \cos^2\theta D_1 + \sin^2\theta D_2 - \sin 2\theta D_3,$$
$$\tilde{D}_2 = \tilde{C}\tilde{C}^T = \sin^2\theta D_1 + \cos^2\theta D_2 + \sin 2\theta D_3,$$
$$\tilde{D}_3 = \tilde{B}\tilde{C}^T = \tfrac{1}{2}\sin 2\theta(D_1 - D_2) + \cos 2\theta D_3.$$

Note that $\tilde{D}_1$, $\tilde{D}_2$, and $\tilde{D}_3$ satisfy equations (7) and (8). As before, we perform an orthogonal equivalence, replacing A by $\tilde{A} = e^{i\theta}A$, and then drop the tildes.

Since a permutation matrix P is a real orthogonal matrix, it is easily seen that there is an orthogonal equivalence (replacing A by $\tilde{A} = PAP^T$) which preserves all of the properties thus far established, and for which BB^T has the form

$$BB^T = D_1 = \begin{bmatrix} \Delta_1 & 0 \\ 0 & 0 \end{bmatrix}$$

where Δ_1 is a non-singular diagonal matrix of order r, the rank of A. Then

(12) $$CC^T = D_2 = \begin{bmatrix} \Delta_2 & 0 \\ 0 & 0 \end{bmatrix}, \qquad BC^T = D_3 = \begin{bmatrix} \Delta_3 & 0 \\ 0 & 0 \end{bmatrix},$$

where Δ_2 and Δ_3 also have order r. Moreover, Δ_1 and Δ_2 are non-negative and

(7′) $$\Delta_1 \Delta_2 = (\Delta_3)^2.$$

We partition A as

(13) $$A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$$

where A_1 is an $r \times n$ matrix and A_2 is an $m - r \times n$ matrix. Then, by equations (5), (12), and (13),

$$AA^* = \begin{bmatrix} A_1 A_1^* & A_1 A_2^* \\ A_2 A_1^* & A_2 A_2^* \end{bmatrix} = \begin{bmatrix} \Delta_1 + \Delta_2 & 0 \\ 0 & 0 \end{bmatrix}$$

and it follows immediately that $A_2 = 0$. Clearly

$$B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \qquad C = \begin{bmatrix} C_1 \\ 0 \end{bmatrix},$$

the partition being the same as in equation (13). Moreover, by the construction in Lemma 3, the rows of B_1 are linearly independent.

LEMMA 4. *There is an orthogonal matrix Q of order n such that*

(14) $$B_1 Q = [\tilde{B}_1 \quad 0], \qquad C_1 Q = [\tilde{C}_1 \quad 0]$$

where $\tilde{B}_1$ and $\tilde{C}_1$ are square matrices of order r and $\tilde{B}_1$ is non-singular.

Proof. Since B_1 has n columns and is of rank r, it follows (**4**, p. 34) that the null space of B_1 has dimension $n - r$. It is only necessary to construct any orthogonal matrix Q in which the last $n - r$ columns form an orthogonal basis for the null space of B_1. Then the first part of equations (14) is immediately satisfied. Set

$$C_1 Q = [\tilde{C}_1 \quad \tilde{C}_2].$$

Then

$$\tilde{A} = AQ = \begin{bmatrix} \tilde{B}_1 + i\tilde{C}_1 & i\tilde{C}_2 \\ 0 & 0 \end{bmatrix}, \quad \tilde{A}^*\tilde{A} = \begin{bmatrix} \tilde{B}_1^T\tilde{B}_1 + \tilde{C}_1^T\tilde{C}_1 & i(\tilde{B}_1^T - i\tilde{C}_1^T)\tilde{C}_2 \\ -i\tilde{C}_2^T(\tilde{B}_1 + i\tilde{C}_1) & \tilde{C}_2^T\tilde{C}_2 \end{bmatrix}.$$

However, $\tilde{A}^*\tilde{A}$ is real, and since $\tilde{B}_1$ is non-singular, it follows that $\tilde{C}_2 = 0$.

As before, there is an orthogonal equivalence (replacing A by $\tilde{A} = AQ$ and then dropping the tildes so that

$$A = \begin{bmatrix} B_1 + iC_1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Set

(15) $$A_1 = B_1 + iC_1.$$

Note that $A_1 A_1^*$ and $A_1^* A_1$ are both real and that

(16) $$B_1 B_1^T = \Delta_1, \qquad C_1 C_1^T = \Delta_2, \qquad B_1 C_1^T = \Delta_3.$$

Proof of Theorem 1. It is sufficient to prove Theorem 1 for A_1, since if

$$A_1 = O_1 D O_2$$

then

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} O_1 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} O_2 & 0 \\ 0 & I \end{bmatrix}$$

as required. Since Δ_1 is a positive definite real diagonal matrix, it has a unique positive definite real diagonal square root, say H. That is,

$$B_1 B_1^T = \Delta_1 = H^2,$$

where H is diagonal. Clearly H is non-singular and hence

(17) $$H^{-1} B_1 B_1^T H^{-1} = I.$$

Set $H^{-1} B_1 = Q$. By (17), Q is a real orthogonal matrix. Multiplying equation (15) on the right by B_1^T and substituting equations (16) yields

$$A_1 B_1^T = \Delta_1 + i\Delta_3.$$

Then

$$A_1 = (\Delta_1 + i\Delta_3)(B_1^T)^{-1} = (\Delta_1 + i\Delta_3) H^{-1} Q,$$

which is the required equation, with $O_1 = I$, $D = (\Delta_1 + i\Delta_3)H^{-1}$, and $O_2 = Q$. This completes our proof.

An interesting special case of Theorem 1 is

COROLLARY 3. *Every unitary matrix U can be expressed as*

$$U = O_1 D O_2,$$

where O_1 and O_2 are real orthogonal matrices and D is a diagonal unitary matrix.

Clearly D of equation (2) can be chosen so that the real part of each diagonal element of Δ of equation (1) is non-negative and so that each pure imaginary diagonal element of Δ (if any) is a positive multiple of i. This is achieved by multiplying the diagonal elements of Δ by -1 when necessary and absorbing a compensating -1 into either O_1 or O_2. Also, the diagonal elements of Δ can be made to appear in any order. In particular, we may assume

$$\Delta = \begin{bmatrix} \rho_1 e^{i\theta_1} & & & & \\ & \rho_2 e^{i\theta_2} & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & \rho_r e^{i\theta_r} \end{bmatrix}$$

where $\rho_i > 0$, $\pi/2 \geqslant \theta_1 \geqslant \theta_2 \geqslant \ldots \geqslant \theta_r > -\pi/2$, and $\rho_{i+1} \geqslant \rho_i$ whenever $\theta_{i+1} = \theta_i$.

THEOREM 2. *With the restrictions of the above paragraph, D of equation* (2) *is unique.*

Proof. Let $A = O_1 D O_2$. Then

$$AA^T = O_1 D^2 O_1^T,$$

that is, the diagonal elements of Δ are the unique (within the restrictions of the last paragraph) square roots of the characteristic roots of AA^T.

If A (and hence D) is square and if the diagonal elements of D are distinct, then O_1 and O_2 are unique up to a diagonal matrix δ, all of whose elements are ± 1. That is, if

$$A = O_1 D O_2 = Q_1 D Q_2$$

then $O_1 = Q_1 \delta$, $O_2 = \delta Q_2$.

COROLLARY 4. *If AA^* and A^*A are real, then AA^T and A^TA are orthogonally similar to a diagonal matrix and hence are normal.*

The normality of A^TA and AA^T could also have been obtained directly from equations (4), (5), (4′), (5′), and Lemma 2.

THEOREM 3. *Let A_1, A_2 be $m \times n$ matrices and let $A_i A_i^*$ and $A_i^* A_i$ be real, $i = 1, 2$. Then a necessary and sufficient condition that there exist orthogonal matrices O_1 and O_2 such that*

$$A_2 = O_1 A_1 O_2 \tag{18}$$

is that the characteristic roots of $A_1 A_1^T$ are the same as the characteristic roots of $A_2 A_2^T$.

Proof. That the condition is necessary is clear, for it follows immediately from equation (18) that $A_1 A_1^T$ is orthogonally similar to $A_2 A_2^T$.

On the other hand, if $A_1 A_1^T$ and $A_2 A_2^T$ have the same characteristic roots, then, by Theorem 2, A_1 and A_2 have expressions of the type of equation (2) with the same D and the theorem follows.

We present a simple example to show that the hypothesis in Theorem 3 that $A_i A_i^*$ and $A_i^* A_i$ both be real cannot be dropped. For example, if

$$A_1 = [1 \quad i], \qquad A_2 = [0 \quad 0],$$

then $A_1 A_1^T = A_2 A_2^T = 0$; but A_1 and A_2 do not satisfy equation (18) for any orthogonal matrices, O_1, O_2.

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The University of Maryland,
College Park, Maryland