STRONG CONVERGENCE THEOREMS FOR FIXED POINTS OF PSEUDO-CONTRACTIVE SEMIGROUP

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1. INTRODUCTION

Let $E$ be a real Banach space with the dual space $E^*$ and $J : E \rightarrow 2^{E^*}$ be a normalised duality mapping defined by

$$J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\},$$

where $\langle \cdot, \cdot \rangle$ denotes the generalised duality pairing. It is well known that (see, for example, [10, pp. 107–113])

(i) $J$ is single-valued if $E^*$ is strictly convex;

(ii) $E$ is uniformly smooth if and only if $J$ is single-valued and uniformly continuous on any bounded subset of $E$.

Let $K$ be a nonempty closed convex subset of $E$. A mapping $T : K \rightarrow K$ is said to be

(i) nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in K;$$

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(ii)  \( L \)-Lipschitzian if there exists a constant \( L > 0 \) such that
\[
\|Tx - Ty\| \leq L\|x - y\|, \quad \forall x, y \in K;
\]

(iii) strongly pseudo-contractive if there exist a constant \( \alpha \in (0,1) \) and \( j(x - y) \in J(x - y) \) such that
\[
\langle Tx - Ty, j(x - y) \rangle \leq \alpha\|x - y\|^2, \quad \forall x, y \in K;
\]

(iv) pseudo-contractive if there exists \( j(x - y) \in J(x - y) \) such that
\[
\langle Tx - Ty, j(x - y) \rangle \leq \|x - y\|^2, \quad \forall x, y \in K.
\]

Obviously, the mapping \( T \) is pseudo-contractive if and only if the mapping \( A = \mathbf{I} - T \) is accretive on \( K \), where \( \mathbf{I} \) is an identity mapping (see [2, 4]); pseudo-contractive mappings are more general than nonexpansive mappings.

A pseudo-contractive semigroup is a family
\[
\Gamma := \{T(t) : t \geq 0\}
\]
of self-mappings on \( K \) such that
\begin{enumerate}
\item \( T(0)x = x \) for all \( x \in K \);
\item \( T(s + t)x = T(s)T(t)x \) for all \( x \in K \) and \( s, t \geq 0 \);
\item \( T(t) \) is pseudo-contractive for each \( t \geq 0 \);
\item for each \( x \in K \), the mapping \( T(\cdot)x \) from \( R^+ \) into \( K \) is continuous.
\end{enumerate}

If the mapping \( T(t) \) in condition (3) is replaced by
\begin{enumerate}
\item[(3)'] \( T(t) \) is nonexpansive for each \( t \geq 0 \);
\end{enumerate}
then \( \Gamma := \{T(t) : t \geq 0\} \) is said to be a nonexpansive semigroup on \( K \).

We denote by \( F(\Gamma) \) the common fixed point set of a pseudo-contractive semigroup \( \Gamma \); that is,
\[
F(\Gamma) = \bigcap_{t \in R^+} F(T(t)) = \{x \in K : T(t)x = x \quad \text{for each} \quad t \geq 0\}.
\]

In the sequel, we always assume \( F(\Gamma) \neq \emptyset \).

Let \( T \) be a nonexpansive mapping from \( K \) into itself. For any given \( u \in K \) and each \( t \in (0,1) \), it follows from Banach’s fixed theorem that the following implicit iteration process is well defined:
\[
x_t = tu + (1 - t)Tx_t
\]

Browder [1] (Reich [8], respectively) proved that as \( t \to 0 \), \( x_t \) converges strongly to some fixed point of \( T \) in Hilbert space (uniformly smooth Banach space, respectively).
An interesting work is to extend Browder's and Reich's results to semigroups. Recently, Suzuki [9] firstly introduced and studied the following implicit iteration sequence constructed from a nonexpansive semigroup in Hilbert space, for any given \( u \in K \),

\[
x_n = \alpha_n u + (1 - \alpha_n) T(t_n)x_n, \quad \forall n \geq 1,
\]

where \( \alpha_n \in (0,1) \), and obtained the convergence theorem under certain appropriate assumptions imposed on the parameters \( \{\alpha_n\} \) and \( \{t_n\} \). Very recently, Xu [12] studied the convergence of (1.2) in a uniformly convex Banach space with a weakly continuous duality mapping and obtained some convergence theorems for nonexpansive semigroup as follows:

**Theorem X.** ([12]) Let \( E \) be a uniformly convex Banach space having a weakly continuous duality map \( J_\varphi \) with gauge \( \varphi \), \( K \) a nonempty closed convex subset of \( E \) and 

\[
\Gamma := \{T(t) : t \geq 0\}
\]

a nonexpansive semigroup on \( K \) such that \( \text{Fix}(\Gamma) \neq \emptyset \). If

\[
\lim_{n \to \infty} t_n = \lim_{n \to \infty} \frac{\alpha_n}{t_n} = 0,
\]

then \( \{x_n\} \) generated by (1.2) converges strongly to a member of \( F \).

Xu [12] also proposed the following problem:

**Problem X.** ([12]) We do not know if Theorem X holds in a uniformly convex and uniformly smooth Banach space (for example, \( L^p \) for \( 1 < p < \infty \)).

On the other hand, Xu and Ori [13] introduced the following implicit iteration process constructed from a finite family of nonexpansive mappings in Hilbert space, for any given \( x_0 \in K \),

\[
x_n = \alpha_n x_{n-1} + (1 - \alpha_n) T_n x_n, \quad \forall n \geq 1,
\]

where \( T_n = T_n (\text{mod} N) \) and \( \alpha_n \in (0,1) \). Furthermore, Osilike [6] extended the results of Xu and Ori [13] from the class of nonexpansive mappings to the more general class of Lipschitzian pseudo-contractive mappings.

It is also an interesting work to extend Osilike's results to semigroup. Thus, we introduce the following implicit iteration process constructed from a pseudo-contractive semigroup \( \Gamma := \{T(t) : t \geq 0\} \) in a real Banach space, for any given \( x_0 \in K \),

\[
x_n = \alpha_n x_{n-1} + (1 - \alpha_n) T(t_n)x_n, \quad \forall n \geq 1,
\]

where \( T(t_n) \in \Gamma \) and \( \alpha_n \in (0,1) \).

In this paper, we study the convergence of the implicit iteration process (1.2) constructed from the pseudo-contractive semigroup \( \Gamma := \{T(t) : t \in R^+\} \) in uniformly convex
Banach spaces with uniformly Gâteaux differential norms. As a special case, we obtain the convergence of the implicit iteration process for approximating the common fixed point of the nonexpansive semigroup in uniformly smooth Banach spaces, which gives a positive answer to Problem X proposed by Xu [12]. We also study the convergence of the implicit iteration process (1.4) constructed from the pseudo-contractive semigroup \( \Gamma \) in uniformly convex Banach spaces with uniformly Gâteaux differential norms. The results presented in this paper generalise some corresponding results in [6, 9, 12].

2. Preliminaries

A real Banach space \( E \) is said to have a weakly continuous duality mapping if \( J \) is single-valued and weak-to-weak* sequentially continuous (that is, if each \( \{x_n\} \) is a sequence in \( E \) weakly convergent to \( x \), then \( \{J(x_n)\} \) converges weakly* to \( J(x) \)). Obviously, if \( E \) has a weakly continuous duality mapping, then \( J \) is norm-to-weak* sequentially continuous. It is well known that \( l^p \) (\( 1 < p < \infty \)) possesses duality mapping which is weakly continuous (see, for example, [12]).

Let \( l^\infty \) be the Banach space of all bounded real-valued sequences. A Banach limit \( \text{LIM} \) (see [10]) is a linear continuous functional on \( l^\infty \) such that

\[
\| \text{LIM} \| = \text{LIM}(1) = 1, \quad \text{LIM}(t_1, t_2, \ldots) = \text{LIM}(t_2, t_3, \ldots)
\]

for each \( t = (t_1, t_2, \ldots) \in l^\infty \). If \( \text{LIM} \) is a Banach limit, then it follows from [10, Theorem 1.4.4] that

\[
\liminf_{n \to \infty} t_n \leq \text{LIM}(t) \leq \limsup_{n \to \infty} t_n
\]

for each \( t = (t_1, t_2, \ldots) \in l^\infty \).

A Banach space \( E \) is said to satisfy Opial's condition if whenever \( \{x_n\} \) is a sequence in \( E \) which converges weakly to \( x \), then

\[
\limsup_{n \to \infty} \|x_n - x\| < \limsup_{n \to \infty} \|x_n - y\|
\]

for all \( y \in E \) with \( y \neq x \). It is well known that every Hilbert space satisfies the Opial's condition (see, for example, [5]).

A mapping \( T \) with domain \( D(T) \) and range \( R(T) \) in \( E \) is said to be demiclosed at a point \( p \in E \) if whenever \( \{x_n\} \) is a sequence in \( D(T) \) which converges weakly to \( x \in D(T) \) and \( \{Tx_n\} \) converges strongly to \( p \), then \( Tx = p \).

For the sake of convenience, we restate the following lemmas that shall be used.

**Lemma 2.1.** ([3]) Let \( E \) be a Banach space, \( K \) be a nonempty closed convex subset of \( E \) and \( T : K \to K \) be a strongly pseudo-contractive and continuous mapping. Then \( T \) has a unique fixed point in \( K \).
**Lemma 2.2.** ([7]) Let $E$ be a Banach space and $J$ be the normalised duality mapping. Then for any $x, y \in E$ and $j(x + y) \in J(x + y)$,

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y) \rangle.$$  

**Lemma 2.3.** ([13]) Let $r > 0$. Then a real Banach space $E$ is uniformly convex if and only if there exists a continuous and strictly increasing convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that

$$\|\lambda x + (1 - \lambda)y\|^2 \leq \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)g(\|x - y\|)$$

for all $x, y \in B_r$, $\lambda \in [0, 1]$, where $B_r = \{x \in E : \|x\| \leq r\}$.

3. Main Results

We first discuss the convergence of implicit iteration process (1.2) constructed from a pseudo-contractive semigroup $\Gamma := \{T(t) : t \geq 0\}$.

**Theorem 3.1.** Let $K$ be a nonempty closed convex subset of a real Banach space $E$ and $\Gamma := \{T(t) : t \in R^+\}$ be a strongly continuous $L$-Lipschitzian semigroup of pseudo-contractive mappings. Then the sequence $\{x_n\}$ generated by (1.2) is well defined. Moreover, if

$$\lim_{n \to \infty} t_n = \lim_{n \to \infty} \frac{\alpha_n}{t_n} = 0,$$

then $\lim_{n \to \infty} \|x_n - T(t)x_n\| = 0$ for any $t \in R^+$.

**Proof:** Let

$$T_nx := \alpha_n u + (1 - \alpha_n)T(t_n)x, \quad \forall n \geq 1.$$  

Since

$$\langle T_nx - T_ny, j(x - y) \rangle = (1 - \alpha_n)\langle T(t_n)x - T(t_n)y, j(x - y) \rangle \leq (1 - \alpha_n)\|x - y\|^2,$$

we know that $T_n$ is strongly pseudo-contractive and strongly continuous. It follows from Lemma 2.1 that $T_n$ has a unique fixed point (say) $x_n \in K$, that is, $\{x_n\}$ generated by (1.2) is well defined.

Taking $p \in F(\Gamma)$, we have

$$\|x_n - p\|^2 = \alpha_n \langle u - p, j(x_n - p) \rangle + (1 - \alpha_n)\langle T(t_n)x_n - T(t_n)p, j(x_n - p) \rangle \leq \alpha_n \|u - p\| \|j(x_n - p)\| + (1 - \alpha_n)\|x_n - p\|^2$$

$$= \alpha_n \|u - p\| \|x_n - p\| + (1 - \alpha_n)\|x_n - p\|^2.$$
and so \(\|x_n - p\| \leq \|u - p\|\). This means \(\{x_n\}\) is bounded. By the Lipschitzian condition of \(\Gamma\), it follows that \(\{T(t_n)x_n\}\) is bounded. Therefore,

\[
(3.1) \quad \|x_n - T(t_n)x_n\| = \alpha_n\|u - T(t_n)x_n\| \to 0.
\]

For any given \(t > 0\),

\[
\|x_n - T(t)x_n\| = \sum_{k=0}^{\lfloor t/t_n \rfloor - 1} \|T((k + 1)t_n)x_n - T(kt_n)x_n\| + \|T(t)x_n - T\left(\frac{t}{t_n}t_n\right)x_n\|
\]

\[
\leq \frac{t}{t_n}L\|x_n - T(t_n)x_n\| + L\|T(t - \frac{t}{t_n}t_n)x_n - x_n\|
\]

\[
\leq tL\frac{\alpha_n}{t_n}\|u - T(t_n)x_n\| + L\max\left\{\|T(s)x_n - x_n\| : 0 \leq s \leq t_n\right\},
\]

where \(\lfloor t/t_n \rfloor\) is the integral part of \(t/t_n\). Since \(\lim_{n \to \infty} (\alpha_n/t_n) = 0\) and \(T(\cdot)x : R^+ \to K\) is continuous for any \(x \in K\), we have

\[
\lim_{n \to \infty} \|x_n - T(t)x_n\| = 0.
\]

This completes the proof. \(\square\)

**Theorem 3.2.** Let \(E\) be a uniformly convex Banach space with the uniformly Gâteaux differential norm and \(K\) be a nonempty closed convex subset of \(E\). Let \(\Gamma := \{T(t) : t \in R^+\}\) be a strongly continuous \(L\)-Lipschitzian semigroup of pseudo-contractive mappings and \(\{x_n\}\) be generated by (1.2). If the following conditions hold:

1. \(\lim_{n \to \infty} (\alpha_n/t_n) = \lim_{n \to \infty} t_n = 0\);
2. \(\lim \|T(t)x_n - T(t)x^*\| \leq \lim \|x_n - x^*\|, \quad \forall x^* \in C, t \in R^+\), where \(C := \{x^* \in K : \Phi(x^*) = \min_{x \in K} \Phi(x)\}\) with \(\Phi(x) = \lim_{n \to \infty} \|x_n - x\|^2\) for all \(x \in K\);

then \(\{x_n\}\) converges strongly to a member of \(F(\Gamma)\).

**Proof:** From Theorem 3.1, we know that \(\{x_n\}\) is bounded and \(\lim_{n \to \infty} \|x_n - T(t)x_n\| = 0\). It is easy to see that \(C\) is a nonempty bounded closed convex subset of \(K\) (see, for example [11]).

Now, we show that there exists a common fixed point of \(\Gamma\) in \(C\). For any \(t \in R^+\) and \(x^* \in C\), it follows from \(\lim_{n \to \infty} \|x_n - T(t)x_n\| = 0\) that

\[
\Phi(T(t)x^*) = \lim_{n \to \infty} \|x_n - T(t)x^*\|^2
\]

\[
= \lim_{n \to \infty} \|T(t)x_n - T(t)x^*\|^2
\]

\[
\leq \lim_{n \to \infty} \|x_n - x^*\|^2
\]

\[
= \Phi(x^*)
\]
and so

\[ T(t)(C) \subset C. \]

Next, we prove that \( C \) is a singleton. In fact, since \( E \) is uniformly convex, by Lemma 2.3 that there exists a continuous and strictly increasing convex function \( g : [0, \infty) \to [0, \infty) \) with \( g(0) = 0 \) such that, for any \( x_1^* \) and \( x_2^* \in C, \)

\[
\left\| x_n - \frac{x_1^* + x_2^*}{2} \right\|^2 \leq \frac{1}{2} \|x_n - x^*\|^2 + \frac{1}{2} \|x_n - x^*_2\|^2 - \frac{1}{4} g(\|x_1^* - x_2^*\|).
\]

Taking Banach limit \( \text{LIM} \) on the above inequality, it follows that

\[
\frac{1}{4} g(\|x_1^* - x_2^*\|) \leq \frac{1}{2} \text{LIM} \|x_n - x_1^*\|^2 + \frac{1}{2} \text{LIM} \|x_n - x_2^*\|^2 - \text{LIM} \left\| x_n - \frac{x_1^* + x_2^*}{2} \right\|^2 
\]

\[
\leq 0.
\]

This implies \( x_1^* = x_2^* \) and so \( C \) is a singleton. Therefore, (3.2) implies that there exists \( x^* \in C \) such that \( x^* \in F(\Gamma) \).

For any \( p \in F(\Gamma) \), from (1.2), we have

\[
\langle x_n - u, j(x_n - p) \rangle = \frac{1 - \alpha_n}{\alpha_n} \langle T(t_n)x_n - x_n, j(x_n - p) \rangle 
\]

\[
= \frac{1 - \alpha_n}{\alpha_n} \left[ \langle T(t_n)x_n - T(t_n)p, j(x_n - p) \rangle - \langle x_n - p, j(x_n - p) \rangle \right] 
\]

\[
\leq 0.
\]

(3.3)

Since \( x^* \in F(\Gamma) \), it follows from (3.3) that

\[ \text{LIM} \langle x_n - u, j(x_n - x^*) \rangle \leq 0. \]

(3.4)

Furthermore, for any \( t \in (0, 1) \), by Lemma 2.2, we have

\[
\left\| x_n - x^* - t(u - x^*) \right\|^2 \leq \|x_n - x^*\|^2 - 2t \langle u - x^*, j(x_n - x^* - t(u - x^*)) \rangle
\]

\[
\leq \|x_n - x^*\|^2 - 2t \langle u - x^*, j(x_n - x^*) \rangle - 2t \langle u - x^*, j(x_n - x^* - t(u - x^*)) \rangle
\]

and

\[
\langle u - x^*, j(x_n - x^*) \rangle \leq \frac{1}{2t} \left[ \|x_n - x^*\|^2 - \|x_n - x^* - t(u - x^*)\|^2 \right] - \langle u - x^*, j(x_n - x^* - t(u - x^*)) \rangle - j(x_n - x^*)\rangle.
\]

For any \( \epsilon > 0 \), since \( E \) has a uniformly Gâteaux differential norm, we know that \( J \) is norm-to-weak* uniformly continuous on any bounded subset of \( E \) (see, for example, [10, pp.107-113]) and so there exists sufficient small \( \delta(\epsilon) > 0 \) such that

\[
\langle u - x^*, j(x_n - x^*) \rangle \leq \frac{1}{2t} \left[ \|x_n - x^*\|^2 - \|x_n - x^* - t(u - x^*)\|^2 \right] + \epsilon, \quad \forall t \in (0, \delta).
\]
This implies that
\[
\lim \langle u - x^*, j(x_n - x^*) \rangle \leq \frac{1}{2t} \left[ \lim \|x_n - x^*\|^2 - \lim \|x_n - x^* - t(u - x^*)\|^2 \right] + \varepsilon < \varepsilon.
\]
By the arbitrariness of \(\varepsilon\), it follows that
\[
\lim \langle u - x^*, j(x_n - x^*) \rangle \leq 0.
\]
Adding inequalities (3.4) and (3.5), we have
\[
\lim \langle x_n - x^*, j(x_n - x^*) \rangle = \lim \|x_n - x^*\|^2 \leq 0.
\]
This implies that there exists subsequence \(\{x_{n_j}\} \subset \{x_n\}\) which converges strongly to \(x^*\). From the proof of (3.6), we know that \(\lim \|x_{n_j} - x^*\|^2 = 0\) for any subsequence \(\{x_{n_j}\} \subset \{x_n\}\) and so there exists subsequence of \(\{x_{n_j}\}\) which converges strongly to \(x^*\). If there exists another subsequence \(\{x_{n_t}\} \subset \{x_n\}\) which converges strongly to \(y^*\), then it follows from Theorem 3.1 that \(y^* \in F(\Gamma)\). From (3.3), we have
\[
\langle x^* - u, j(x^* - y^*) \rangle \leq 0, \quad \langle y^* - u, j(y^* - x^*) \rangle \leq 0.
\]
This implies that \(\|x^* - y^*\|^2 \leq 0\) and so \(x^* = y^*\). Therefore, \(\{x_n\}\) converges strongly to \(x^* \in F(\Gamma)\). This completes the proof. \(\square\)

If \(\Gamma := \{T(t) : t \in R^+\}\) is a nonexpansive semigroup, then \(\Gamma := \{T(t) : t \in R^+\}\) is a strongly continuous \(L\)-Lipschitzian semigroup of pseudo-contractive mappings. From Theorem 3.2, we have the following result.

**Corollary 3.1.** Let \(E\) be a uniformly convex Banach space with the uniformly Gâteaux differential norm and \(K\) be a nonempty closed convex subset of \(E\). Let \(\Gamma := \{T(t) : t \in R^+\}\) be a nonexpansive semigroup and \(\{x_n\}\) be generated by (1.2). If
\[
\lim \frac{\alpha_n}{t_n} = \lim \frac{\alpha_n}{t_n} = 0,
\]
then \(\{x_n\}\) converges strongly to a member of \(F(\Gamma)\).

**Remark 3.1.** If \(E\) is a uniformly smooth Banach space, then \(E\) has the uniformly Gâteaux differential norm. Thus, Corollary 3.1 gives a positive answer to Problem X proposed by Xu [12].

**Theorem 3.3.** Let \(E\) be a uniformly smooth Banach space and \(K\) be a nonempty closed convex subset of \(E\). Let \(\Gamma := \{T(t) : t \in R^+\}\) be a nonexpansive semigroup and \(\{x_n\}\) be generated by (1.2). If
\[
\lim \frac{\alpha_n}{t_n} = \lim \frac{\alpha_n}{t_n} = 0,
\]
then \(\{x_n\}\) converges strongly to a member of \(F(\Gamma)\).
PROOF: For the nonexpansive semigroup $\Gamma$, condition (2) of Theorem 3.2 is trivial and so formula (3.2) holds. Since uniformly smooth Banach space $E$ has the fixed point property for nonexpansive mapping $T(t)$ (see, for example, [11]), $T(t)$ has a fixed point $x^* \in C \cap F(\Gamma)$. The rest proof is similar to the proof of Theorem 3.2 and so we omit it. This completes the proof. 

REMARK 3.2. Theorem 3.3 gives a positive answer to Problem X proposed by Xu [12] in a uniformly smooth Banach space $E$ without the uniform convexity.

**Theorem 3.4.** Let $E$ be a real Hilbert space and $K$ be a nonempty closed convex subset of $E$. Let $\Gamma := \{T(t) : t \in R^+\}$ be a strongly continuous $L$-Lipschitzian semigroup of pseudo-contractive mappings and $\{x_n\}$ be generated by (1.2). If
\[
\lim_{n \to \infty} \frac{\alpha_n}{t_n} = \lim_{n \to \infty} t_n = 0,
\]
then $\{x_n\}$ converges strongly to a member of $F(\Gamma)$.

PROOF: From the proof of Theorem 3.1, we know that $\{x_n\}$ is bounded and so there exists subsequence $\{x_{n_j}\} \subset \{x_n\}$ which converges weakly to some point $x^* \in K$. By Theorem 3.1, we have
\[
\lim_{n \to \infty} \|x_n - T(t)x_n\| = 0.
\]
It follows from [2, Theorem 3.18b] that $I - T(t)$ is demiclosed at zero for each $t \in R^+$, where $I$ is an identity mapping. This implies that $x^* \in F(\Gamma)$.

In addition, from (1.2), we have
\[
\|x_n - x^*\|^2 = \alpha_n \langle u - x^*, x_n - x^* \rangle + (1 - \alpha_n) \langle T(t_n)x_n - x^*, x_n - x^* \rangle \\
\leq \alpha_n \langle u - x^*, x_n - x^* \rangle + (1 - \alpha_n) \|x_n - x^*\|^2
\]
and so
\[
\|x_n - x^*\|^2 \leq \langle u - x^*, x_n - x^* \rangle.
\]
This implies that $\{x_{n_j}\}$ converges strongly to $x^* \in F(\Gamma)$. Similar to the proof of Theorem 3.2, it is easy to show that $\{x_n\}$ converges strongly to $x^* \in F(\Gamma)$. This completes the proof. 

REMARK 3.3. Theorem 3.4 extends the results of Suzuki [9] and Xu [12] from nonexpansive semigroup to pseudo-contractive semigroup in real Hilbert spaces.

Now we turn to discuss the convergence of implicit iteration process (1.4) for approximating the common fixed point of the pseudo-contractive semigroup $\Gamma := \{T(t) : t \geq 0\}$.

**Theorem 3.5.** Let $K$ be a nonempty closed convex subset of a real Banach space $E$ and $\Gamma := \{T(t) : t \in R^+\}$ be a strongly continuous $L$-Lipschitzian semigroup of
pseudo-contractive mappings. Then the sequence \( \{x_n\} \) generated by (1.4) is well defined. Moreover, if

\[
\lim_{n \to \infty} \frac{\alpha_n}{t_n} = \lim_{n \to \infty} t_n = 0,
\]

then \( \lim_{n \to \infty} \|x_n - T(t)x_n\| = 0 \) for any \( t \in R^+ \) and \( \lim \|x_n - p\| \) exists for any \( p \in F(\Gamma) \). Furthermore, if \( E \) is a uniformly convex Banach space with the uniformly Gâteaux differential norm and condition (2) of Theorem 3.2 holds, then \( \{x_n\} \) converges strongly to a member of \( F(\Gamma) \).

**Proof:** Let

\[
T_n x := \alpha_n x_{n-1} + (1 - \alpha_n)T(t_n)x, \quad \forall n \geq 1.
\]

It is obvious that \( T_n \) is strongly pseudo-contractive and strongly continuous. From Lemma 2.1, it is easy to know that \( \{x_n\} \) generated by (1.4) is well defined.

Taking \( p \in F(\Gamma) \), it follows that

\[
\|x_n - p\|^2 = \alpha_n \langle x_{n-1} - p, j(x_n - p) \rangle + (1 - \alpha_n) \langle T(t_n)x_n - T(t_n)p, j(x_n - p) \rangle \\
\leq \alpha_n \|x_{n-1} - p\| \|j(x_n - p)\| + (1 - \alpha_n) \|x_n - p\|^2 \\
= \alpha_n \|x_{n-1} - p\| \|x_n - p\| + (1 - \alpha_n) \|x_n - p\|^2
\]

and so \( \|x_n - p\| \leq \|x_{n-1} - p\| \). This means \( \lim_{n \to \infty} \|x_n - p\| \) exists for any \( p \in F(\Gamma) \). Similar to the proof of Theorem 3.1, we have \( \lim_{n \to \infty} \|x_n - T(t)x_n\| = 0 \) for any \( t \in R^+ \).

From (1.4) and (3.4), for any \( x^* \in C \cap F(\Gamma) \), we have

\[
\lim \langle x_n - x_{n-1}, j(x_n - x^*) \rangle \leq 0.
\]

Since \( \|x_n - p\| \leq \|x_{n-1} - p\| \), we know that \( \{x_n\} \) is bounded. Similar to the proof of (3.5), we have

\[
\lim \langle x_{n-1} - x^*, j(x_n - x^*) \rangle \leq 0.
\]

Adding inequalities (3.7) and (3.8), we get

\[
\lim \langle x_n - x^*, j(x_n - x^*) \rangle = \lim \|x_n - x^*\|^2 \leq 0.
\]

This implies that there exists subsequence \( \{x_{n_j}\} \subset \{x_n\} \) which converges strongly to \( x^* \). Since \( \lim \|x_n - x^*\| \) exists, it follows that \( \{x_n\} \) converges strongly to \( x^* \in F(\Gamma) \). This completes the proof.

If \( \Gamma := \{T(t) : t \in R^+\} \) is a nonexpansive semigroup, then \( \Gamma := \{T(t) : t \in R^+\} \) is a strongly continuous \( L \)-Lipschitzian semigroup of pseudo-contractive mappings. Therefore, Theorem 3.5 gives the following result.
**Corollary 3.2.** Let $E$ be a uniformly convex Banach space with the uniformly Gâteaux differential norm and $K$ be a nonempty closed convex subset of $E$. Let $\Gamma := \{T(t) : t \in R^+\}$ be a nonexpansive semigroup and $\{x_n\}$ be generated by (1.4). If
\[
\lim_{n \to \infty} \frac{\alpha_n}{t_n} = \lim_{n \to \infty} t_n = 0,
\]
then $\{x_n\}$ converges strongly to a member of $F(\Gamma)$.

**Theorem 3.6.** Let $E$ be a uniformly smooth Banach space and $K$ be a nonempty closed convex subset of $E$. Let $\Gamma := \{T(t) : t \in R^+\}$ be a nonexpansive semigroup and $\{x_n\}$ be generated by (1.4). If
\[
\lim_{n \to \infty} \frac{\alpha_n}{t_n} = \lim_{n \to \infty} t_n = 0,
\]
then $\{x_n\}$ converges strongly to a member of $F(\Gamma)$.

**Proof:** For the nonexpansive semigroup $\Gamma$, condition (2) of Theorem 3.2 is trivial and so formula (3.2) holds. Since uniformly smooth Banach space $E$ has the fixed point property for nonexpansive mapping $T(t)$ (see, for example, [11]), $T(t)$ has a fixed point $x^* \in C \cap F(\Gamma)$. The rest proof is similar to the proof of Theorem 3.5 and so we omit it. This completes the proof. \qed

**Theorem 3.7.** Let $E$ be a real Hilbert space and $K$ be a nonempty closed convex subset of $E$. Let $\Gamma := \{T(t) : t \in R^+\}$ be a strongly continuous $L$-Lipschitzian semigroup of pseudo-contractive mappings and $\{x_n\}$ be generated by (1.4). If
\[
\lim_{n \to \infty} \frac{\alpha_n}{t_n} = \lim_{n \to \infty} t_n = 0,
\]
then $\{x_n\}$ converges weakly to a member of $F(\Gamma)$.

**Proof:** From Theorem 3.5 and the fact that $I - T(t)$ is demiclosed at zero for each $t \in R^+$, we know that $\{x_n\}$ is bounded and so there exists subsequence $\{x_{n_j}\} \subset \{x_n\}$ which converges weakly to some fixed point $x^* \in F(\Gamma)$. Suppose there exists another subsequence $\{x_{n_k}\} \subset \{x_n\}$ which converges weakly to $y^* \in F(\Gamma)$ with $y^* \neq x^*$. Since
\[
\lim_{n \to \infty} \|x_n - y^*\|
\]
exists and Hilbert space $E$ satisfies Opial's condition, it follows from the standard argument that $y^* = x^*$. Thus, $\{x_n\}$ converges weakly to a member of $F(\Gamma)$. This completes the proof. \qed

**Remark 3.4.** Theorems 3.5 and 3.7 extend the corresponding results of Osilike [6] from the finite family of pseudo-contractive mappings to the pseudo-contractive semigroup.

**References**


