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SPHERICAL MEAN AND THE FUNDAMENTAL GROUP

Dedicated to Professor Akihiko Morimoto for his 60th birthday

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ABSTRACT. We investigate some properties of spherical means on the universal covering space of a compact Riemannian manifold. If the fundamental group is amenable then the greatest lower bounds of the spectrum of spherical Laplacians are equal to zero. If the fundamental group is non-transient so are geodesic random walks. We also give an isoperimetric inequality for spherical means.

Introduction. Let N be a complete Riemannian manifold. Given a positive r we define the *spherical mean* L_r on N with radius r by

$$L_rf(x) = \int_{U_xN} f(\exp_x rv)\, dS_x(v)$$

for a continuous function f. Here dS_x is the normalized canonical density on the unit tangent sphere U_xN. It has a continuous selfadjoint extension $L^2(N) \longrightarrow L^2(N)$, and is the generator of the r-geodesic random walk (see [10]). Many properties of L_r of course depend on the geometry of the underlying manifold and on the radius r. For example, r-geodesic random walk on a standard sphere is not transitive if and only if r is a multiple of the diameter. In this paper we mainly treat the case that underlying manifold is a covering space of a compact manifold, and point out some properties of the covering transformation group give information on spherical means.

We shall call $\Delta_r = I - L_r$ as the *spherical Laplacian*, and set

$$\lambda_0(N;r) = \inf \frac{\int_N f\Delta_r f\, d\,\mathrm{vol}}{\int_N f^2\, d\,\mathrm{vol}},$$

where f runs over all continuous functions on N with compact support. Then $\lambda_0(N;r)$ is the greatest lower bound of the spectrum of Δ_r. When N is a normal covering of a compact manifold, the amenability of the deck transformation group is reflected on $\lambda_0(N;r)$. We show the following in Section 1.

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THEOREM 1. *If the fundamental group of a compact manifold M is amenable then $\lambda_0(\tilde{M}; r)$ of the universal covering space $\tilde{M}$ of M equals to zero for every r.*

We are now interested in the alternative direction of the above theorem. For the first step to attack this problem we estimate in Section 2 $\lambda_0(N; r)$ by the geometric constant $h(N; r)$ defined in the following manner. Let $d_r\colon L^p(N) \longrightarrow L^p(UN)$ denote the r-difference operator defined by

$$d_r f(v) = f(\exp_{\pi v} rv) - f(\pi v),$$

where $\pi\colon UN \longrightarrow N$ is the projection of the unit tangent bundle. Given a bounded domain D in N we set

$$h_{r,D} = \inf_E \left(\mathrm{vol}(E)^{-1} \int_{UD} |d_r \chi_E| \, dS \right);$$

here E runs over all measurable subset of D, and χ_E denotes the characteristic function of E. Although it seems a bit complicated, as $\int_{UD} |d_r\chi_E| \, dS$ can be regarded as the volume of *r-boundary* of E, it naturally corresponds to Cheeger's isoperimetric constant (see [4]). We shall therefore call

$$h(N; r) = \inf_D h_{r,D}$$

as the r-isoperimetric constant of N. On the analogue of Cheeger's isoperimetric inequality we can get

THEOREM 2. $\lambda_0(N; r) \geq h(N; r)^2 / 8$.

We remark that if N has finite volume then trivially $\lambda_0(N, r) = h(N; r) = 0$. The above inequality makes sense when the volume of N is infinite. For a compact manifold we give an estimate of the minimum of non-trivial eigenvalues of Δ_r.

In the final section we deal with the transiency of geodesic random walks. We call an r-geodesic random walk *transient* if $\sum_{n=0}^{\infty} L_r f < \infty$ for some positive $f \in L^2(N)$. When N is a normal covering of a compact manifold, the non-transiency of the deck transformation group is also reflected on random walks on N.

THEOREM 3. *Let M be a compact Riemannian manifold. If the fundamental group $\pi_1(M)$ is non-transient then transitive r-geodesic random walk on the universal covering space $\tilde{M}$ is non-transient.*

The reader should compare some works on the Brownian motion and on the Laplace-Beltrami operator. Brooks [2] and Varopoulos [13] proved that the fundamental group of a compact manifold is amenable if and only if the greatest lower bound of the spectrum of the Laplace-Beltrami operator equals to zero. For the transience of the Brownian motion on the universal covering space Valopoulos [13] showed by use of S-operators that it is transient if and only if the fundamental group is transient. In some sense our work can be regarded as a discrete version of their results. From the interpolation of the Laplacian

$$\Delta f = \lim_{r \to 0} \frac{1}{r^2}(f - L_r f)$$

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one may guess for small r properties of the spherical Laplacian Δ_r are similar to that of the Laplacian. But since r is not necessarily small and there are no relations between L_{2r} and L_r, our situation is not trivial. We provide quite elementary proofs without any aid of these results.

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1. **Amenability and the spectrum of spherical Laplacian.** A countably generated discrete group G is said to be *amenable* if there is a bounded linear functional $\nu: L^\infty(G) \longrightarrow \mathbb{R}$ having the following properties;

(A1) $\inf_{\gamma\in G} f(\gamma) \le \nu(f) \le \sup_{\gamma\in G} f(\gamma)$,

(A2) $\nu(\gamma\cdot f) = \nu(f)$, where $\gamma\cdot f(\sigma) = f(\gamma^{-1}\sigma)$.

In this section we prove Theorem 1 by using the following combinatorial characterization.

THEOREM (FØLNER [6]). *A group G is amenable if and only if, for every finite subset A of G and arbitrary k, $0 < k < 1$, there exists a finite subset E of G such that*

$$\#(E \cap E\cdot a) \ge k\#E$$

for every $a \in A$.

Given a domain D in the universal covering $\tilde{M}$ of a compact manifold M we set

$$D^r = \{x \in D \mid \exp_x rv \in D \text{ for every } v \in U_x\tilde{M}\},$$
$$\partial^r D = D \setminus D^r.$$

Let F be a connected bounded fundamental domain of $\tilde{M} \longrightarrow M$ with piecewise smooth boundary. Since r is not necessarily small, we need to choose a bounded domain $\hat{F}$ with nice boundary $\partial^r\hat{F}$ in the following manner. We denote by $B_r(F)$ the set of all points $x \in \tilde{M}$ with $d(x,F) = \inf_{y\in F} d(x,y) \le r$, and set

$$\Gamma = \{\gamma \in \pi_1(M) \mid \gamma(F) \cap B_r(F) \ne \emptyset\}.$$

We show

LEMMA 4. *The bounded domain $\hat{F} = \cup_{\gamma\in\Gamma}\gamma(F)$ satisfies*

$$\partial^r\Big(\cup_{a\in A} a(\hat{F})\Big) \cap \hat{F} = \emptyset,$$

where $A = \{\gamma \in \pi_1(M) \mid \gamma(\bar{\hat{F}}) \cap \bar{\hat{F}} \ne \emptyset\}$.

PROOF. Put $y = \exp_x rv$ for a point $x \in \hat{F}$ and a vector $v \in U_xM$. If $x \in \gamma(F)$ for $\gamma \in \Gamma$ then $\gamma^{-1}(y)$ is contained in $\hat{F}$, because $\gamma^{-1}(x) \in F$ and $d\big(\gamma^{-1}(x), \gamma^{-1}(y)\big) = d(x,y) \le r$. On the other hand since $\gamma(\hat{F}) \cap \hat{F}$ contains $\gamma(F)$, γ is an element of A. Therefore $y \in \gamma(\hat{F}) \subset \cup_{a\in A} a(\hat{F})$, hence $\big(\cup_{a\in A} a(\hat{F})\big)^r$ contains $\hat{F}$.

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Given k, $0 < k < 1$, pick a finite subset E of $\pi_1(M)$ as in Følner's theorem. We use the characteristic function χ_H of the set $H = \cup_{\sigma \in E} \sigma(\hat{F})$ as a test function. Since the support of $\chi_H \Delta_r \chi_H$ is contained in $\partial^r H$, we have

$$\begin{aligned}\int_{\tilde{M}} \chi_H \Delta_r \chi_H \, d\,\mathrm{vol} &= \mathrm{vol}(H) - \int_H L_r \chi_H \, d\,\mathrm{vol} \\ &= \mathrm{vol}(H) - \mathrm{vol}(H^r) - \int_{U\tilde{M}|\partial^r H} \chi_H(\exp rv)\, dS \\ &\leq \mathrm{vol}(H) - \mathrm{vol}(H^r),\end{aligned}$$

hence get

$$\frac{\int_{\tilde{M}} \chi_H \Delta_r \chi_H \, d\,\mathrm{vol}}{\int_{\tilde{M}} \chi_H^2 \, d\,\mathrm{vol}} \leq 1 - \frac{\mathrm{vol}(H^r)}{\mathrm{vol}(H)}.$$

In order to give an upper estimate of the left-hand side of the above inequality, it is enough to estimate $\mathrm{vol}(H^r)/\ \mathrm{vol}(H)$ from below.

Suppose $\sigma \in \pi_1(M)$ satisfies $\sigma a \in E$ for every $a \in A$. Then

$$\sigma(\hat{F}) \cap \partial^r H \subset \sigma(\hat{F}) \cap \Big(\partial^r\big(\cup_{a\in A} \sigma a(\hat{F})\big) \cup \big(H \setminus \cup_{a\in A} \sigma a(\hat{F})\big)\Big).$$

Since A contains the unit element, the right-hand side coincides with

$$\sigma(\hat{F}) \cap \partial^r\big(\cup_{a\in A} \gamma a(\hat{F})\big) = \sigma\Big(\hat{F} \cap \partial^r\big(\cup_{a\in A} a(\hat{F})\big)\Big),$$

which is empty as we have seen in Lemma 4. Thus H^r contains

$$H \setminus \cup_\sigma \partial^r\big(\sigma(\hat{F})\big) = H \setminus \cup_\sigma (\partial^r \hat{F}),$$

where in the union σ runs over all elements in E with $\sigma a \notin E$ for some $a \in A$. Therefore we have

$$\mathrm{vol}(H^r) \geq \mathrm{vol}(H) - \mathrm{vol}(\partial^r \hat{F})\, \#\{\sigma \in E \mid \sigma a \notin E \text{ for some } a \in A\}.$$

Since A is symmetric, i.e., $A = A^{-1} = \{a^{-1} \mid a \in A\}$, and

$$\#\{\sigma \in E \mid \sigma a \notin E\} = \#(E \setminus E \cdot a^{-1}) \leq (1-k)\,\#E,$$

we get

$$\mathrm{vol}(H^r) \geq \mathrm{vol}(H) - (1-k)\,\#A\,\#E\,\mathrm{vol}(\partial^r \hat{F}).$$

Thus the inequality $\mathrm{vol}(H) \geq \#E\,\mathrm{vol}(F)$ leads us to

$$\begin{aligned}\lambda_0(\tilde{M}; r) &\leq \frac{\int_{\tilde{M}} \chi_H \Delta_r \chi_H \, d\,\mathrm{vol}}{\int_{\tilde{M}} \chi_H^2 \, d\,\mathrm{vol}} \leq 1 - \frac{\mathrm{vol}(H^r)}{\mathrm{vol}(H)} \\ &\leq (1-k)\,\#A\,\#E \frac{\mathrm{vol}(\partial^r \hat{F})}{\mathrm{vol}(H)} \leq (1-k)\,\#A \frac{\mathrm{vol}(\partial^r \hat{F})}{\mathrm{vol}(F)}.\end{aligned}$$

Letting $k \longrightarrow 1$, we get the conclusion of Theorem 1.

2. **Isoperimetric inequality for the spherical Laplacian.** In this section we estimate from below the greatest lower bound of the spectrum of spherical Laplacians. What we have to do is to construct a discrete version of Cheeger's proof. On this line some difficulties arise from the fact that the r-difference d_rf of f is a function on the unit tangent bundle. Let $\varphi_t\colon UN \longrightarrow UN$ denote the geodesic flow on the unit tangent bundle of a complete Riemannian manifold N. The r-difference operator can be written as $d_rf = f \circ \pi \circ \varphi_r - f \circ \pi$. Since the Liouville measure dS is φ_t-invariant, we get by easy calculations the following.

LEMMA 5. *Let D be an arbitrary domain in N. For L^2-functions f and g we have*

$$\begin{aligned}\int_{UD} d_rf \cdot d_rg\, dS = \int_D f \cdot \Delta_r g\, d\,\mathrm{vol} + \int_D g \cdot \Delta_r f\, d\,\mathrm{vol} \\ + \int_{UD} f\circ\pi\circ\varphi_r \cdot g\circ\pi\circ\varphi_r\, dS - \int_D f\cdot g\, d\,\mathrm{vol},\end{aligned} \tag{1}$$

in particular

$$\begin{aligned}&\int_{UN} d_rf \cdot d_rg\, dS = \int_N f \cdot \Delta_r g\, d\,\mathrm{vol} + \int_N g\cdot \Delta_r f\, d\,\mathrm{vol}. \\ &\int_{UD} |d_r(f^2)|\, dS \le \sqrt{2}\left(\int_{UD} (d_rf)^2\, dS\right)^{1/2} \times \left(\int_{UD} f^2\circ\pi\circ\varphi_r + f^2\circ\pi\, dS\right)^{1/2}.\end{aligned} \tag{2}$$

The following lemma, which combines a function on N with its r-difference in terms of their L^1-norms, is a key to prove Theorem 2.

LEMMA 6. *Let f be a non-negative function whose support is contained in a bounded domain D. Then we have*

$$\int_{UD} |d_rf|\, dS \ge h_{r,D} \int_D f\, d\,\mathrm{vol}.$$

PROOF. Since d_r is continuous, we may only treat the case f as a step function:

$$f = \sum_{i=0}^{K} a_i \chi_{E_i},$$

where $a_i > 0$, $i = 1,\dots,K$, and $D = E_0 \supset E_1 \supset \cdots \supset E_K$, with proper containment for $i = 1,\dots,K$.

Let $x \in E_j \setminus E_{j+1}$, and suppose $v \in U_xN$ satisfies $\pi\circ\varphi_r(v) \in E_k\setminus E_{k+1}$. Then we find

$$d_r\chi_{E_i}(v) = \begin{cases} -1 & i = k+1,\dots,j \\ 0 & \text{otherwise,}\end{cases} \quad \text{when } k < j,$$

$$d_r\chi_{E_i}(v) = 0 \text{ for every i, when } k = j,$$

and

$$d_r\chi_{E_i}(v) = \begin{cases} 1 & i = j+1,\dots,k \\ 0 & \text{otherwise,}\end{cases} \quad \text{when } k > j.$$

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Thus we get

$$|d_r f| = \sum_{i=0}^{K} a_i |d_r \chi_{E_i}|,$$

which leads us to the conclusion.

Let f be a continuous function on N with compact support and $\int_N f^2\, d\,\mathrm{vol} = 1$. We put $D = \mathrm{Int}\Big(B_r\big(\mathrm{supp}(f)\big)\Big)$ when N is not compact, and $D = N$ when N is compact. Using Lemmas 5 and 6 we find

$$\begin{aligned} 2\int_N f \cdot \Delta_r f\, d\,\mathrm{vol} &= \int_{UN} (d_r f)^2\, dS = \int_{UD} (d_r f)^2\, dS \\ &\geq \frac{1}{2} \frac{\left(\int_{UD} |d_r(f^2)|\, dS\right)^2}{\int_{UD} f^2 \circ \pi \circ \varphi_r + f^2 \circ \pi\, dS} \\ &\geq \frac{h_{r,D}^2}{2} \frac{\int_D f^2\, dS}{\int_{UD} f^2 \circ \pi \circ \varphi_r + f^2 \circ \pi\, dS} = \frac{h_{r,D}^2}{4}, \end{aligned}$$

hence we get the conclusion of Theorem 2:

$$\lambda_0(N; r) = \inf \frac{\int_N f \Delta_r f\, d\,\mathrm{vol}}{\int_N f^2\, d\,\mathrm{vol}} \geq \frac{h(N; r)^2}{8}.$$

If N has finite volume, constant functions are eigenfunctions, hence $\lambda_0(N; r) = 0$. To avoid this triviality, we consider for a compact manifold M the minimum of non-trivial eigenvalues of Δ_r. We set

$$h_c(M; r) = \inf_D h_{r,D},$$

where D runs over all domain in M with $\mathrm{vol}(D) \leq \frac{1}{2}\,\mathrm{vol}(M)$. This constant may equal to zero. For example $h_c(S^n; r) = 0$ when r is a multiple of the diameter of a standard sphere S^n. Using this constant we estimate

$$\lambda_1(M; r) = \inf\{\, \lambda \geq 0 \mid \Delta_r f = \lambda f \text{ for some non-constant } f \in L^2(M)\}$$

from below.

Let f be a non-constant eigenfunction associated to λ. We may suppose the volume of $D = f^{-1}([0, \infty))$ is positive and is not greater than $\frac{1}{2}\,\mathrm{vol}(M)$. Define a function g by

$$g(x) = \begin{cases} f(x) & \text{if } f(x) \geq 0 \\ 0 & \text{if } f(x) < 0. \end{cases}$$

By the inequality $f \leq g$ we find $L_r f \leq L_r g$. Therefore we get

$$\lambda \int_D f^2\, d\,\mathrm{vol} = \int_D f \Delta_r f\, d\,\mathrm{vol} \geq \int_D g \Delta_r g\, d\,\mathrm{vol}.$$

Since the support of g coincides with D, we have

$$\int_D g^2\, d\,\mathrm{vol} \geq \int_X g^2 \circ \pi \circ \varphi_r\, dS,$$

where $X = UM|_D$. This inequality and Lemmas 5 and 6 imply

$$\begin{aligned} 2\int_D g\Delta_r g\, d\,\mathrm{vol} &\geq \int_X (d_r g)^2\, dS \\ &\geq \frac{1}{2}\frac{\left(\int_X |d_r(g^2)|\, dS\right)^2}{\int_X g^2\circ\pi\circ\varphi_r + \int_D g^2\, d\,\mathrm{vol}} \\ &\geq \frac{h_{r,D}^2}{4}\int_D g^2\, d\,\mathrm{vol} = \frac{h_{r,D}^2}{4}\int_D f^2\, d\,\mathrm{vol}. \end{aligned}$$

Summarizing up we conclude

PROPOSITION 7. *For a compact manifold M, the following isoperimetric inequality holds;*

$$\lambda_1(M;r) \geq h_c(M;r)^2/8.$$

We close this section by posing a question: is it true that $h_c(M;r) = 0$ if and only if M is a $C_{2\ell}$-manifold (see [1] for the definition) and r is a multiple of ℓ?

3. **Transiency of random walks.** A finitely generated discrete group G is said to be *transient* if there exists a probability Borel measure μ with the following properties:

(T1) μ is symmetric; $\mu(g) = \mu(g^{-1}), g \in G$,

(T2) the support supp(μ) is finite and generates G,

(T3) the random walk defined by the transition probability $p_\mu(\gamma,\sigma) = \mu(\gamma^{-1}\sigma)$ is transient.

It is known that (T3) is equivalent to $\sum_{n=0}^{\infty}\mu^n(e) < \infty$, where μ^n denotes the n-th convolution of μ and e is the unit element. Moreover if G is transient then every probability measure with properties (T1) and (T2) satisfies (T3) (see [8]).

We define a combinatorial mean $L_\mu\colon L^2(G) \longrightarrow L^2(G)$ associated to a probability measure μ by

$$L_\mu h(\gamma) = \sum_{\sigma\in G}\mu(\sigma)h(\gamma\sigma).$$

If μ satisifes (T1) and (T2), by general theory (see [7] and [9]), G is non-transient if and only if there is a sequence (h_n) in $L^2(G)$ such that

(R1) $h_n \longrightarrow 1$,

(R2) $\langle \Delta_\mu h_n, h_n\rangle = \sum_{\gamma\in G} h_n(\gamma)\cdot\Delta_\mu h_n(\gamma) \longrightarrow 0$, where $\Delta_\mu = I - L_\mu$.

Let M be a compact Riemannian manifold. We denote by $\varphi_t\colon U\tilde{M} \longrightarrow U\tilde{M}$ the geodesic flow on the unit tangent bundle $\pi\colon U\tilde{M} \longrightarrow \tilde{M}$ of the universal covering space. Throughout this section we suppose r-geodesic random walk on $\tilde{M}$ is transitive: for any open sets U and V of $\tilde{M}$ there exist unit tangent vectors $v_j, j = 0,\ldots,K$, such that $\pi(v_0) \in U$, $\pi(v_{j+1}) = \pi\circ\varphi_r(v_j)$, $j = 1,\ldots,K-1$, and $\pi\circ\varphi_r(v_K) \in V$.

Choose a bounded fundamental domain F in $\tilde{M}$ and define a probability measure μ_r on $\pi_1(M)$ by

$$\mu_r(\gamma) = \frac{1}{\text{vol}(M)} \int_F L_r \chi_{\gamma(F)} \, d\,\text{vol},$$

We first point out this measure satisfies (T1) and (T2).

Let $\tau\colon U\tilde{M} \longrightarrow U\tilde{M}$ denote the canonical involution given by $\tau(v) = -v$. For each $\gamma \in \pi_1(M)$ we define a measure preserving diffeomorphism $\Phi_\gamma\colon U\tilde{M} \longrightarrow U\tilde{M}$ by $\Phi_\gamma = d\tau^{-1} \circ \tau \circ \varphi_r$. Since $\varphi_t \circ \tau = \tau \circ \varphi_{-t}$, if $v \in U\tilde{M}|_F$ and $\varphi_r(v) \in U\tilde{M}|\gamma(F)$ then $\Phi_\gamma(v) \in U\tilde{M}|_F$ and $\varphi_r \circ \Phi_\gamma(v) \in U\tilde{M}|_{\gamma^{-1}(F)}$, hence $\chi_{\gamma^{-1}(F)} \circ \pi \circ \varphi_r \circ \Phi_\gamma = \chi_{\gamma(F)} \circ \pi \circ \varphi_r$ on $U\tilde{M}|_F$. This guarantees μ_r satisfies (T1).

Since F is bounded it is clear that $\text{supp}(\mu_r)$ is finite. By the transitivity of the r-geodesic random walk on $\tilde{M}$, one can find for each $\gamma \in \pi_1(M)$ unit tangent vectors v_j and $\gamma_j \in \pi_1(M)$, $j = 0, \ldots, K$, having the properties

(i) $\gamma_0 = \text{Id}$,

(ii) $\pi(v_j) \in \text{Int}\big(\gamma_j(F)\big)$, $\pi \circ \varphi_r(v_k) \in \text{Int}\big(\gamma(F)\big)$,

(iii) $\pi \circ \varphi_r(v_j) = \pi(v_{j+1})$.

By (ii) and (iii) we get $\mu_r(\gamma_j^{-1} \cdot \gamma_{j+1}) > 0$. This implies $\text{supp}(\mu_r)$ generates $\pi_1(M)$.

We now show Theorem 3. Let (h_n) be a sequence in $L^2\big(\pi_1(M)\big)$ having the properties $h_n \longrightarrow 1$ and $\langle \Delta_{\mu_r} h_n, h_n \rangle \longrightarrow 0$. Define L^2-functions f_n on $\tilde{M}$ by

$$f_n(x) = h_n(\gamma), \text{ if } x \in \gamma(F).$$

Then

$$\begin{aligned} \int_{\gamma(F)} L_r f_n \, d\,\text{vol} &= \sum_{\sigma \in \pi_1(M)} h_n(\sigma) \int_{\gamma(F)} L_r \chi_{\sigma(F)} \, d\,\text{vol} \\ &= \text{vol}(M) \sum_\sigma \mu_r(\gamma^{-1}\sigma) h_n(\sigma) = \text{vol}(M) L_{\mu_r} h_n(\gamma). \end{aligned}$$

Therefore we get

$$\begin{aligned} \int_{\tilde{M}} f_n \cdot \Delta_r f_n \, d\,\text{vol} &= \sum_{\gamma \in \pi_1(M)} \left\{ \text{vol}(M) h_n(\gamma)^2 - h_n(\gamma) \int_{\gamma(F)} L_r f_n \, d\,\text{vol} \right\} \\ &= \text{vol}(M) \langle \Delta_{\mu_r} h_n, h_n \rangle. \end{aligned}$$

The existence of a sequence (f_n) in $L^2(\tilde{M})$ having the properties (R1) $f_n \longrightarrow 1$ and (R2) $\int_{\tilde{M}} f_n \cdot \Delta_r f_n \, d\,\text{vol} \longrightarrow 0$ leads us to the conclusion.

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