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On the Chiral Ring of Calabi–Yau Hypersurfaces in Toric Varieties

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Abstract. This paper is devoted to the calculation of the B-model chiral ring, used in physics, for semiample Calabi–Yau hypersurfaces. We also study the cohomology of semiample hypersurfaces.

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Introduction

In nonlinear sigma models (see Appendix B.2 in [CK] on physical theories) there are two twisted theories, the A-model and the B-model. Mirror symmetry is an isomorphism between the A-model and the B-model for a pair of two distinct Calabi–Yau threefolds V and V° with Kähler structures. One consequence of mirror symmetry is an isomorphism between the quantum cohomology on $\bigoplus_{p,q} H^{p,q}(V)$ and the chiral ring of the B-model $\bigoplus_{p,q} H^p(V^\circ, \wedge^q \mathcal{T}_{V^\circ})$, which implies the equality of the corresponding correlation functions (Yukawa couplings). These correlation functions describe interactions between strings. From a mathematical point of view, knowledge about the B-model Yukawa coupling and the equality with the A-model Yukawa coupling of the mirror manifold produces enumerative information on this mirror manifold. One important construction widely used in physics and mathematics is the Batyrev mirror construction in toric varieties (see [B2]).

In this paper we study the chiral ring $\bigoplus_p H^p(X, \wedge^p \mathcal{T}_X)$ (this is actually a subring of the whole chiral ring) for quasismooth hypersurfaces X in complete simplicial toric varieties. In particular, we completely describe the chiral ring $\bigoplus_p H^p(X, \wedge^p \mathcal{T}_X)$ in the case of three-dimensional Calabi–Yau hypersurfaces. This applies to the mirror symmetric hypersurfaces in Batyrev's construction.

The following is an outline of the paper. We begin in Section 1 with a review of notation and general facts from toric geometry. For complete toric varieties, the notions of semiample, nef (numerically effective) and generated by global sections are equivalent for invertible sheaves (divisors). Geometry and intersection theory

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associated with big (the self-intersection number is positive) and nef divisors was studied in [M]. Here, we generalize those results to all semiample divisors. Such divisors on complete toric varieties naturally produce a surjective morphism of the ambient space onto another complete toric variety. Moreover, this construction is unique with certain conditions relating semiample divisors to ample divisors on the new toric variety. We also show that a proper birational morphism of toric varieties induces a natural graded homomorphism of the coordinate rings. For a semiample divisor, this gives an isomorphism of rings in the degree of the divisor.

Section 2 uses the results of Section 1 to describe the geometry of semiample nondegenerate (transversal to orbits) hypersurfaces in complete toric varieties. We get a stratification of such hypersurfaces in terms of nondegenerate affine hypersurfaces cohomology of which has been studied in [B1]. We also review some facts about hypersurfaces in complete simplicial toric varieties. In particular, we recall from [M] the relationship between the Jacobian ring $R(f)$ (resp., $R_1(f)$) and the middle cohomology $H^{d-1}(X)$ of a quasismooth (resp., big and nef nondegenerate) hypersurface X in a d-dimensional complete simplicial toric variety. The ring $R_1(f)$ has been used in [M] to describe the middle cohomology of a three-dimensional big and nef nondegenerate hypersurface completely.

In Section 3, we introduce the (*Zariski*) pth *exterior power* $\wedge^p\mathcal{T}_X$ *of the tangent sheaf* for an arbitrary orbifold X, which is defined similarly to the sheaf Ω^p_X of Zariski p-forms (see [CK, A.3]). Then we show that for a quasismooth hypersurface X of degree β there is a ring homomorphism $R(f)_{*\beta} \to H^*(X, \wedge^*\mathcal{T}_X)$ (the latter is our notation for $\bigoplus_p H^p(X, \wedge^p\mathcal{T}_X)$). Also, with respect to this homomorphism the map between $R(f)$ and the middle cohomology of a quasismooth hypersurface is a morphism of modules. In the Calabi–Yau case the situation is especially nice because we get an injective ring homomorphism $R_1(f)_{*\beta} \to H^*(X, \wedge^*\mathcal{T}_X)$ (we call $R_1(f)_{*\beta}$ the polynomial part of the chiral ring because its graded piece in $H^1(X, \mathcal{T}_X)$ should correspond to polynomial infinitesimal deformations for a minimal Calabi–Yau X (see [CK])).

According to the above terminology, in Section 4, we study the nonpolynomial part of the chiral ring complementary to the polynomial part. We construct new elements in $H^*(X, \wedge^*\mathcal{T}_X)$ for a big and nef quasismooth hypersurface X, and in the case of a minimal Calabi–Yau these elements in $H^1(X, \mathcal{T}_X)$ should correspond to nonpolynomial deformations. The new elements are represented by a map from some quotient $R^\sigma(f)$ of the Jacobian ring to $H^*(X, \wedge^*\mathcal{T}_X)$, and this map is actually a morphism of modules with respect to the ring homomorphism $R(f)_{*\beta} \to H^*(X, \wedge^*\mathcal{T}_X)$. We also calculate some vanishing cup products of the new elements. The new part of $H^*(X, \wedge^*\mathcal{T}_X)$ has its analogue in the middle cohomology $H^{d-1}(X)$ of the hypersurface. This is also given by a map from certain graded pieces of $R^\sigma(f)$ to $H^{d-1}(X)$. We show that this map is morphism of modules with respect to $R(f)_{*\beta} \to H^*(X, \wedge^*\mathcal{T}_X)$.

In Section 5, we describe the *toric part* of cohomology of a semiample nondegenerate hypersurface. This part is the image of cohomology of the ambient space, while

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its complement, called the *residue part*, comes from the residues of rational differential forms with poles along the hypersurface. We show that the cohomology of a semiample nondegenerate hypersurface is a direct sum of its toric and residue parts.

Section 6 studies the middle cohomology of a big and nef nondegenerate hypersurface. We provide a better and more general description of the middle cohomology than the one given for three-dimensional hypersurfaces in [M]. Here, we use a new ring $R_1^{\sigma}(f)$, analogous to the ring $R_1(f)$. An algebraic description of the middle cohomology can be used in the Calabi–Yau case to compute the product structure on the chiral ring.

In Section 7, we consider semiample anticanonical nondegenerate hypersurfaces. Such hypersurfaces are Calabi–Yau, implying that their chiral ring is isomorphic to the middle cohomology. Using the description of Section 6, we have a partial description of the space $H^*(X, \wedge^*\mathcal{T}_X)$ in terms of $R_1(f)$ and $R_1^{\sigma}(f)$. We show that this part is a subring of the chiral ring. This subring is the whole $H^*(X, \wedge^*\mathcal{T}_X)$ in the case of Calabi–Yau threefolds. The product structure of the polynomial part $R_1(f)$ is in Section 3, while the product of two different elements from $R_1(f)$ and $R_1^{\sigma}(f)$ is in Section 4. We describe the nontrivial product structure on the spaces $R_1^{\sigma}(f)$ in terms of triple products. Since $H^*(X, \wedge^*\mathcal{T}_X)$ and the described subring have a nondegenerate pairing, induced by the cup product on the middle cohomology, one can recover the chiral ring structure completely on these spaces.

1. Semiampleness

In this section we first review some basic facts and notation, and then generalize the geometric construction of [M] associated with semiample divisors on complete toric varieties. We show that a semiample divisor naturally produces a surjective morphism of the ambient space onto another complete toric variety. This construction is unique with certain conditions which relate the semiample divisor to an ample divisor on the new toric variety. At the end of this section we show that a proper birational morphism of toric varieties gives a natural graded homomorphism of the homogeneous coordinate rings of the varieties. We apply this to the maps associated with semiample divisors.

Let M be a lattice of rank d, then $N = \mathrm{Hom}(M, \mathbb{Z})$ is the dual lattice; $M_{\mathbb{R}}$ (resp. $N_{\mathbb{R}}$) denotes the $\mathbb{R}$-scalar extension of M (resp. of N). The symbol $\mathbf{P}_{\Sigma}$ stands for a d-dimensional toric variety associated with a finite rational fan Σ in $N_{\mathbb{R}}$. A toric variety $\mathbf{P}_{\Sigma}$ is a disjoint union of its orbits by the action of the torus $\mathbf{T} = N \otimes \mathbb{C}^*$ that sits naturally inside $\mathbf{P}_{\Sigma}$. Each orbit $\mathbf{T}_{\sigma}$ is a torus corresponding to a cone $\sigma \in \Sigma$. The closure of each orbit $\mathbf{T}_{\sigma}$ is again a toric variety denoted $V(\sigma)$.

We use $\Sigma(k)$ for the set of all k-dimensional cones in Σ; in particular, $\Sigma(1) = \{\rho_1, \dots, \rho_n\}$ is the set of one-dimensional cones in Σ with the minimal integral generators $e_1, \dots, e_n$, respectively. Each one-dimensional cone ρ_i corresponds to a torus invariant divisor D_i in $\mathbf{P}_{\Sigma}$.

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A torus invariant Weil divisor $D = \sum_{i=1}^{n} a_i D_i$ determines a convex polyhedron

$$\Delta_D = \{m \in M_{\mathbb{R}} : \langle m, e_i \rangle \geqslant -a_i \text{ for all } i\} \subset M_{\mathbb{R}}.$$

Each Weil divisor D gives a reflexive sheaf $O_{\mathbf{P}_\Sigma}(D)$, whose sections over $U \subset \mathbf{P}_\Sigma$ are the rational functions f such that $\operatorname{div}(f) + D \geqslant 0$ on U. When $D = \sum_{i=1}^{n} a_i D_i$ is Cartier, there is a support function ψ_D: $N_{\mathbb{R}} \to \mathbb{R}$ that is linear on each cone $\sigma \in \Sigma$ and determined by some $m_\sigma \in M$:

$$\psi_D(e_i) = \langle m_\sigma, e_i \rangle = -a_i \qquad \text{for all } e_i \in \sigma.$$

When $\mathbf{P}_\Sigma$ is complete, the polyhedron Δ_D of a torus invariant Weil divisor is bounded and called polytope. Also, the line bundle $O_{\mathbf{P}_\Sigma}(D)$, corresponding to a Cartier divisor D, is generated by global sections if and only if ψ_D is convex.

We call a Cartier divisor D on a complete toric variety $\mathbf{P}_\Sigma$ *semiample* if $O_{\mathbf{P}_\Sigma}(D)$ is generated by global sections.

Remark 1.1. This definition is consistent with the one in [EV, Section 5] used in a non-toric context for projective varieties, because an invertible sheaf $\mathcal{L}$ on a complete toric variety is generated by global sections iff some positive power $\mathcal{L}^k$ is generated by global sections.

Theorem 1.6 in [M] Shows:#$O_{\mathbf{P}_\Sigma}(D)$ is generated by global sections is equivalent to the condition that the divisor D is nef (numerically effective). Therefore, the notions of semiample and nef are equivalent for divisors on complete toric varieties.

Following [EV, Section 5], a semiample divisor D on $\mathbf{P}_\Sigma$ also has the Iitaka dimension:

$$\kappa(D) := \kappa(O_{\mathbf{P}_\Sigma}(D)) = \dim \phi_D(\mathbf{P}_\Sigma),$$

where ϕ_D: $\mathbf{P}_\Sigma \to \mathbb{P}(H^0(\mathbf{P}_\Sigma, O_{\mathbf{P}_\Sigma}(D)))$ is the map defined by the sections of the line bundle $O_{\mathbf{P}_\Sigma}(D)$. The possible values for this characteristic are $\kappa(D) = 0, \ldots, \dim \mathbf{P}_\Sigma$. Moreover, the Exercise on page 73 in [F1, Section 3.4] shows that $\kappa(D)$ for a torus invariant D is exactly the dimension of the associated polytope Δ_D. It will be convenient for us to introduce the following notion.

DEFINITION 1.2. A semiample divisor D on a complete toric variety $\mathbf{P}_\Sigma$ is called *i-semiample* if the Iitaka dimension $\kappa(D) = i$.

Remark 1.3. In [M] we called a Cartier divisor D semiample if $O_{\mathbf{P}_\Sigma}(D)$ is generated by global sections and the intersection number $(D^d) > 0$. In fact, such divisors have the maximal Iitaka dimension $\kappa(D) = \dim \mathbf{P}_\Sigma$. In the common terminology, they correspond to big $((D^d) > 0)$ and nef, and, according to the above definition, we should call them d-semiample with $d = \dim \mathbf{P}_\Sigma$.

All ample divisors on $\mathbf{P}_\Sigma$ are semiample and have the Iitaka dimension equal to $\dim \mathbf{P}_\Sigma$. Our goal is to show that semiample divisors give rise to a natural geometric

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construction connected with ample divisors. Let $D = \sum_{k=1}^{n} a_k D_k$ be an i-semiample divisor on $\mathbf{P}_\Sigma$ with the convex support function ψ_D. For each d-dimensional cone $\sigma \in \Sigma$ there is a unique $m_\sigma \in M$ such that $\psi_D(v) = \langle m_\sigma, v\rangle$ for all $v \in \sigma$. Glue together the maximal dimensional cones in Σ with the same value m_σ. The glued set $\tau(m_\sigma)$ is a convex rational polyhedral cone. Indeed, let v be in the convex hull of $\tau(m_\sigma)$, then $\psi_D(v) \leqslant \langle m_\sigma, v\rangle$, by the convexity of the support function. On the other hand, v is lying in some d-dimensional cone, where the value of ψ_D is determined by $m' \in M$. Hence, $\psi_D(e_k) = \langle m_\sigma, e_k\rangle \leqslant \langle m', e_k\rangle$ for all generators e_k from the set $\tau(m_\sigma)$. Since v is a positive linear combination of some generators lying in $\tau(m_\sigma)$, we get $\langle m_\sigma, v\rangle \leqslant \langle m', v\rangle = \psi_D(v)$. Therefore, the glued set $\tau(m_\sigma)$ coincides with its convex hull. The new cones $\tau(m_\sigma)$ are not necessarily strongly convex, but they all contain the same linear subspace

$$\tau(m_\sigma) \cap (-\tau(m_\sigma)) = \{v \in N_{\mathbb{R}} : \psi_D(-v) = -\psi_D(v)\} \tag{1}$$

To see the equality note that $\langle m_\sigma, w\rangle \geqslant \psi_D(w)$ for any w, by the convexity of the support function. Therefore, for v in the right-hand side of (1), we have $\langle m_\sigma, -v\rangle \geqslant \psi_D(-v) = -\psi_D(v) \geqslant -\langle m_\sigma, v\rangle$ implying that $v \in \tau(m_\sigma) \cap (-\tau(m_\sigma))$. The other way is obvious. From here we get that the linear space in (1) consists of $v \in N_{\mathbb{R}}$ such that $\langle m_\sigma, v\rangle$ is the same for all σ. Since $\mathrm{O}_{\mathbf{P}_\Sigma}(D)$ is generated by global sections, the polytope Δ_D is the convex hull of m_σ. Therefore, the dimension of (1) is exactly $d - i$. If Δ_D contains the origin, this linear space can be obtained as the orthogonal complement of the polytope.

Denote by $N' = \{v \in N : \psi_D(-v) = -\psi_D(v)\}$ a sublattice of N, we also get the quotient lattice $N_D := N/N'$. Then the i-dimensional linear space $N'_{\mathbb{R}}$ is a support of a complete fan Σ' filled up by the cones of the fan Σ contained in $N'_{\mathbb{R}}$. The quotient sets $\tau(m_\sigma)/N'_{\mathbb{R}}$ in $(N_D)_{\mathbb{R}}$ are strongly convex polyhedral cones and form another complete fan Σ_D. Thus, we get the following picture: there is a natural exact sequence of lattices $0 \to N' \to N \to N_D \to 0$ compatible with the fans Σ', Σ and Σ_D, giving rise to toric morphisms $\mathbf{P}_{\Sigma'} \stackrel{v}{\to} \mathbf{P}_\Sigma \stackrel{\pi}{\to} \mathbf{P}_{\Sigma_D}$. Let us note that linearly equivalent semiample divisors D produce the same construction. The complete toric variety $\mathbf{P}_{\Sigma'}$ is mapped into an open toric subvariety $\mathbf{P}_{\tilde{\Sigma}'} \subset \mathbf{P}_\Sigma$ given by the subfan $\tilde{\Sigma}' \subset \Sigma$ of all cones that lie in $N'_{\mathbb{R}}$. Section 2.1 in [F1] shows that the above sequence of toric morphisms induces a trivial fibration over the maximal dimensional torus $\mathbf{T}_{\Sigma_D} := N_D \otimes \mathbb{C}^*$ of $\mathbf{P}_{\Sigma_D}$: $\mathbf{P}_{\Sigma'} \stackrel{v}{\to} \mathbf{P}_{\tilde{\Sigma}'} \stackrel{\pi}{\to} \mathbf{T}_{\Sigma_D}$. We next show that the above construction is unique in a certain sense. Using a standard description of a toric morphism, we can see that the toric subvarieties $V(\gamma) \subset \mathbf{P}_\Sigma$ of dimension i, such that $\gamma \in \Sigma(d - i)$ and $\gamma \subset N'_{\mathbb{R}}$, map birationally onto $\mathbf{P}_{\Sigma_D}$. As in [F1, Section], let us restrict the semiample divisor $D = \sum_{k=1}^{n} a_k D_k$ to $V(\gamma)$. Using the linear equivalence, we can assume that the origin is one of the vertices of the polytope Δ_D. In this case, Equation (1) implies that $a_k = 0$ for $\rho_k \subset N'_{\mathbb{R}}$ (equivalently, $\psi_D = 0$ on $N'_{\mathbb{R}}$), whence $V(\gamma)$ is not contained in the support of D. Therefore, we get a Weil divisor $D \cdot V(\gamma)$ in the Chow group

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$A_{i-1}(V(\gamma))$ representing the Cartier divisor $D|_{V(\gamma)}$. Its support function $\psi_{D \cdot V(\gamma)}$ is represented by ψ_D which descends to the quotient space $(N_D)_{\mathbb{R}} = N_{\mathbb{R}}/N'_{\mathbb{R}}$. The lattice $M_D := N'^{\perp} \cap M$ is the dual of N_D, and the polytope Δ_D contained in $(M_D)_{\mathbb{R}}$ is exactly the polytope of the Weil divisor $D \cdot V(\gamma)$. By construction, the function $\psi_{D \cdot V(\gamma)}$ is strictly convex with respect to the fan Σ_D. Now the arguments of [M, Section 1] show that Σ_D is the normal fan of Δ_D, and the pushforward $\pi_*(D \cdot V(\gamma))$ is an ample divisor. We also get a commutative diagram (see F[2]):

$$\begin{array}{ccc} A_{i-1}(V(\gamma)) & \xrightarrow{\pi_*} & A_{i-1}(\mathbf{P}_{\Sigma_D}) \\ \uparrow & & \uparrow \\ \mathrm{Pic}(\mathbf{P}_\Sigma) & \xleftarrow[\pi^*]{} & \mathrm{Pic}(\mathbf{P}_{\Sigma_D}), \end{array}$$

where the right vertical arrow is injective and the left is the composition $\mathrm{Pic}(\mathbf{P}_\Sigma) \to \mathrm{Pic}(V(\gamma)) \to A_{i-1}(V(\gamma))$ of the restriction map and the inclusion. Since the support function of $\pi_*(D \cdot V(\gamma))$ is induced by ψ_D, we have the equality $\pi^*\pi_*[D \cdot V(\gamma)] = [D]$ in the Chow group $A_{d-1}(\mathbf{P}_\Sigma)$.

Now we prove that the conditions on the divisor D deduced in the previous paragraph uniquely determine the constructed morphism. Let $p\colon \mathbf{P}_\Sigma \to \mathbf{P}_{\Sigma_1}$ be a surjective morphism of complete toric varieties arising from a surjective homomorphism of lattices $\tilde{p}\colon N \to N_1$ which maps the fan Σ into Σ_1. The kernel of $\tilde{p}$ is a sublattice $N_2 \subset N$. It is not difficult to see that a cone of Σ is either lying in the space $(N_2)_{\mathbb{R}}$ or its relative interior has no intersection with this space. Hence, the space $(N_2)_{\mathbb{R}}$ is a support of a complete fan Σ_2 filled up by those cones of Σ lying in $(N_2)_{\mathbb{R}}$. The toric subvarieties $V(\gamma)$ corresponding to $\gamma \in \Sigma(d-k)$ ($k := \dim \mathbf{P}_{\Sigma_1}$), contained in $(N_2)_{\mathbb{R}}$, are the only ones mapping birationally onto $\mathbf{P}_{\Sigma_1}$. Suppose now that we have an i-semiample (torus invariant) divisor D on $\mathbf{P}_\Sigma$ such that $p_*[D \cdot V(\gamma)]$ is ample and $p^*p_*[D \cdot V(\gamma)] = [D]$ for some $V(\gamma)$, $\gamma \in \Sigma(d-k)$, which maps birationally onto $\mathbf{P}_{\Sigma_1}$. Then the polytope of the divisor $p_*(D \cdot V(\gamma))$ has dimension equal to $\dim \mathbf{P}_{\Sigma_1}$. On the other hand, the support function of D is induced by the support function of $p_*(D \cdot V(\gamma))$, implying that the polytopes of these divisors is the same set in $M \cap N_2^{\perp}$. Therefore, the dimension of $\mathbf{P}_{\Sigma_1}$ is i, and the fan Σ_1 coincides with Σ_D constructed before. Thus, we proved the following.

THEOREM 1.4. *Let $[D] \in A_{d-1}(\mathbf{P}_\Sigma)$ be an i-semiample divisor class on a complete toric variety $\mathbf{P}_\Sigma$ of dimension d. Then, there exists a unique complete toric variety $\mathbf{P}_{\Sigma_D}$ with a surjective morphism $\pi\colon \mathbf{P}_\Sigma \to \mathbf{P}_{\Sigma_D}$, corresponding to a map of Σ into Σ_D, such that $\pi_*[D \cdot V(\gamma)]$ is ample and $\pi^*\pi_*[D \cdot V(\gamma)] = [D]$ for some closed toric subvariety $V(\gamma) \subset \mathbf{P}_\Sigma$, $\gamma \in \Sigma$, which maps birationally onto $\mathbf{P}_{\Sigma_D}$. Moreover, $\dim \mathbf{P}_{\Sigma_D} = i$, and the fan Σ_D is the normal fan of Δ_D for a torus invariant D.*

Remark 1.5. The fan Σ_D is canonical with respect to the equivalence relation on the divisors. Therefore, it will sometimes be convenient for us to use the notation $\Sigma_\beta := \Sigma_D$ for a semiample divisor class $\beta = [D] \in A_{d-1}(\mathbf{P}_\Sigma)$.

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While a restriction of a semiample divisor D on $\mathbf{P}_\Sigma$ to a closed toric subvariety is again a semiample divisor, the Iitaka dimension of the restricted divisor may change. Let us investigate this problem. If D is an i-semiample divisor on $\mathbf{P}_{\Sigma_1}$ then, by the above theorem, we have a unique toric morphism $\pi\colon \mathbf{P}_\Sigma \to \mathbf{P}_{\Sigma_D}$, arising from a homomorphism $\tilde{\pi}\colon N_{\mathbb{R}} \to (N_D)_{\mathbb{R}}$ mapping Σ into Σ_D. This morphism encodes information about the structure of the variety $\mathbf{P}_\Sigma$. The Iitaka dimension of the semiample divisor $D \cdot V(\sigma)$ on $V(\sigma)$, $\sigma \in \Sigma$, can be determined in the following way. The complete toric variety $V(\sigma)$ is mapped onto a closed subvariety $V(\sigma_0) \subset \mathbf{P}_{\Sigma_D}$ such that the cone $\sigma_0 \in \Sigma_D$ is the smallest that contains $\tilde{\pi}(\sigma)$. We claim that this induced map $\pi\colon V(\sigma) \to V(\sigma_0)$ is exactly the one associated with the semiample divisor $D \cdot V(\sigma)$. To prove this we will verify the conditions which uniquely determine such a morphism. As in the theorem above, let $V(\gamma)$ be such that $\pi_*[D \cdot V(\gamma)]$ is ample and $\pi^*\pi_*[D \cdot V(\gamma)] = [D]$, and let $V(\gamma') \subset V(\sigma)$ be a closed toric subvariety mapping birationally onto $V(\sigma_0)$. By the projection formula (see [F2]), we get

$$\begin{aligned}\pi_*[D \cdot V(\gamma')] &= \pi_*[(\pi^*\pi_*[D \cdot V(\gamma)]) \cdot V(\gamma')]\\ &= \pi_*[D \cdot V(\gamma)] \cdot \pi_*[V(\gamma')] = \pi_*[D \cdot V(\gamma)] \cdot V(\sigma_0)\end{aligned}$$

in the Chow group of the toric variety $V(\sigma_0)$. Since $\pi_*[D \cdot V(\gamma)]$ is ample, the divisor class $\pi_*[D \cdot V(\gamma')]$ is ample as well. The other condition for the semiample divisor $D \cdot V(\sigma)$ also follows:

$$\pi^*\pi_*[D \cdot V(\gamma')] = \pi^*[\pi_*[D \cdot V(\gamma)] \cdot V(\sigma_0)] = \pi^*\pi_*[D \cdot V(\gamma)] \cdot V(\sigma) = [D \cdot V(\sigma)],$$

where we used the commutative diagram

$$\begin{array}{ccc} \mathrm{Pic}(\mathbf{P}_{\Sigma_D}) & \xrightarrow{\pi^*} & \mathrm{Pic}(\mathbf{P}_\Sigma) \\ \downarrow & & \downarrow \\ \mathrm{Pic}(V(\sigma_0)) & \xrightarrow{\pi^*} & \mathrm{Pic}(V(\sigma)). \end{array}$$

Thus, by the uniqueness part of Theorem 1.4, we get the next result.

PROPOSITION 1.6. *Let $[D] \in A_{d-1}(\mathbf{P}_\Sigma)$ be an i-semiample divisor class on $\mathbf{P}_\Sigma$ with the associated morphism π: $\mathbf{P}_\Sigma \to \mathbf{P}_{\Sigma_D}$ arising from a map of the fan Σ into Σ_D. Then, for $\sigma \in \Sigma$, the restriction $[D \cdot V(\sigma)]$ is a k-semiample divisor class on $V(\sigma)$ with $k = i - \dim(\sigma_0) = \dim V(\sigma_0)$, where $\sigma_0 \in \Sigma_D$ is the smallest cone that contains the image of σ. Moreover, the induced map π: $V(\sigma) \to V(\sigma_0)$ is the one associated with the semiample divisor class $[D \cdot V(\sigma)]$.*

This proposition says that the maps associated with the semiample divisors are compatible with the restrictions.

Any toric variety $\mathbf{P}_\Sigma$ has a homogeneous coordinate ring $S(\Sigma) = \mathbb{C}[x_1, \dots, x_n]$ with variables $x_1, \dots, x_n$ corresponding to the irreducible torus invariant divisors $D_1, \dots, D_n$. This ring is graded by the Chow group $A_{d-1}(\mathbf{P}_\Sigma)$, assigning $[\sum_{i=1}^n a_i D_i]$ to $\deg\left(\prod_{i=1}^n x_i^{a_i}\right)$. For a Weil divisor D on $\mathbf{P}_\Sigma$, there is an isomorphism

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$H^0(\mathbf{P}_\Sigma, \mathrm{O}_{\mathbf{P}_\Sigma}(D)) \cong S(\Sigma)_\alpha$, where $\alpha = [D] \in A_{d-1}(\mathbf{P}_\Sigma)$. If D is torus invariant, the monomials in $S(\Sigma)_\alpha$ correspond to the lattice points of the associated polyhedron Δ_D.

Now consider a proper birational morphism $\pi: \mathbf{P}_{\Sigma_1} \to \mathbf{P}_{\Sigma_2}$ of toric varieties, associated with a subdivision Σ_1 of Σ_2. In this situation, the one-dimensional cones of the two fans are related by $\Sigma_2(1) \subset \Sigma_1(1)$, and there is a natural relation of the coordinate rings $S(\Sigma_1) = \mathbb{C}[x_k: \rho_k \in \Sigma_1(1)]$ and $S(\Sigma_2) = \mathbb{C}[y_k: \rho_k \in \Sigma_2(1)]$ of the toric varieties. For $\alpha = [\mathrm{O}_{\mathbf{P}_{\Sigma_1}}(D)] \in A_{d-1}(\mathbf{P}_{\Sigma_1})$ we have a commutative diagram:

$$\begin{array}{ccc} S(\Sigma_1)_\alpha & \cong & H^0(\mathbf{P}_{\Sigma_1}, \mathrm{O}_{\mathbf{P}_{\Sigma_1}}(D)) \\ \downarrow & & \downarrow \\ S(\Sigma_2)_{\pi_*\alpha} & \cong & H^0(\mathbf{P}_{\Sigma_2}, \mathrm{O}_{\mathbf{P}_{\Sigma_2}}(\pi_* D)), \end{array}$$

where the left vertical arrow sends a monomial $\prod_{\rho_k \in \Sigma_1(1)} x_k^{a_k + \langle m, e_k \rangle}$ in $S(\Sigma_1)_\alpha$ to $\prod_{\rho_k \in \Sigma_2(1)} y_k^{a_k + \langle m, e_k \rangle}$, and the right vertical arrow is induced by the natural morphism of sheaves $\pi_* \mathrm{O}_{\mathbf{P}_{\Sigma_1}}(D) \to \mathrm{O}_{\mathbf{P}_{\Sigma_2}}(\pi_* D)$. This gives a graded ring homomorphism $\pi_*: S(\Sigma_1) \to S(\Sigma_2)$ which sends x_k to y_k, if $\rho_k \in \Sigma_2(1)$, and sends x_k to 1, otherwise.

We now apply the above to semiample divisors. Let D be a semiample (torus invariant) divisor on a complete toric variety $\mathbf{P}_\Sigma$ in degree $\beta \in A_{d-1}(\mathbf{P}_\Sigma)$. on a complete toric variety $\mathbf{P}_\Sigma$, By Theorem 1.4, we get the associated toric morphism $\pi: \mathbf{P}_\Sigma \to \mathbf{P}_{\Sigma_D}$ such that $\pi_*[D \cdot V(\gamma)]$ is ample and $\pi^*\pi_*[D \cdot V(\gamma)] = [D]$ for some closed toric subvariety $V(\gamma) \subset \mathbf{P}_\Sigma$, $\gamma \in \Sigma$, which maps birationally onto $\mathbf{P}_{\Sigma_D}$. In this situation, there is the following natural diagram:

$$\begin{array}{ccccc} S(\Sigma)_{p\beta} & \longrightarrow & S(V(\gamma))_{p\bar\beta} & \longrightarrow & S(\Sigma_D)_{p\pi_*\bar\beta} \\ \downarrow & & \downarrow & & \downarrow \\ H^0(\mathbf{P}_\Sigma, \mathrm{O}_{\mathbf{P}_\Sigma}(pD)) & \longrightarrow & H^0(V(\gamma), O_{V(\gamma)}(pD_\gamma)) & \longrightarrow & H^0(\mathbf{P}_{\Sigma_D}, \mathrm{O}_{\mathbf{P}_{\Sigma_D}}(p\pi_* D_\gamma)), \end{array}$$

where $\bar\beta = [D_\gamma]$, $D_\gamma := D \cdot V(\gamma)$, in the Chow group of $V(\gamma)$, and the vertical arrows are isomorphisms. Since the monomials in $S(\Sigma)_{p\beta}$ and $S(\Sigma_D)_{p\pi_*\bar\beta}$ correspond to the lattice points of the same polytope $p\Delta_D$, we get the isomorphisms

$$S(\Sigma)_{p\beta} \cong S(V(\gamma))_{p\bar\beta} \cong S(\Sigma_D)_{p\pi_*\bar\beta}.$$

2. Toric Hypersurfaces

Here, we apply the results of the previous section to semiample hypersurfaces in a complete toric variety $\mathbf{P}_\Sigma$, which have only transversal intersections with the torus-orbits. We also review some results about hypersurfaces in complete simplicial toric varieties. As a reference we use [M] and [BC].

A hypersurface $X \subset \mathbf{P}_\Sigma$ is called Σ-regular (or simply *nondegenerate*) if $X \cap T_\sigma$ is empty or a smooth subvariety of codimension 1 in each torus T_σ for $\sigma \in \Sigma$.

By [D, Proposition 6.8], a generic hypersurface $X \subset \mathbf{P}_\Sigma$ of a given semiample degree is Σ-regular.

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LEMMA 2.1. *Let X be an i-semiample hypersurface in a complete toric variety $\mathbf{P}_\Sigma$ with $i > 1$. Then X is connected, and X is irreducible if X is Σ-regular.*

Proof. The arguments are the same as for Lemma 2.3 in [M]. □

Remark 2.2. Let us note that a 0-semiample hypersurface is always empty because its divisor class is trivial.

PROPOSITION 2.3. *Let X be a Σ-regular semiample hypersurface in a complete toric variety $\mathbf{P}_\Sigma$, and let $\pi\colon \mathbf{P}_\Sigma \to \mathbf{P}_{\Sigma_X}$ be the associated morphism for $[X] \in A_{d-1}(\mathbf{P}_\Sigma)$, then $Y = \pi(X)$ is a Σ_X-regular ample hypersurface, and $X = \pi^{-1}(Y)$.*

Proof. Start with the case of an i-semiample hypersurface with $i > 1$. From Theorem 1.4 we have a closed toric subvariety $V(\gamma) \subset \mathbf{P}_\Sigma$, for $\gamma \in \Sigma$, which maps birationally onto $\mathbf{P}_{\Sigma_X}$ such that $\pi_*[X \cdot V(\gamma)]$ is ample and $\pi^*\pi_*[X \cdot V(\gamma)] = [X]$. Since X is transversal to the orbits of $\mathbf{P}_\Sigma$, the divisor class of the hypersurface $X \cap V(\gamma)$ in $V(\gamma)$ is exactly $[X \cdot V(\gamma)]$. Proposition 1.6 implies that $[X \cdot V(\gamma)]$ is an i-semiample divisor class in $A_{i-1}(V(\gamma))$. The value i is the maximum for the possible Iitaka dimensions of semiample divisors on the toric variety $V(\gamma)$. Applying Remark 1.3 of the previous section and Proposition 2.4 in [M], we get that $\pi(X \cap V(\gamma))$ is a Σ_X-regular ample hypersurface. On the other hand, by Lemma 2.1, the hypersurface X is irreducible. Therefore, its image $Y = \pi(X)$ is also irreducible. Since $\dim \pi(X) \leqslant i$ and $\pi(X \cap V(\gamma)) \subset \pi(X)$, the hypersurface $\pi(X)$ coincides with $\pi(X \cap V(\gamma))$. The hypersurface Y is ample nondegenerate and does not intersect the zero-dimensional orbits. Together with the facts that X and $\pi(X)$ are irreducible this implies the property $X = \pi^{-1}(Y)$.

The case of a 1-semiample hypersurface is special because such a hypersurface is not necessarily connected. In this situation, we have a closed toric subvariety $V(\gamma) \subset \mathbf{P}_\Sigma$, for $\gamma \in \Sigma$, which maps isomorphically onto $\mathbf{P}_{\Sigma_X} \cong \mathbb{P}^1$ such that $\pi_*[X \cdot V(\gamma)]$ is ample and $\pi^*\pi_*[X \cdot V(\gamma)] = [X]$. It follows from Proposition 1.6 and Remark 2.2 that the image $Y = \pi(X)$ is contained in the one dimensional torus of $\mathbf{P}_{\Sigma_X} \cong \mathbb{P}^1$. The preimage $\pi^{-1}(Y)$ of this finite set can be easily seen from the description of the toric morphism π in Section 1. This morphism is a trivial fibration over the one-dimensional torus of $\mathbf{P}_{\Sigma_X}$, and each point of $\pi(X)$ gives exactly one irreducible component of $\pi^{-1}(Y)$ which is actually a complete toric variety.

On the other hand, each point of $\pi(X)$ came from an irreducible component of $X \subset \pi^{-1}(Y)$. Hence, $X = \pi^{-1}(Y)$. This gives an isomorphism $\pi\colon X \cap V(\gamma) \cong \pi(X)$. Thus, $\pi(X)$ is a Σ_X-regular ample hypersurface. □

Remark 2.4. If, in addition, we assume in this proposition that X is an anticanonical hypersurface in $\mathbf{P}_\Sigma$, then X is big and $\mathbf{P}_{\Sigma_X}$ is a Fano toric variety associated to a reflexive polytope, and this corresponds to the construction in [B2].

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Let Y be an ample nondegenerate hypersurface in a complete toric variety $\mathbf{P}_\Sigma$. A hypersurface in the torus $\mathbf{T} \subset \mathbf{P}_\Sigma$ isomorphic to the affine hypersurface $Y \cap \mathbf{T}$ in $\mathbf{T}$ is called *nondegenerate*. Cohomology of such hypersurfaces has been studied in [DK] and [B2].

LEMMA 2.5 ([DK]). *Let Z be a nondegenerate affine hypersurface in the torus $\mathbf{T}$, then the natural map $H^i(\mathbf{T}) \to H^i(Z)$, induced by the inclusion, is an isomorphism of Hodge structures for $i < \dim \mathbf{T} - 1$ and an injection for $i = \dim \mathbf{T} - 1$.*

Using the standard description of a toric morphism, from Proposition 2.3 we get a stratification of an i-semiample nondegenerate hypersurface $X \subset \mathbf{P}_\Sigma$ in terms of nondegenerate affine hypersurfaces:

$$X \cap \mathbf{T}_\sigma \cong (\pi(X) \cap \mathbf{T}_{\sigma_0}) \times (\mathbb{C}^*)^l, \tag{2}$$

where $\pi\colon \mathbf{P}_\Sigma \to \mathbf{P}_{\Sigma_X}$ is the associated morphism, $l = d - i + \dim \sigma_0 - \dim \sigma$, $d = \dim \mathbf{P}_\Sigma$, and $\sigma_0 \in \Sigma_X$ is the smallest cone containing the image of $\sigma \in \Sigma$.

From here on, we assume that $\mathbf{P} := \mathbf{P}_\Sigma$ denotes a complete simplicial toric variety. In this case, [BC] shows that homogeneous polynomials in $S := S(\Sigma)$ determine hypersurfaces in $\mathbf{P}$. In terms of the coordinate ring S, a nondegenerate hypersurface in $\mathbf{P}$ defined by a homogeneous polynomial $f \in S_\beta$ is characterized by the condition that $x_1(\partial f/\partial x_1), \dots, x_n(\partial f/\partial x_n)$ do not vanish simultaneously on $\mathbf{P}$ (see [C2, Proposition 5.3]). A more general class of hypersurfaces in $\mathbf{P}$ called *quasismooth* is defined by a similar condition that $\partial f/\partial x_1, \dots, \partial f/\partial x_n$ do not vanish simultaneously on $\mathbf{P}$ (see [BC]).

We also like to mention the following fact.

PROPOSITION 2.6. *An anticanonical quasismooth hypersurface X in a Gorenstein complete simplicial toric variety $\mathbf{P}$ is Calabi–Yau.*

Proof. A quasismooth hypersurface is an orbifold (see [BC]), and for a $(d-1)$-dimensional orbifold X Calabi–Yau means that $\Omega_X^{d-1} \simeq O_X$ and $H^i(X, O_X) = 0$ for $i = 1, \dots, d-2$ (see [CK, A.2]). The arguments of the proof that anticanonical implies Calabi–Yau are the same as in [C3]: use the adjunction formula $\Omega_X^{d-1} \simeq \Omega_{\mathbf{P}}^d(X) \otimes O_X$, the isomorphism $O_{\mathbf{P}}(-X) \simeq \Omega_{\mathbf{P}}^d$ and the exact sequence $0 \to O_{\mathbf{P}}(-X) \to O_{\mathbf{P}} \to O_X \to 0$. □

DEFINITION 2.7 ([BC]). Fix an integer basis $m_1, \dots, m_d$ for the lattice M. Then given subset $I = \{i_1, \dots, i_d\} \subset \{1, \dots, n\}$, denote $\det(e_I) = \det(\langle m_j, e_{i_k}\rangle_{1 \leqslant j,k \leqslant d})$, $\mathrm{d}x_I = \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_d}$ and $\hat{x}_I = \prod_{i \notin I} x_i$. Define the d-form Ω by the formula $\Omega = \sum_{|I|=d} \det(e_I)\hat{x}_I \mathrm{d}x_I$, where the sum is over all d element subsets $I \subset \{1, \dots, n\}$.

Let $X \subset \mathbf{P}$ be a quasismooth (not necessarily Cartier) hypersurface defined by

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$f \in S_\beta$. For $A \in S_{(a+1)\beta-\beta_0}$ (here, $\beta_0 = \sum_{i=1}^n \deg(x_i)$), consider a rational d-form

$$\omega_A := A\Omega/f^{a+1} \in H^0(\mathbf{P}, \Omega^d_{\mathbf{P}}((a+1)X)).$$

This form gives a class in $H^d(\mathbf{P} \setminus X)$, and by the residue map $\mathrm{Res} : H^d(\mathbf{P} \setminus X) \to H^{d-1}(X)$ we get $\mathrm{Res}(\omega_A) \in H^{d-1}(X)$.

Remark 2.8. The residue map and the residues of rational differential forms with poles along a nondegenerate hypersurface are well defined even if the toric variety is not simplicial (see the proof of Theorem 3.7 in [DK] and Remark 6.4 in [B2]).

DEFINITION 2.9 ([BC]). Given $f \in S_\beta$, we have the *Jacobian ideal* $J(f)$ in S generated by the partial derivatives $\partial f/\partial x_1, \ldots, \partial f/\partial x_n$, the ideal

$$J_0(f) = \langle x_1(\partial f/\partial x_1), \ldots, x_n(\partial f/\partial x_n)\rangle$$

and the ideal quotient (see [CLO, p. 193]) $J_1(f) = J_0(f) : x_1 \cdots x_n$. These give the *Jacobian ring* $R(f) = S/J(f)$, $R_0(f) = S/J_0(f)$ and $R_1(f) = S/J_1(f)$ graded by the Chow group $A_{d-1}(\mathbf{P})$.

In [M] we have shown that the induced maps

$$\mathrm{Res}(\omega_\_)^{d-1-q,q} : R(f)_{(q+1)\beta-\beta_0} \to H^q(X, \Omega_X^{d-1-q})$$

(sending A to the Hodge component $\mathrm{Res}(\omega_A)^{d-1-q,q}$) for a quasismooth hypersurface $X \subset \mathbf{P}$ and, respectively,

$$\mathrm{Res}(\omega_\_)^{d-1-q,q} : R_1(f)_{(q+1)\beta-\beta_0} \to H^q(X, \Omega_X^{d-1-q})$$

for a big and nef nondegenerate hypersurface are well defined. There we also studied the relationship between the multiplicative structure on $R(f)$ (resp., $R_1(f)$) and the cup product on the middle cohomology of a quasismooth (resp., big and nef nondegenerate) hypersurface in $\mathbf{P}$. From Theorem 4.4 [M] we have the following description of the middle cohomology of big and nef nondegenerate hypersurfaces $X \subset \mathbf{P}_\Sigma$:

$$H^{d-1-q,q}(X) \cong R_1(f)_{(q+1)\beta-\beta_0} \bigoplus \left(\sum_{i=1}^n \varphi_{i!} H^{d-2-q,q-1}(X \cap D_i)\right), \tag{3}$$

where $\varphi_{i!}$ are the Gysin maps for $\varphi_i : X \cap D_i \hookrightarrow X$. In the case, when the dimension of the ambient space is 4 we have (see [M, Theorem 5.2]):

THEOREM 2.10. *Let* $X \subset \mathbf{P}_\Sigma$, $\dim \mathbf{P}_\Sigma = 4$, *be a big and nef nondegenerate hypersurface defined by* $f \in S_\beta$. *Then there is a natural isomorphism*

$$H^{3-q,q}(X) \cong R_1(f)_{(q+1)\beta-\beta_0} \bigoplus \left(\bigoplus_{\sigma \in \Sigma_X(2)} (R_1(f_\sigma)_{q\beta^\sigma - \beta_0^\sigma})^{n(\sigma)} \right),$$

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where $n(\sigma)$ is the number of cones ρ_i such that $\rho_i \subset \sigma$ and $\rho_i \notin \Sigma_X(1)$, and where f_σ is the polynomial of degree β^σ, defining the ample hypersurface $\pi(X) \cap V(\sigma) \subset V(\sigma)$ (here, π: $\mathbf{P}_\Sigma \to \mathbf{P}_{\Sigma_X}$ is the associated morphism), and β_0^σ is the degree of the anticanonical divisor on the two-dimensional toric variety $V(\sigma)$.

3. Polynomial Part of the Chiral Ring

Here we show that for a quasismooth hypersurface X of degree β there is a homomorphism between $R(f)_{*\beta}$ and the chiral ring $H^*(X, \wedge^* \mathcal{T}_X)$. We will also show that $R_1(f)_{*\beta}$ is a subring of the chiral ring for a semiample anticanonical nondegenerate hypersurface $X \subset \mathbf{P}$ (which is Calabi–Yau). This subring may be called 'polynomial' because its graded piece in $H^1(X, \mathcal{T}_X)$ should correspond to polynomial infinitesimal deformations of X performed in the toric variety $\mathbf{P}$ (see [CK]).

Let Ω_X^p be the sheaf of *Zariski p-forms* on an orbifold X (see Appendix A.3 in [CK]). We can also define $\wedge^p \mathcal{T}_X := (\Omega_X^p)^* = \mathrm{Hom}_{O_X}(\Omega_X^p, O_X)$ for an orbifold X. We call this the (*Zariski*) pth exterior power of the tangent sheaf of X. For $p = 1$ this sheaf is isomorphic to the usual tangent sheaf Θ_X, by Proposition A.4.1 in [CK]. When X is smooth, $\wedge^p \mathcal{T}_X$ coincides with the standard exterior power sheaf. Moreover, if $j : X_o \subset X$ is the inclusion of the smooth locus of X, then the argument in the proof of Proposition 3.10 in [Od] shows that $j_*(\bigwedge^p \Theta_{X_o}) = \wedge^p \mathcal{T}_X$. One can use the same argument to prove $\Omega_X^p \simeq (\wedge^p \mathcal{T}_X)^*$ and that Ω_X^p is isomorphic to the dual $(\bigwedge^p \Theta_X)^*$ of the usual pth exterior power of Θ_X, whence $\wedge^p \mathcal{T}_X \simeq (\bigwedge^p \Theta_X)^{**}$. In particular, we also have the natural maps of sheaves $\wedge^p \mathcal{T}_X \otimes \wedge^q \mathcal{T}_X \to \wedge^{p+q} \mathcal{T}_X$ and $\wedge^p \mathcal{T}_X \otimes \Omega_X^q \to \Omega_X^{q-p}$.

Let $X \subset \mathbf{P}$ be a quasismooth hypersurface defined by $f \in S_\beta$, which is an orbifold as we know from [BC]. By definition of quasismooth, we get an open cover $\mathcal{U} = \{U_i\}_{i=1}^n$ of $\mathbf{P}$, where $U_i = \{x \in \mathbf{P} : f_i(x) \neq 0\}$ and f_i denotes the partial derivative $\partial f / \partial x_i$.

DEFINITION 3.1. Denote $\partial_{i_0 \cdots i_p} = \partial / \partial x_{i_0} \wedge \cdots \wedge \partial / \partial x_{i_p}$ for an ordered subset $\{i_0, \ldots, i_p\}$ in $\{1, \ldots, n\}$. Then given $A \in S_{p\beta}$, set

$$(\gamma_A)_{i_0 \ldots i_p} = \left\{ \frac{(-1)^{p^2/2} A \langle \partial_{i_0 \ldots i_p}, \mathrm{d}f \rangle}{f_{i_0} \cdots f_{i_p}} \right\}_{i_0 \ldots i_p},$$

where $\langle\, ,\, \rangle$ denotes the contraction (the extra factor of $(-1)^{p^2/2}$ which is $\sqrt{-1}$ for odd p is added to make convenient commutative diagrams later).

This defines a Čech cocycle, giving its class in $\check{H}^p(\mathcal{U}|_X, \wedge^p \mathcal{T}_X)$. Indeed, $(\gamma_A)_{i_0 \ldots i_p}$ is homogeneous of degree 0 and is a cochain in $C^p(\mathcal{U}|_X, \wedge^p \mathcal{T}_X)$ by the exact sequence

$$0 \to \wedge^p \mathcal{T}_{X_o} \to i^* \wedge^p \mathcal{T}_{\mathbf{P}_o} \stackrel{i^* \mathrm{d}f}{\to} \wedge^{p-1} \mathcal{T}_{X_o} \otimes O_{X_o}(X)$$

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(where $i: X \subset \mathbf{P}$ is the inclusion, $\mathbf{P}_o$ is the smooth locus of $\mathbf{P}$ such that $X_o = \mathbf{P}_o \cap X$ (see [Hi, Section 4, p. 55])), and because of $\langle\langle\partial_{i_0\dots i_p}, \mathrm{d}f\rangle, \mathrm{d}f\rangle = 0$ since $\mathrm{d}f \wedge \mathrm{d}f = 0$. On the other hand, it is straightforward to verify that

$$(\gamma_A)_{i_0\dots i_p} = \left\{\frac{(-1)^{p^2/2}A}{f_{i_0}\cdots f_{i_p}}\sum_{j=0}^{p}(-1)^j f_{i_j}\partial_{i_0\dots\widehat{i_j}\dots i_p}\right\}_{i_0\dots i_p}$$

vanishes under the Čech coboundary map $C^p(\mathcal{U}|_X, \wedge^p\mathcal{T}_X) \to C^{p+1}(\mathcal{U}|_X, \wedge^p\mathcal{T}_X)$. One can actually show that $(\gamma_A)_{i_0\dots i_p}$ is a coboundary in $C^p(\mathcal{U}|_X, i^* \wedge^p \mathcal{T}_\mathbf{P})$.

For $A \in S_{p\beta}$ let $\gamma_A \in H^p(X, \wedge^p\mathcal{T}_X)$ be the image of the Čech cocycle $(\gamma_A)_{i_0\dots i_p}$ under the natural map $\check{H}^p(\mathcal{U}|_X, \wedge^p\mathcal{T}_X) \to H^p(X, \wedge^p\mathcal{T}_X)$. And we get a well defined map $\gamma: R(f)_{*\beta} \to H^*(X, \wedge^*\mathcal{T}_X)$ because of the following statement.

LEMMA 3.2. *If $A \in J(f)_{p\beta}$, then the cocycle $(\gamma_A)_{i_0\dots i_p}$ is a Čech coboundary in $C^p(\mathcal{U}|_X, \wedge^p\mathcal{T}_X)$.*

Proof. If $A \in J(f)_{p\beta}$, then we can assume that A is a multiple of one of the partial derivatives $f_k = \partial f/\partial x_k$. We have

$$f_k\frac{\langle\partial_{i_0\dots i_p}, \mathrm{d}f\rangle}{f_{i_0}\dots f_{i_p}} = \sum_{j=0}^{p}(-1)^j\frac{f_k\partial_{i_0\dots\widehat{i_j}\dots i_p}}{f_{i_0}\dots\widehat{f_{i_j}}\dots f_{i_p}} = \sum_{j=0}^{p}(-1)^j\frac{\langle\partial_k, \mathrm{d}f\rangle\partial_{i_0\dots\widehat{i_j}\dots i_p}}{f_{i_0}\dots\widehat{f_{i_j}}\dots f_{i_p}} -$$
$$- \sum_{j=0}^{p}(-1)^j\frac{\partial_k \wedge \left\langle\partial_{i_0\dots\widehat{i_j}\dots i_p}, \mathrm{d}f\right\rangle}{f_{i_0}\dots\widehat{f_{i_j}}\dots f_{i_p}} = \sum_{j=0}^{p}(-1)^j\frac{\left\langle\partial_{ki_0\dots\widehat{i_j}\dots i_p}, \mathrm{d}f\right\rangle}{f_{i_0}\dots\widehat{f_{i_j}}\dots f_{i_p}},$$

where the second sum after the second equality is identically zero. Hence, it follows that $(\gamma_A)_{i_0\dots i_p}$ is in the image of the Čech coboundary map $C^{p-1}(\mathcal{U}|_X, \wedge^p\mathcal{T}_X) \to C^p(\mathcal{U}|_X, \wedge^p\mathcal{T}_X)$. □

We now study the compatibility of the multiplication in the Jacobian ring $R(f)$ and the cohomology ring $H^*(X, \wedge^*\mathcal{T}_X)$. The cocycle $(\gamma_A)_{i_0\dots i_p}$ (up to an extra factor) and the calculations in the next two theorems are essentially due to D. Cox and D. Morrison.

THEOREM 3.3. *Let $X \subset \mathrm{P}$ be a quasismooth hypersurface defined by $f \in S_\beta$. The map $R(f)_{*\beta} \to H^*(X, \wedge^*\mathcal{T}_X)$, assigning γ_A to a polynomial A, is a ring homomorphism.*

Proof. We need to show that $\gamma_A \cup \gamma_B = \gamma_{AB}$ for $A \in S_{p\beta}$ and $B \in S_{q\beta}$. Similar to [CaG, page 63], the cup product $\gamma_A \cup \gamma_B$ is represented by the Čech cocycle

$$(-1)^{pq}\left\{\frac{(-1)^{p^2/2}A\langle\partial_{i_0\dots i_p}, \mathrm{d}f\rangle \wedge (-1)^{q^2/2}B\langle\partial_{i_p\dots i_{p+q}}, \mathrm{d}f\rangle}{f_{i_0}\cdots f_{i_{p+q}}\cdot f_{i_p}}\right\}_{i_0\dots i_{p+q}}.$$

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Note that

$$\langle\partial_{i_0\ldots i_p},\mathrm{d}f\rangle\wedge\langle\partial_{i_p\ldots i_{p+q}},\mathrm{d}f\rangle=\sum_{j=0}^{p}(-1)^j f_{i_j}\partial_{i_0\ldots\widehat{i_j}\ldots i_p}\wedge\sum_{l=p}^{p+q}(-1)^{l-p}f_{i_l}\partial_{i_p\ldots\widehat{i_l}\ldots i_{p+q}}$$
$$=f_{i_p}\sum_{j=0}^{p+q}(-1)^j f_{i_j}\partial_{i_0\ldots\widehat{i_j}\ldots i_p}=f_{i_p}\langle\partial_{i_o\ldots i_{p+q}},\mathrm{d}f\rangle,\qquad(4)$$

where we used $\partial_{i_p}\wedge\partial_{i_p}=0$. Hence, the result follows. □

The middle cohomology of a quasismooth hypersurface $X\subset \mathbf{P}$ is a module over $H^*(X,\wedge^*\mathcal{T}_X)$ with respect to the natural cup product

$$H^p(X,\wedge^p\mathcal{T}_X)\otimes H^q(X,\Omega_X^{d-1-q})\xrightarrow{\cup}H^{p+q}(X,\Omega_X^{d-1-p-q}).$$

From the previous section we know that there is a natural map

$$\mathrm{Res}(\omega)^{d-1-q,q}\colon R(f)_{(q+1)\beta-\beta_0}\to H^q(X,\Omega_X^{d-1-q}).$$

We normalize this map as $[\omega_A]=(-1)^{q/2}q!\mathrm{Res}(\omega_A)^{d-1-q,q}$ (where we assume $(-1)^{q/2}=(\sqrt{-1})^q$) to show that this gives a morphism of modules $R(f)_{(*+1)\beta-\beta_0}\to H^*(X,\Omega_X^{d-1-*})$.

THEOREM 3.4. *Let $X\subset\mathbf{P}$ be a quasismooth hypersurface defined by $f\in S_\beta$. Then the diagram*

$$\begin{array}{ccc}R(f)_{p\beta}\otimes R(f)_{(q+1)\beta-\beta_0} & \longrightarrow & R(f)_{(p+q+1)\beta-\beta_0}\\ \big\downarrow{\scriptstyle\gamma\_\otimes[\omega\_]} & & \big\downarrow{\scriptstyle[\omega\_]}\\ H^p(X,\wedge^p\mathcal{T}_X)\otimes H^q(X,\Omega_X^{d-1-q}) & \xrightarrow{\cup} & H^{p+q}(X,\Omega_X^{d-1-p-q})\end{array}$$

commutes, where the top arrow is induced by the multiplication. When $X\subset\mathbf{P}$ is a d-semiample nondegenerate hypersurface the same diagram commutes with $R_1(f)_{(*+1)\beta-\beta_0}$ in place of $R(f)_{(*+1)\beta-\beta_0}$.

Proof. From Theorem 3.3 in [M] we know that $[\omega_B]=(-1)^{q/2}q!\mathrm{Res}(\omega_B)^{d-1-q,q}$, for $B\in S_{(q+1)\beta-\beta_0}$, is represented by the Čech cocycle

$$(-1)^{d-1+(q(q+2)/2)}\left\{\frac{BK_{i_q}\cdots K_{i_0}\Omega}{f_{i_0}\cdots f_{i_q}}\right\}_{i_0\ldots i_q}\in\check{H}^q(\mathcal{U}|_X,\Omega_X^{d-1-q}),$$

where K_i is the contraction operator $(\partial/\partial x_i)\lrcorner$. Therefore, for $A\in S_{p\beta}$ the cup product $\gamma_A\cup[\omega_B]$ is represented by the Čech cocycle

$$\left\{\frac{(-1)^{p^2/2}A\langle\partial_{i_0\ldots i_p},\mathrm{d}f\rangle}{f_{i_0}\cdots f_{i_p}}\lrcorner\ \frac{(-1)^{d-1+(q(q+2)/2)}BK_{i_{p+q}}\cdots K_{i_p}\Omega}{f_{i_p}\cdots f_{i_{p+q}}}\right\}_{i_0\ldots i_{p+q}}.$$

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But note that

$$\langle\partial_{i_0\ldots i_p},\mathrm{d}f\rangle \lrcorner\, K_{i_{p+q}}\cdots K_{i_p}\Omega = \sum_{j=0}^{p}(-1)^j f_{i_j}\partial_{i_0\ldots\widehat{i_j}\ldots i_p}\lrcorner\, K_{i_{p+q}}\cdots K_{i_p}\Omega$$
$$= (-1)^p f_{i_p}\partial_{i_0\ldots i_{p-1}}\lrcorner\, K_{i_{p+q}}\cdots K_{i_p}\Omega = (-1)^{p+pq} f_{i_p}K_{i_{p+q}}\cdots K_{i_0}\Omega.$$

Since $(-1)^{p^2/2}\cdot(-1)^{q(q+2)/2}\cdot(-1)^{p+pq} = (-1)^{(p+q)(p+q+2)/2}$ we obtain $\gamma_A\cup[\omega_B] = [\omega_{AB}]$, whence the diagram commutes. □

For an anticanonical quasismooth hypersurface X in a Gorenstein toric variety $\mathbf{P}$ (by Proposition 2.6, X is Calabi–Yau) the situation is especially nice. In this case the natural product $\wedge^p\mathcal{T}_X\otimes\Omega_X^{d-1}\to\Omega_X^{d-1-p}$ induced by the contraction is an isomorphism since $\Omega_X^{d-1}\simeq O_X$ and $\Omega_X^{d-1-p}\simeq \mathrm{Hom}_{O_X}(\Omega_X^p,\Omega_X^{d-1})$ (see [CK, A.3]), so that the cup product with $[\omega_1]$ corresponding to $1\in S_0$ ($\beta=\beta_0$ because of anticanonical) gives

$$\cup[\omega_1] : H^p(X,\wedge^p\mathcal{T}_X)\cong H^p(X,\Omega_X^{d-1-p}). \tag{5}$$

For nondegenerate hypersurfaces this implies:

THEOREM 3.5. *Let $X\subset\mathbf{P}$ be a semiample anticanonical nondegenerate hypersurface defined by $f\in S_\beta$. Then the map γ: $R_1(f)_{*\beta}\to H^*(X,\wedge^*\mathcal{T}_X)$ is an injective ring homomorphism.*

Proof. The map is a well defined ring homomorphism by Theorems 3.3, 3.4 and (5), while the injectivity follows from Theorem 4.4 in [M]. □

4. Nonpolynomial Part of the Chiral Ring

This section studies the non-polynomial part of the chiral ring which is complementary to the polynomial part. We will construct new cocycles representing elements in $H^*(X,\wedge^*\mathcal{T}_X)$ for a big and nef quasismooth hypersurface $X\subset\mathbf{P}_\Sigma$. In Section 7 we will see that these elements with $R_1(f)_\beta$ span $H^1(X,\mathcal{T}_X)$ for a semiample anticanonical nondegenerate hypersurface $X\subset\mathbf{P}_\Sigma$ ($\dim\mathbf{P}_\Sigma\neq 1,3$). This means that we have found all cocycles corresponding to nonpolynomial infinitesimal deformations for a minimal Calabi–Yau X (see [CK]).

Let X be a d-semiample quasismooth hypersurface, defined by $f\in S_\beta$, in a complete simplicial toric variety $\mathbf{P}_\Sigma$ of dimension d. Then, from Proposition 2.3 we get the associated toric morphism π: $\mathbf{P}_\Sigma\to\mathbf{P}_{\Sigma_X}$. Take a two-dimensional cone $\sigma\in\Sigma_X$ with at least one one-dimensional cone $\rho_i\subset\sigma$ such that $\rho_i\notin\Sigma_X(1)$. Using such a cone σ we can form a new cover of the toric variety $\mathbf{P}_\Sigma$ by the open sets

$$U_{\sigma'} = \left\{x\in\mathbf{P}_\Sigma : \prod_{\rho_k\subset\sigma\setminus\sigma'} x_k\neq 0\right\}$$

for all two-dimensional cones $\sigma'\in\Sigma$ that lie in σ. Let us fix one order for this open cover corresponding to as the cones lie inside σ:

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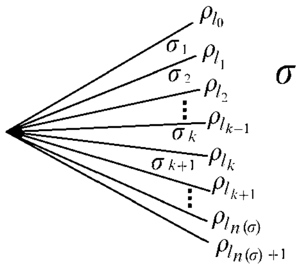

(6)

where $n(\sigma)$ is the number of cones ρ_i such that $\rho_i \subset \sigma$ and $\rho_i \notin \Sigma_X(1)$.

Now we take a refinement $U_{i,\sigma_j} = U_i \cap U_{\sigma_j}$ of this open cover and the open cover $\mathcal{U} = \{U_i\}_{i=1}^n$ from the previous section. Denote the refined cover $\mathcal{U}^\sigma$, considering the order on this cover as the lexicographic order for the pairs of indices (i, j).

DEFINITION 4.1. Given $\rho_i \subset \sigma \in \Sigma_X(2)$ such that $\rho_i \notin \Sigma_X(1)$, then, as in (6), $i = l_k$ for some k, and we set

$$\partial^i_k = \frac{x_{l_{k-1}} \partial_{l_{k-1}}}{\mathrm{mult}(\sigma_k)}, \qquad \partial^i_{k+1} = -\frac{x_{l_{k+1}} \partial_{l_{k+1}}}{\mathrm{mult}(\sigma_{k+1})}, \quad \text{and} \quad \partial^i_j = 0 \ \text{ for } j \neq k, k+1,$$

where mult denotes multiplicity of a cone as in [F1, page 48]. For $A \in S_{\beta_1^\sigma}$ (here, $\beta_1^\sigma := \sum_{\rho_k \subset \sigma} \deg(x_k)$), define

$$(\gamma^i_A)_{(i_0,j_0),(i_1,j_1)} = \left\{ \frac{A}{\prod_{\rho_k \subset \sigma} x_k} \left(\frac{\langle \partial_{i_1} \wedge \partial^i_{j_1}, \mathrm{d}f \rangle}{f_{i_1}} - \frac{\langle \partial_{i_0} \wedge \partial^i_{j_0}, \mathrm{d}f \rangle}{f_{i_0}} \right) \right\}_{(i_0,j_0),(i_1,j_1)}.$$

LEMMA 4.2. *In the definition,* $(\gamma^i_A)_{(i_0,j_0),(i_1,j_1)}$ *is a Čech cocycle in* $C^1(\mathcal{U}^\sigma|_X, \mathcal{T}_X)$.

Proof. By the arguments after Definition 3.1, $(\gamma^i_A)_{(i_0,j_0),(i_1,j_1)}$ is a cocycle class in $\check{H}^1(\mathcal{U}^\sigma|_X, \mathcal{T}_X)$. The only thing that we need to check in addition is that it is well defined on the given cover, which follows easily from the following two observations. Let X be equivalent to a torus invariant divisor $D = \sum_{k=1}^n a_k D_k$ with the associated polytope Δ_D and the support function ψ_D. Since ψ_D is linear on σ and determines a_k, a monomial $\prod_{k=1}^n x_k^{a_k + \langle m, e_k \rangle}$ (in $x_{l_j} f_{l_j}$) with $m \in \Delta_D$ is divisible by x_{l_j} implies that $a_k + \langle m, e_k \rangle > 0$ for all $\rho_k \subset \sigma$ such that $\rho_k \notin \Sigma_X(1)$. In particular, such a monomial is divisible by x_i. On the other hand, we have an identity on $\mathbf{P}_\Sigma$:

$$\frac{x_{l_{k-1}} \partial_{l_{k-1}}}{\mathrm{mult}(\sigma_k)} + \frac{x_{l_{k+1}} \partial_{l_{k+1}}}{\mathrm{mult}(\sigma_{k+1})} = \frac{\mathrm{mult}(\sigma_k + \sigma_{k+1})}{\mathrm{mult}(\sigma_k)\mathrm{mult}(\sigma_{k+1})} x_{l_k} \partial_{l_k}, \tag{7}$$

where σ_k and σ_{k+1} are the two cones contained in σ and containing ρ_i (the identity corresponds to an Euler vector field (see [BC, Remark 3.10]) coming from the relation of the cone generators $\mathrm{mult}(\sigma_{k+1}) e_{l_{k-1}} + \mathrm{mult}(\sigma_k) e_{l_{k+1}} = \mathrm{mult}(\sigma_k + \sigma_{k+1}) e_{l_k}$ (see [D, Section 8.2])). $\square$

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Remark 4.3. Finding the above cocycle is far from obvious, but Propositions 6.3, 6.4, 6.6 with Theorem 4.11 and Equation (5) show how this comes up in the case of Calabi–Yau threefolds from the description of the middle cohomology in Theorem 2.10.

Next we generalize the cocycles from Definition 4.1.

DEFINITION 4.4. Let $\rho_i \subset \sigma \in \Sigma_X(2)$ be such that $\rho_i \notin \Sigma_X(1)$. Given $A \in S_{(p-1)\beta+\beta_1^\sigma}$, $\beta_1^\sigma = \sum_{\rho_k\subset\sigma}\deg(x_k)$, and an index set $I = \{(i_0, j_0), \ldots, (i_p, j_p)\}$, define

$$(\gamma_A^i)_I = \left\{ \frac{(-1)^{(p-1)^2/2}A}{\prod_{\rho_k\subset\sigma} x_k} \sum_{\tilde I = I\setminus\{(i_k,j_k)\}} (-1)^k \frac{\langle \partial_{\tilde i_0\ldots\tilde i_{p-1}} \wedge \partial^i_{\tilde j_{p-1}}, \mathrm{d}f\rangle}{f_{\tilde i_0}\cdots f_{\tilde i_{p-1}}} \right\}_I,$$

where the sum is over the ordered sets

$$\tilde I = \{(\tilde i_0, \tilde j_0), \ldots, (\tilde i_{p-1}, \tilde j_{p-1})\} = \{(i_0, j_0), \ldots, \widehat{(i_k, j_k)}, \ldots, (i_p, j_p)\}.$$

As in the proof of Lemma 4.2, $(\gamma_A^i)_I$ also determines a Čech cocycle class in $\check H^p(\mathcal{U}^\sigma|_X, \wedge^p\mathcal{T}_X)$. Denoting its image in $H^p(X, \wedge^p\mathcal{T}_X)$ by γ_A^i, we get a map

$$\gamma_-^i : S_{(p-1)\beta+\beta_1^\sigma} \to H^p(X, \wedge^p\mathcal{T}_X),$$

when $\rho_i\setminus\{0\}$ lies in the relative interior of a two-dimensional cone $\sigma \in \Sigma_X$.

LEMMA 4.5. *If* $A \in \langle J(f), x_i\rangle_{(p-1)\beta+\beta_1^\sigma}$ *and* $p > 1$ *or* $A \in \langle x_i\rangle_{\beta_1^\sigma}$, *then* $\gamma_A^i = 0$.

Proof. If A is divisible by x_i, then $(\gamma_A^i)_I$ is clearly a Čech coboundary, by Definition 4.4. Assume $p > 1$ and $A \in J(f)_{(p-1)\beta+\beta_1^\sigma}$ is a multiple of one of the partial derivatives f_s. Similar to the proof of Lemma 3.2, we have

$$\begin{aligned}\frac{f_s\left\langle \partial_{\tilde i_0\ldots\tilde i_{p-1}} \wedge \partial^i_{\tilde j_{p-1}}, \mathrm{d}f\right\rangle}{(\prod_{\rho_k\subset\sigma} x_k) f_{\tilde i_0}\cdots f_{\tilde i_{p-1}}} &\equiv \sum_{l=0}^{p-1}(-1)^l \frac{\left\langle \partial_{s\tilde i_0\ldots\widehat{\tilde i_l}\ldots\tilde i_{p-1}} \wedge \partial^i_{\tilde j_{p-1}}, \mathrm{d}f\right\rangle}{(\prod_{\rho_k\subset\sigma} x_k) f_{\tilde i_0}\cdots\widehat{f_{\tilde i_l}}\cdots f_{\tilde i_{p-1}}} \\ &\equiv \sum_{\tilde{\tilde I} = \tilde I\setminus\{(\tilde i_l,\tilde j_l)\}} (-1)^l \frac{\left\langle \partial_{s\tilde{\tilde i}_0\ldots\tilde{\tilde i}_{p-2}} \wedge \partial^i_{\tilde{\tilde j}_{p-2}}, \mathrm{d}f\right\rangle}{(\prod_{\rho_k\subset\sigma} x_k) f_{\tilde{\tilde i}_0}\cdots f_{\tilde{\tilde i}_{p-2}}}\end{aligned}$$

(the sum is over the ordered sets $\tilde{\tilde I} = \{(\tilde i_0, \tilde j_0), \ldots, \widehat{(\tilde i_l, \tilde j_l)}, \ldots, (\tilde i_{p-1}, \tilde j_{p-1})\}$) modulo well defined expressions on the open set $U_{\tilde i_0,\sigma_{\tilde j_0}} \cap \cdots \cap U_{\tilde i_{p-1},\sigma_{\tilde j_{p-1}}}$, because $\langle \partial^i_{\tilde j_{p-1}}, \mathrm{d}f\rangle$ is divisible by x_i and because of Equation (7). On the other hand, there is an identity

$$\sum_{\tilde I = I\setminus\{(i_k,j_k)\}} (-1)^k \sum_{\tilde{\tilde I} = \tilde I\setminus\{(\tilde i_l,\tilde j_l)\}} (-1)^l \frac{\left\langle \partial_{s\tilde{\tilde i}_0\ldots\tilde{\tilde i}_{p-2}} \wedge \partial^i_{\tilde{\tilde j}_{p-2}}, \mathrm{d}f\right\rangle}{f_{\tilde{\tilde i}_0}\cdots f_{\tilde{\tilde i}_{p-2}}} = 0,$$

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since the square of a coboundary map is zero. This shows that $(\gamma^i_A)_I$ is a Čech coboundary for $A \in J(f)$. □

DEFINITION 4.6. Given $f \in S_\beta$, let $J^i(f)$ be the ideal in S generated by the Jacobian ideal $J(f)$ and x_i. Then we get the quotient ring $R^i(f) = S/J^i(f)$ graded by the Chow group $A_{d-1}(\mathbf{P}_\Sigma)$.

Lemma 4.5 shows that there are well defined maps $\gamma^i_- : R^i(f)_{(p-1)\beta+\beta^\sigma_1} \to H^p(X, \wedge^p \mathcal{T}_X)$, for $p > 1$, and $\gamma^i_- : (S/\langle x_i\rangle)_{\beta^\sigma_1} \to H^1(X, \mathcal{T}_X)$. Note, however, that a monomial $\prod_{\rho_l \not\subset \sigma} x_l \prod_{l=1}^n x_l^{(p-1)a_l+\langle m,e_l\rangle}$ in $\langle x_k\rangle_{(p-1)\beta+\beta^\sigma_1}$ (with $\rho_k \subset \sigma$) corresponds to $m \in M$ satisfying the inequalities $(p-1)a_l + \langle m, e_l\rangle \geqslant -1$ for $\rho_l \subset \sigma$, $l \neq k$, and $(p-1)a_k + \langle m, e_k\rangle \geqslant 0$. Since the support function, corresponding to $\beta = [\sum_{i=1}^n a_i D_i]$, is linear on σ and determines a_i, it follows from a relation of the cone generators that $(p-1)a_i + \langle m, e_i\rangle \geqslant 0$ and, consequently, the above monomial is divisible by x_i, for all $\rho_i \subset \sigma$ such that $\rho_i \notin \Sigma_X(1)$. Therefore, for all such ρ_i the ideal $J^i(f)$ is the same as

$$J^\sigma(f) := \langle J(f), x_k : \rho_k \subset \sigma\rangle$$

in the degree $(p-1)\beta + \beta^\sigma_1$. Hence, we define $R^\sigma(f) = S/J^\sigma(f)$.

The cocycle $(\gamma^i_A)_I$ in Definition 4.4 came from the proof of the following theorem.

THEOREM 4.7. *Let $X \subset \mathbf{P}_\Sigma$ be a d-semiample quasismooth hypersurface defined by $f \in S_\beta$. Then, for $q > 1$, the diagram*

$$\begin{array}{ccc} R(f)_{p\beta} \otimes R^\sigma(f)_{(q-1)\beta+\beta^\sigma_1} & \longrightarrow & R^\sigma(f)_{(p+q-1)\beta+\beta^\sigma_1} \\ \Big\downarrow{\scriptstyle \gamma_- \otimes \gamma^i_-} & & \Big\downarrow{\scriptstyle \gamma^i_-} \\ H^p(X, \wedge^p \mathcal{T}_X) \otimes H^q(X, \wedge^q \mathcal{T}_X) & \stackrel{\cup}{\longrightarrow} & H^{p+q}(X, \wedge^{p+q}\mathcal{T}_X) \end{array}$$

commutes, where $\beta^\sigma_1 = \sum_{\rho_k \subset \sigma} \deg(x_k)$ and the top arrow is induced by the multiplication. For $q = 1$ the diagram commutes with $(S/\langle x_i\rangle)_{\beta^\sigma_1}$ in place of $R^\sigma(f)_{\beta^\sigma_1}$.

Proof. For simplicity, we just show that if $A \in S_{p\beta}$ and $B \in S_{\beta^\sigma_1}$, then $\gamma_A \cup \gamma^i_B = \gamma^i_{AB}$ (the general case is similar though more complicated to write out). For such A and B the cup product $\gamma_A \cup \gamma^i_B$ is represented by the Čech cocycle

$$(-1)^p \left\{ \frac{(-1)^{p^2/2} AB}{\prod_{\rho_k \subset \sigma} x_k} \frac{\langle \partial_{i_0\ldots i_p}, \mathrm{d}f\rangle}{f_{i_0}\cdots f_{i_p}} \wedge \left(\frac{\langle \partial_{i_{p+1}} \wedge \partial^i_{j_{p+1}}, \mathrm{d}f\rangle}{f_{i_{p+1}}} - \frac{\langle \partial_{i_p} \wedge \partial^i_{j_p}, \mathrm{d}f\rangle}{f_{i_p}} \right) \right\}_I,$$

where $I = \{(i_0, j_0), \ldots, (i_{p+1}, j_{p+1})\}$. Compute

$$\begin{aligned} &\langle \partial_{i_0\ldots i_p}, \mathrm{d}f\rangle \wedge \langle \partial_{i_{p+1}} \wedge \partial^i_{j_{p+1}}, \mathrm{d}f\rangle \\ &\quad = \sum_{k=0}^p (-1)^k f_{i_k} \Big(\partial_{i_0\ldots\widehat{i_k}\ldots i_p} \wedge \langle \partial_{i_{p+1}} \wedge \partial^i_{j_{p+1}}, \mathrm{d}f\rangle + \end{aligned}$$

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$$+(-1)^p\Big\langle\partial_{i_0\ldots\widehat{i_k}\ldots i_p}, \mathrm{d}f\Big\rangle\wedge\partial_{i_{p+1}}\wedge\partial^i_{j_{p+1}}\Big)$$
$$=(-1)^p\sum_{k=0}^{p}(-1)^k f_{i_k}\Big\langle\partial_{i_0\ldots\widehat{i_k}\ldots i_{p+1}}\wedge\partial^i_{j_{p+1}}, \mathrm{d}f\Big\rangle,$$

where the sum of the second terms in the first equality is identically equal to zero. On the other hand, similar to (4),

$$\langle\partial_{i_0\ldots i_p}, \mathrm{d}f\rangle\wedge\langle\partial_{i_p}\wedge\partial^i_{j_p}, \mathrm{d}f\rangle = f_{i_p}\langle\partial_{i_0\ldots i_p}\wedge\partial^i_{j_p}, \mathrm{d}f\rangle.$$

Hence, the result follows easily. □

We next show when the cup product of two cocycles $(\gamma^i_A)_I$ and $(\gamma^j_B)_J$ vanishes.

LEMMA 4.8. *The cup product* $\gamma^i_A\cup\gamma^j_B=0$, *for* $A\in S_{(p-1)\beta+\beta^\sigma_1}$ *and* $B\in S_{(q-1)\beta+\beta^\sigma_1}$, *if* $\rho_i,\rho_j\subset\sigma\in\Sigma_X(2)$ *with* $i\neq j$ *do not span a two-dimensional cone of the fan* Σ.

Proof. For simplicity, we assume that γ^i_A and γ^j_B are from $H^1(X,\mathcal{T}_X)$.

The cup product $\gamma^i_A\cup\gamma^j_B$ is represented by the Čech cocycle

$$(-1)\left\{\frac{AB}{(\prod_{\rho_k\subset\sigma}x_k)^2}\left(\frac{u^i_{i_1,j_1}}{f_{i_1}}-\frac{u^i_{i_0,j_0}}{f_{i_0}}\right)\wedge\left(\frac{u^j_{i_2,j_2}}{f_{i_2}}-\frac{u^j_{i_1,j_1}}{f_{i_1}}\right)\right\}_I,$$

where $u^s_{i_k,j_k}$ denotes $\langle\partial_{i_k}\wedge\partial^s_{j_k}, \mathrm{d}f\rangle$ for $s\in\{i,j\}$, and $I=\{(i_0,j_0),(i_1,j_1),(i_2,j_2)\}$. Note that $u^i_{i_1,j_1}\wedge u^j_{i_1,j_1}=0$ because either $\partial^i_{j_1}$ or $\partial^j_{j_1}$ vanishes (see Definition 4.1) since the corresponding cone $\sigma_{j_1}\subset\sigma$ can not contain both ρ_i and ρ_j, by the given condition.

The above cocycle vanishes in the cohomology, being the image of

$$(-1)\left\{\frac{ABu^i_{i_0,j_0}\wedge u^j_{i_1,j_1}}{(\prod_{\rho_k\subset\sigma}x_k)^2 f_{i_0}\cdot f_{i_1}}\right\}_{(i_0,j_0),(i_1,j_1)}$$

under the Čech coboundary map $C^1(\mathcal{U}^\sigma|_X,\wedge^2\mathcal{T}_X)\to C^2(\mathcal{U}^\sigma|_X,\wedge^2\mathcal{T}_X)$. The latter cocycle is well defined since

$$\frac{AB\langle\partial_{i_0}\wedge\partial^i_{j_0}, \mathrm{d}f\rangle\wedge\langle\partial_{i_1}\wedge\partial^j_{j_1}, \mathrm{d}f\rangle}{(\prod_{\rho_k\subset\sigma}x_k)^2 f_{i_0}\cdot f_{i_1}}$$

has no poles on the open set $U_{i_0}\cap U_{\sigma_{j_0}}\cap U_{i_1}\cap U_{\sigma_{j_1}}$, which follows from the condition of the lemma. □

We created the cocycles $(\gamma^i_A)_I$, now we define the corresponding elements in the middle cohomology $H^{d-1}(X)$ of a d-semiample quasismooth hypersurface X.

DEFINITION 4.9. Let $\rho_i\subset\sigma\in\Sigma_X(2)$ be such that $\rho_i\notin\Sigma_X(1)$. Given $A\in S_{p\beta-\beta_0+\beta^\sigma_1}$ (where $\beta_0=\sum_{k=1}^n\deg(x_k)$, $\beta^\sigma_1=\sum_{\rho_k\subset\sigma}\deg(x_k)$) and an index set $I=\{(i_0,j_0),$

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$\dots,(i_p,j_p)\}$, define

$$(\omega^i_A)_I=\left\{\frac{(-1)^{d+((p-1)^2/2)}A}{\prod_{\rho_k\subset\sigma}x_k}\sum_{\tilde I=I\setminus\{(i_k,j_k)\}}(-1)^k\frac{K_{\tilde i_{p-1}}\cdots K_{\tilde i_0}(\partial^i_{\tilde j_0}\lrcorner\,\Omega)}{f_{\tilde i_0}\cdots f_{\tilde i_{p-1}}}\right\}_I,$$

where the sum is over the ordered sets

$$\tilde I=\{(\tilde i_0,\tilde j_0),\dots,(\tilde i_{p-1},\tilde j_{p-1})\}=\{(i_0,j_0),\dots,\widehat{(i_k,j_k)},\dots,(i_p,j_p)\}.$$

This determines a Čech cocycle class in $\check H^p(\mathcal U^\sigma|_X,\Omega_X^{d-1-p})$, whose image in $H^p(X,\Omega_X^{d-1-p})$ is denoted by ω^i_A.

LEMMA 4.10. *If* $A\in J^i(f)_{p\beta-\beta_0+\beta_1^\sigma}$, *then* $\omega^i_A=0$.

Proof. If A is divisible by x_i, then, by Definition 4.9, $(\omega^i_A)_I$ is a Čech coboundary. Assume that $A\in J(f)$ is a multiple of one of the partial derivatives f_s.

First, consider the case $p=1$. If $\rho_s\subset\sigma$ and $s\neq i$, then, by the argument after Definition 4.1, f_s is divisible by x_i, implying $(\omega^i_A)_I$ is a Čech coboundary. The case $f_s=f_i$ is impossible, because of $S_{\beta_i-\beta_0+\beta}{}^{\sigma_1}=0$ ($\beta_i:=\deg(x_i)$), following from the completeness of the fan Σ. The same is true if $\rho_s\not\subset\sigma$ and $\dim\mathbf P_\Sigma>2$. Notice

$$\frac{f_sK_{\tilde i_0}(\partial^i_{\tilde j_0}\lrcorner\,\Omega)}{(\prod_{\rho_k\subset\sigma}x_k)f_{\tilde i_0}}=\frac{K_sK_{\tilde i_0}\partial^i_{\tilde j_0}\lrcorner\,(\mathrm df\wedge\Omega)}{(\prod_{\rho_k\subset\sigma}x_k)f_{\tilde i_0}}-\frac{\langle\partial^i_{\tilde j_0},\mathrm df\rangle K_sK_{\tilde i_0}\Omega}{(\prod_{\rho_k\subset\sigma}x_k)f_{\tilde i_0}}+\frac{K_s(\partial^i_{\tilde j_0}\lrcorner\,\Omega)}{(\prod_{\rho_k\subset\sigma}x_k)}.$$

Also, note that if $\dim\mathrm P_\Sigma=2$ and $\rho_s\not\subset\sigma$, then $K_s(\partial^i_{\tilde j_0}\lrcorner\,\Omega)$ is a multiple of x_i, by the definition of the form Ω. Since $\mathrm df\wedge\Omega\equiv 0$ modulo multiples of f, by Equation (3) in [M], and since $\langle\partial^i_{\tilde j_0},\mathrm df\rangle$ is divisible by x_i, it follows that $(\omega^i_A)_I$ is a Čech coboundary in this case.

The case left is $p>1$. We have

$$\begin{aligned}
&f_s\frac{K_{\tilde i_{p-1}}\cdots K_{\tilde i_0}(\partial^i_{\tilde j_0}\lrcorner\,\Omega)}{(\prod_{\rho_k\subset\sigma}x_k)f_{\tilde i_0}\cdots f_{\tilde i_{p-1}}}\\
&=(-1)^{p+1}\frac{K_sK_{\tilde i_{p-1}}\cdots K_{\tilde i_0}\partial^i_{\tilde j_0}\lrcorner\,(\mathrm df\wedge\Omega)}{(\prod_{\rho_k\subset\sigma}x_k)f_{\tilde i_0}\cdots f_{\tilde i_{p-1}}}+\\
&\quad+(-1)^p\frac{\langle\partial^i_{\tilde j_0},\mathrm df\rangle K_sK_{\tilde i_{p-1}}\cdots K_{\tilde i_0}\Omega}{(\prod_{\rho_k\subset\sigma}x_k)f_{\tilde i_0}\cdots f_{\tilde i_{p-1}}}-\sum_{l=0}^{p-1}(-1)^{p+l}\frac{K_sK_{\tilde i_{p-1}}\cdots\widehat{K_{\tilde i_l}}\cdots K_{\tilde i_0}(\partial^i_{\tilde j_0}\lrcorner\,\Omega)}{(\prod_{\rho_k\subset\sigma}x_k)f_{\tilde i_0}\cdots\widehat{f_{\tilde i_l}}\cdots f_{\tilde i_{p-1}}}\\
&\equiv(-1)^{p+1}\sum_{\tilde{\tilde I}=\tilde I\setminus\{(\tilde i_l,\tilde j_l)\}}(-1)^l\frac{K_sK_{\tilde{\tilde i}_{p-2}}\cdots K_{\tilde{\tilde i}_0}(\partial^i_{\tilde{\tilde j}_0}\lrcorner\,\Omega)}{(\prod_{\rho_k\subset\sigma}x_k)f_{\tilde{\tilde i}_0}\cdots f_{\tilde{\tilde i}_{p-2}}}
\end{aligned}$$

(the sum is over the ordered sets $\tilde{\tilde I}=\{(\tilde i_0,\tilde j_0),\dots,\widehat{(\tilde i_l,\tilde j_l)},\dots,(\tilde i_{p-1},\tilde j_{p-1})\}$) modulo well defined expressions on the open set $U_{\tilde i_0,\sigma_{\tilde j_0}}\cap\cdots\cap U_{\tilde i_{p-1},\sigma_{\tilde j_{p-1}}}\cap X$, because $\mathrm df\wedge\Omega\equiv 0$

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modulo multiples of f, $\langle \partial^i_{\tilde{j}_{p-1}}, \mathrm{d}f\rangle$ is divisible by x_i and because of equation (7). And, we also have an identity

$$\sum_{\tilde{I}=I\setminus\{(i_k,j_k)\}} (-1)^k \sum_{\tilde{\tilde{I}}=\tilde{I}\setminus\{(\tilde{i}_l,\tilde{j}_l)\}} (-1)^l \frac{K_s K_{\tilde{\tilde{i}}_{p-2}} \cdots K_{\tilde{\tilde{i}}_0}(\partial^i_{\tilde{\tilde{j}}_0} \lrcorner\, \Omega)}{f_{\tilde{\tilde{i}}_0} \cdots f_{\tilde{\tilde{i}}_{p-2}}} = 0,$$

since the square of a coboundary map is zero. $(\gamma^i_{\mathcal{A}})_I$ is a Čech coboundary if $\mathcal{A} \neq J(f)$. □

The last lemma shows that there is a well defined map

$$\omega^i_- : R^i(f)_{p\beta-\beta_0+\beta_1^\sigma} \to H^p(X, \Omega_X^{d-1-p}).$$

Since $p\beta$ is d-semiample, multiplying a monomial in $\langle x_k\rangle_{p\beta-\beta_0+\beta_1^\sigma}$ (for $\rho_k \subset \sigma$) by $\prod_{\rho_l \not\subset \sigma} x_l$ and applying the argument in the proof of Lemma 4.2, we get a monomial divisible by all x_i corresponding to $\rho_i \subset \sigma$ such that $\rho_i \notin \Sigma_X(1)$.Therefore, for all such ρ_i the ideal $J^i(f)$ is the same as $J^\sigma(f)$ in the degree $p\beta - \beta_0 + \beta_1^\sigma$.

The cocycle $(\omega^i_A)_I$ came from the proof of the following result.

THEOREM 4.11. *Let $X \subset \mathrm{P}_\Sigma$ be a d-semiample quasismooth hypersurface defined by $f \in S_\beta$. Then, for $p > 1$, the diagram*

$$\begin{array}{ccc} R^\sigma(f)_{(p-1)\beta+\beta_1^\sigma} \otimes R(f)_{(q+1)\beta+\beta_0} & \longrightarrow & R^\sigma(f)_{(p+q)\beta-\beta_0+\beta_1^\sigma} \\ \big\downarrow{\scriptstyle \gamma^i_- \otimes [\omega_-]} & & \big\downarrow{\scriptstyle \omega^I_-} \\ H^p(X, \Lambda^p \mathcal{T}_X) \otimes H^q(X, \Omega_X^{d-1-q}) & \xrightarrow{\cup} & H^{p+q}(X, \Omega_X^{d-1-p-q}) \end{array}$$

commutes, where the top arrow is the multiplication (for $p = 1$ the diagram commutes with $(S/\langle x_i\rangle)_{\beta_1^\sigma}$ in place of $R^\sigma(f)_{\beta_1^\sigma}$).

Proof. For simplicity, we only show that $\gamma^i_A \cup [\omega_B] = \omega^i_{AB}$ for $A \in S_{\beta_1^\sigma}$ and $B \in S_{(q+1)\beta-\beta_0}$ (as in the proof of Theorem 4.7, the general case is similar, but more complicated to write out). Similar to the proof of Theorem 3.4, the cup product $\gamma^i_A \cup [\omega_B]$ is represented by the Čech cocycle

$$\left\{ \frac{(-1)^{d-1+(q(q+2)/2)} AB}{\prod_{\rho_k \subset \sigma} x_k} \left(\frac{\langle \partial_{i_1} \wedge \partial^i_{j_1}, \mathrm{d}f\rangle}{f_{i_1}} - \frac{\langle \partial_{i_0} \wedge \partial^i_{j_0}, \mathrm{d}f\rangle}{f_{i_0}} \right) \lrcorner\, \frac{K_{i_{q+1}} \cdots K_{i_1}\Omega}{f_{i_1} \cdots f_{i_{q+1}}} \right\}_I,$$

where I is the index set $\{(i_0, j_0), \ldots, (i_{q+1}, j_{q+1})\}$, corresponding to the cover $\mathcal{U}^\sigma|_X$. Compute

$$\begin{aligned} &\left(\frac{\langle \partial_{i_1} \wedge \partial^i_{j_1}, \mathrm{d}f\rangle}{f_{i_1}} - \frac{\langle \partial_{i_0} \wedge \partial^i_{j_0}, \mathrm{d}f\rangle}{f_{i_0}} \right) \lrcorner\, \frac{K_{i_{q+1}} \cdots K_{i_1}\Omega}{f_{i_1} \cdots f_{i_{q+1}}} \\ &\quad = (-1)^{q+1} \frac{K_{i_{q+1}} \cdots K_{i_1}(\partial^i_{j_1} \lrcorner\, \Omega)}{f_{i_1} \cdots f_{i_{q+1}}} - (-1)^{q+1} \frac{K_{i_{q+1}} \cdots K_{i_1}(\partial^i_{j_0} \lrcorner\, \Omega)}{f_{i_1} \cdots f_{i_{q+1}}} + \\ &\qquad + (-1)^{q+1} \frac{\langle \partial^i_{j_0}, \mathrm{d}f\rangle K_{i_{q+1}} \cdots K_{i_0}\Omega}{f_{i_0} \cdots f_{i_{q+1}}} \end{aligned}$$

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Also, notice

$$
\begin{aligned}
&K_{i_{q+1}}\cdots K_{i_0}\partial^i_{j_0}\lrcorner\,(\mathrm{d}f\wedge\Omega)\\
&\quad=\langle\partial^i_{j_0},\mathrm{d}f\rangle K_{i_{q+1}}\cdots K_{i_0}\Omega-f_{i_0}K_{i_{q+1}}\cdots K_{i_1}(\partial^i_{j_0}\lrcorner\,\Omega)+\\
&\qquad+\sum_{k=1}^{q+1}(-1)^{k-1}f_{i_k}K_{i_{q+1}}\cdots\widehat{K_{i_k}}\cdots K_{i_1}K_{i_0}(\partial^i_{j_0}\lrcorner\,\Omega).
\end{aligned}
$$

Since $\mathrm{d}f\wedge\Omega\equiv 0$ modulo multiples of f, as in Lemma 4.10, we can see that $\gamma^i_A\cup[\omega_B]$ is actually represented by the Čech cocycle

$$
\left\{\frac{(-1)^{d+(q^2/2)}AB}{\prod_{\rho_k\subset\sigma}x_k}\left(\sum_{\tilde I=I\setminus\{(i_k,j_k)\}}(-1)^k\frac{K_{\tilde i_q}\cdots K_{\tilde i_0}(\partial^i_{\tilde j_0}\lrcorner\,\Omega)}{f_{\tilde i_0}\cdots f_{\tilde i_q}}\right)\right\}_I,
$$

where the sum is over the ordered sets

$$
\tilde I=\{(\tilde i_0,\tilde j_0),\ldots,(\tilde i_q,\tilde j_q)\}=\{(i_0,j_0),\ldots,\widehat{(i_k,j_k)},\ldots,(i_{q+1},j_{q+1})\}. \qquad \square
$$

The next result (a proof of which is similar to the above) shows that the map $\omega^i_-:R^\sigma(f)_{*\beta-\beta_0+\beta^\sigma_1}\to H^*(X,\Omega^{d-1-*}_X)$ is a morphism of modules with respect to the ring homomorphism $R(f)_{*\beta}\to H^*(X,\wedge^*\mathcal{T}_X)$.

THEOREM 4.12. *Let $X\subset \mathrm{P}_\Sigma$ be a d-semiample quasismooth hypersurface defined by $f\in S_\beta$. Then the diagram*

$$
\begin{array}{ccc}
R(f)_{p\beta}\otimes R^\sigma(f)_{q\beta-\beta_0\,\beta+\beta^\sigma_1} & \longrightarrow & R^\sigma(f)_{(p-q)\beta-\beta_0\,\beta+\beta^\sigma_1}\\
\Big\downarrow{\scriptstyle \gamma_-\otimes[\omega^i_-]} & & \Big\downarrow{\scriptstyle \omega_i}\\
H^p(X,\Lambda^p\mathcal{T}_X)\otimes H^q(X,\Omega^{d-1-q}_X) & \xrightarrow{\ \cup\ } & H^{p+q}(X,\Omega^{d-1-p-q}_X)
\end{array}
$$

commutes, where the top arrow is induced by the multiplication.

Similar to Lemma 4.8, we also get when the cup product of two cocycles $(\gamma^I_A)_I$ and $(\omega^j_B)_J$ vanishes.

LEMMA 4.13. *The cup product $\gamma^I_A\cup\omega^j_B=0$, for $A\in S_{(p-1)\beta+\beta^\sigma_1}$ and $B\in S_{(q-1)\beta+\beta^\sigma_1}$, if $\rho_i,\rho_j\subset\sigma\in\Sigma_X(2)$ with $I\neq j$ do not span a 2-dimensional cone of the fan Σ.*

5. Toric and Residue Parts of Cohomology

In this section we describe the toric part of cohomology of a semiample nondegenerate hypersurface in a complete simplicial toric variety P_Σ. This part is the image of cohomology of the ambient space induced by the inclusion of the hypersurface. In

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this case, we also show that cohomology has a natural decomposition into a direct sum of the toric part and the residue part which comes from the residues of rational differential forms with poles along the hypersurface.

Since P_Σ is simplicial, we know from [F1] that the cohomology ring $H^*(\mathrm{P}_\Sigma)$ (with complex coefficients) is isomorphic to

$$\mathbb{C}[D_1, \dots, D_n]/(P(\Sigma) + SR(\Sigma)),$$

where the generators correspond to the torus invariant divisors of P_Σ, and where

$$P(\Sigma) = \left\langle \sum_{i=1}^{n} \langle m, e_i \rangle D_i : m \in M \right\rangle,$$

$$SR(\Sigma) = \langle D_{i_1} \cdots D_{i_k} : \{e_{i_1}, \dots, e_{i_k}\} \not\subset \sigma \text{ for all } \sigma \in \Sigma \rangle$$

($SR(\Sigma)$ is the *Stanley–Reisner* ideal of Σ) The *toric part* $H^*_{\mathrm{toric}}(X)$ of cohomology of a hypersurface X in P_Σ is defined as the image of the restriction map i^*: $H^*(\mathrm{P}_\Sigma) \to H^*(X)$ induced by the inclusion i: $X \subset \mathrm{P}_\Sigma$.

THEOREM 5.1. *Let X be a semiample nondegenerate hypersurface in a complete simplicial toric variety P_Σ. Then*

$$H^*_{\mathrm{toric}}(X) \cong H^*(\mathrm{P}_\Sigma)/\mathrm{Ann}([X]) \cong \mathbb{C}[D_1, \dots, D_n]/I,$$

where $\mathrm{Ann}([X])$ *is the annihilator of the class* $[X] \in H^2(\mathrm{P}_\Sigma)$, *and where* $I = (P(\Sigma) + SR(\Sigma)) : [X]$ *is the ideal quotient.*

Proof. We need to show that $\ker(i^*: H^*(\mathrm{P}_\Sigma) \to H^*(X))$ coincides with $\ker(\cup[X] : H^*(\mathrm{P}_\Sigma) \to H^{*+2}(\mathrm{P}_\Sigma))$. Since $\cup[X] = i_! i^*$ (where $i_!$ is the Gysin map), this is equivalent to $\ker(i_!) \cap \mathrm{im}(i^*) = 0$ in $H^p(X)$ for all p. Using an induction on the dimension of the hypersurface, we will show a stronger statement:

$$H^p(X) = \mathrm{im}(i^*) \bigoplus \ker(i_!) \quad \text{for all } p. \tag{8}$$

If $\dim X = 0$, then $\mathrm{P}_\Sigma = \mathbb{P}^1$. In this case, the composition $H^0(\mathbb{P}^1) \xrightarrow{i^*} H^0(X) \xrightarrow{I_1} H^2(\mathbb{P}^1)$ is clearly an isomorphism, and (8) follows.

Let $\dim X = d - 1 > 0$. For all odd p, $H^p(X) = \ker(i_!)$ and Equation (8) holds because $H^{\mathrm{odd}}(\mathbf{P}_\Sigma)$ vanishes. So we can assume that p is even.

We show first that $H^p(X) = \mathrm{im}(i^*) + \ker(i_!)$. The Gysin spectral sequence (see [M, Section 4]) gives an exact sequence

$$\bigoplus_{k=1}^{n} H^{p-2}(X \cap D_k) \to H^p(X) \to \mathrm{Gr}_p^W H^p(X \cap \mathbf{T}) \to 0.$$

Also, by the Gysin exact sequence (see [DK, Theorem 3.7]), we get

$$0 \to H^{p+1}(\mathbf{P}_\Sigma \setminus X) \xrightarrow{\mathrm{Res}} H^p(X) \xrightarrow{i_i} H^{p+2}(\mathbf{P}_\Sigma) \tag{9}$$

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for even p. Hence, $\mathrm{Res}(H^{p+1}(\mathbf{P}_\Sigma \setminus X)) = \ker(i_!)$. We claim that the composition

$$H^{p+1}(\mathbf{P}_\Sigma \setminus X) \xrightarrow{\mathrm{Res}} H^p(X) \to \mathrm{Gr}_p^W H^p(X \cap \mathbf{T}) \tag{10}$$

is a surjective map for $p > 0$. If $[X]$ is an i-semiample divisor class, then we get the associated morphism $\pi: \mathbf{P}_\Sigma \to \mathbf{P}_{\Sigma_X}$, and the ample nondegenerate hypersurface $Y = \pi(X)$ in $\mathbf{P}_{\Sigma_X}$, by Proposition 2.3. The statement is trivial for $p \neq i-1$ because, in this case,

$$\mathrm{Gr}_p^W H^p(X \cap \mathbf{T}) \cong \mathrm{Gr}_p^W H^p((Y \cap \mathbf{T}_{\Sigma_X}) \times (\mathbb{C}^*)^{d-i}) = 0 \tag{11}$$

(where $\mathbf{T}_{\Sigma_X}$ is the maximal torus of $\mathbf{P}_{\Sigma_X}$), by Equation (2) and the Künneth isomor phism theorem with Lemma 2.5. For $p = i-1$, consider the following commutative diagram:

$$\begin{array}{ccccc}
H^i(\mathbf{P}_\Sigma \setminus X) & \xrightarrow{\mathrm{Res}} & H^{i-1}(X) & \longrightarrow & H^{i-1}(X \cap \mathbf{T}) \\
\uparrow & & \uparrow & & \uparrow \\
H^i(\mathbf{P}_{\Sigma X} \setminus Y) & \xrightarrow{\mathrm{Res}} & H^{i-1}(Y) & \longrightarrow & H^{i-1}(Y \cap \mathbf{T}_{\Sigma_X}),
\end{array}$$

where the vertical arrows are induced by the morphism π. The right vertical arrow descends to an isomorphism

$$\pi^*: \mathrm{Gr}_{i-1}^W H^{i-1}(Y \cap \mathbf{T}_{\Sigma_X}) \cong \mathrm{Gr}_{i-1}^W H^{i-1}(X \cap \mathbf{T}) \tag{12}$$

which follows from Equation (2), the Künneth isomorphism and Lemma 2.5. On the other hand, the proof of Theorem 4.4 in [M] and Remark 2.8 show that the weight space $W_{i-1}H^{i-1}(Y \cap \mathbf{T}_{\Sigma_X})$ lies in the image of the composition of maps on the bottom of the diagram. Thus, we have shown that the composition (10) is surjective for all $p > 0$. Hence, $\ker(i_!)$ in $H^p(X)$ maps onto $\mathrm{Gr}_p^W H^p(X \cap \mathbf{T})$. Since $\mathrm{Gr}_p^W H^p(\mathbf{T}) = 0$ for $p > 0$, we get the commutative diagram:

$$\begin{array}{ccccccc}
\bigoplus_{k=1}^n H^p(D_k) & \longrightarrow & H^{p+2}(\mathbf{P}_\Sigma) & \longrightarrow & 0 & & \\
\uparrow{\scriptstyle i_!} & & \uparrow{\scriptstyle i_!} & & & & \\
\bigoplus_{k=1}^n H^{p-2}(X \cap D_k) & \longrightarrow & H^p(X) & \longrightarrow & \mathrm{Gr}_p^W H^p(X \cap \mathbf{T}) & \longrightarrow & 0 \\
\uparrow{\scriptstyle i^*} & & \uparrow{\scriptstyle i^*} & & & & \\
\bigoplus_{k=1}^n H^{p-2}(D_k) & \longrightarrow & H^p(\mathbf{P}_\Sigma) & \longrightarrow & 0, & &
\end{array}$$

where the rows are exact sequences arising from the Gysin spectral sequence. Chasing this diagram and using the induction assumption (8) for the semiample nondegenerate hypersurfaces $X \cap D_k \subset D_k$, we can see that $H^p(X)$ is spanned by $\ker(i_!)$ and $\mathrm{im}(i^*)$ for all $p > 0$. Let us show this in the case $p = 0$. If X is connected, then $i^*: H^0(\mathbf{P}_\Sigma) \to H^0(X)$ is an isomorphism of one-dimensional spaces, whence

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$H^0(X) = \mathrm{im}(i^*)$. By Lemma 2.1, we are left to consider the case when X is a 1-semi-ample hypersurface. We use another commutative diagram:

$$\begin{array}{ccccc} H^0(\mathbf{P}_\Sigma) & \xrightarrow{i^*} & H^0(X) & \xrightarrow{i_!} & H^2(\mathbf{P}_\Sigma) \\ \uparrow \pi^* & & \uparrow \pi^* & & \uparrow \pi^* \\ H^0(\mathbf{P}_{\Sigma X}) & \longrightarrow & H^0(Y) & \longrightarrow & H^2(\mathbf{P}_{\Sigma X}). \end{array}$$

The property $X = \pi^{-1}(Y)$ from Proposition 2.3 gives an isomorphism $\pi^* : H^0(Y) \to H^0(X)$. Using the diagram and the fact $\mathbf{P}_{\Sigma_x} \cong \mathbb{P}^1$, we deduce $H^0(X) = \mathrm{im}(i^*) + \ker(i_!)$.

To prove (8) it suffices now to show that $\mathrm{im}(i^*)$ and $\ker(i_!)$ have complementary dimensions in $H^p(X)$. From Equation (9) we get $\dim\ker(i_!) = h^{p+1}(\mathbf{P}_\Sigma \setminus X)$. The exact sequence of cohomology with compact supports

$$H^p(\mathbf{P}_\Sigma) \xrightarrow{i^*} H^p(X) \to H_c^{p+1}(\mathbf{P}_\Sigma \setminus X) \to 0$$

also gives $\dim\mathrm{im}(i^*) = h^p(X) - h_c^{p+1}(\mathbf{P}_\Sigma \setminus X)$ for even p. Since $H^p(X) = \mathrm{im}(i^*) + \ker(i_!)$, the inequalities

$$h_c^{p+1}(\mathbf{P}_\Sigma \setminus X) \leqslant h^{p+1}(\mathbf{P}_\Sigma \setminus X) \tag{13}$$

hold for all even p. By Poincaré duality, we have the equalities

$$h_c^{p+1}(\mathbf{P}_\Sigma \setminus X) = h^{2d-p-1}(\mathbf{P}_\Sigma \setminus X), h^{p+1}(\mathbf{P}_\Sigma \setminus X) = h_c^{2d-p-1}(\mathbf{P}_\Sigma \setminus X).$$

Applying them to (13), we get

$$h^{2d-p-1}(\mathbf{P}_\Sigma \setminus X) \leqslant h_c^{2d-p-1}(\mathbf{P}_\Sigma \setminus X)$$

for all even p. Hence, all these inequalities are equalities, and Equation (8) follows. The proof by induction is finished. □

Remark 5.2. We should note that the above nontrivial result or its equivalent has been used without a proof for smooth Calabi–Yau hypersurfaces (complete intersections) in many papers (e.g., [B3, Proposition 8.1], [HLY, Section 3.4], [St, Section 9]; cup product induces a nondegenerate pairing on the toric part [CK, Lemma 8.6.11], [Gi, Introduction]). In the case of ample quasismooth hypersurfaces, this follows directly from the Hard–Lefschetz theorem. It is an open question whether Theorem 5.1 holds in general for smooth or quasismooth semiample hypersurfaces.

Remark 5.3. An interesting equality follows from the proof of Theorem 5.1:

$$h^p(\mathbf{P}_\Sigma \setminus X) = h_c^p(\mathbf{P}_\Sigma \setminus X) \text{ for odd } p.$$

If X is ample, these Hodge numbers vanish for p away from the middle dimension d. But in the semiample case they are nontrivial in general.

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As a consequence of the above proof, we have a direct sum decomposition $H^p(X) = \mathrm{im}(i^*) \bigoplus \ker(i_!)$ for a semiample nondegenerate hypersurface. By the Gysin exact sequence, the kernel of the Gysin map is exactly the image of the residue map. Therefore, it is natural to introduce the following.

DEFINITION 5.4. The *residue part* $H^*_{\mathrm{res}}(X)$ of cohomology of a quasismooth hypersurface X in a complete simplicial toric variety $\mathbf{P}_\Sigma$ is defined as the image of the residue map $\mathrm{Res} : H^{*+1}(\mathbf{P}_\Sigma \setminus X) \to H^*(X)$.

Remark 5.5. The residue part $H^*_{\mathrm{res}}(X)$ is isomorphic to the primitive cohomology $PH^*(X)$ defined in [BC] by the exact sequence

$$H^*(\mathbf{P}_\Sigma) \to H^*(X) \to PH^*(X) \to 0.$$

By the definitions of the toric and residue parts of cohomology introduced earlier, we get the next result.

THEOREM 5.6. *For a semiample nondegenerate hypersurface X in a complete simplicial toric variety* $\mathbf{P}_\Sigma$*, there is a natural decomposition:*

$$H^*(X) = H^*_{\mathrm{toric}}(X) \bigoplus H^*_{\mathrm{res}}(X).$$

Theorem 5.1 described the toric part. Note that

$$H^*_{\mathrm{toric}}(X) \cup H^*_{\mathrm{res}}(X) \subset H^*_{\mathrm{res}}(X),$$

since $i_!(i^*a \cup b) = a \cup i_! b = 0$ for $b \in \ker(i_!)$, by the projection formula. Therefore, the residue part is a submodule of $H^*(X)$ over the ring $H^*_{\mathrm{toric}}(X)$.

Finally, we suggest an algorithmic approach to computing the residue part of cohomology. As in the proof of Theorem 5.1, the Gysin spectral sequence gives the commutative diagram:

$$\begin{array}{ccccccc}
0 & & 0 & & 0 & & \\
\uparrow & & \uparrow & & \uparrow & & \\
\bigoplus_{k=1}^n H^{p-2}_{\mathrm{res}}(X \cap D_k) & \to & H^p_{\mathrm{res}}(X) & \to & \mathrm{Gr}^W_p PH^p(X \cap \mathbf{T}) & \to & 0 \\
\uparrow & & \uparrow & & \uparrow & & \\
\bigoplus_{k=1}^n H^{p-2}(X \cap D_k) & \to & H^p(X) & \to & \mathrm{Gr}^W_p H^p(X \cap \mathbf{T}) & \to & 0 \\
\uparrow & & \uparrow & & \uparrow & & \\
\bigoplus_{k=1}^n H^{p-2}(D_k) & \to & H^p(\mathbf{P}_\Sigma) & \to & \mathrm{Gr}^W_p H^p(\mathbf{T}) & \to & 0,
\end{array} \tag{14}$$

where the columns and the rows are exact, and where $PH^p(X \cap \mathbf{T})$ is defined, as in

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[B1, Definition 3.13], by the exact sequence

$$H^*(\mathbf{T}) \to H^*(X \cap \mathbf{T}) \to PH^*(X \cap \mathbf{T}) \to 0.$$

The hypersurfaces $X \cap D_k$ in D_k are semiample nondegenerate of lower dimension, and the space $\mathrm{Gr}_p^W PH^p(X \cap \mathbf{T})$ can be described in terms of cohomology of a nondegenerate affine hypersurface, again, using the proof of Theorem 5.1. Therefore, this provides a way to calculate $H^p_{\mathrm{res}}(X)$.

6. Cohomology of Semiample Nondegenerate Hypersurfaces

In this section we continue the study of the cohomology of semiample nondegenerate hypersurfaces which was initiated in [M, Section 4]. Applying the algorithmic approach of the previous section, we will compute the residue part of the middle cohomology of a big and nef nondegenerate hypersurface X. In particular, we will generalize the description in Equation (3) and Theorem 2.10. An algebraic description of the middle cohomology is important because, in the Calabi–Yau case, this is isomorphic to the chiral ring $H^*(X, \wedge^* \mathcal{T}_X)$, by Equation (5). In terms of this description, one should be able to compute the product structure of the chiral ring. Here, we also compute the nontrivial cup products $\gamma_A^i \cup \omega_B^j$ of elements constructed in Section 4.

Let X be a d-semiample nondegenerate hypersurface, defined by $f \in S_\beta$, in a complete simplicial toric variety $\mathbf{P}_\Sigma$. Our goal is to relate ω_A^i, defined in Section 4, to the description of the middle cohomology of X given in Equation (3). First, we define new Cech cocycles, representing elements in $H^{d-3}(X \cap D_i)$.

DEFINITION 6.1. Given $\sigma \in \Sigma_X(2)$ with the ordered integral generators e_{l_0} and $e_{l_{n(\sigma)+1}}$ as in (6), introduce a $(d-2)$-form

$$\Omega_\sigma = \frac{x_{l_0} x_{l_{n(\sigma)+1}} K_{l_{n(\sigma)+1}} K_{l_0} \Omega}{\mathrm{mult}(\sigma) \prod_{\rho_k \subset \sigma} x_k}.$$

Then, for $A \in S_{(p+1)\beta - \beta_0 + \beta_1^\sigma}$ and $\rho_i \subset \sigma$ such that $\rho_i \notin \Sigma_X(1)$, define

$$(\tilde{\omega}_A^i)_I = (-1)^{p^2/2} \left\{ \frac{A K_{i_p} \cdots K_{i_0} \Omega_\sigma}{f_{i_0} \cdots f_{i_p}} \right\}_I,$$

where I is the index set $\{i_0, \ldots, i_p\}$, representing the intersection of open sets $U_{i_0} \cap \cdots \cap U_{i_p} \cap X \cap D_i$ in $X \cap D_i$.

Consider a rational $(d-2)$-form

$$(A\Omega_\sigma / f^{p+1}) \in H^0(D_i, \Omega_{D_i}^{d-2}((p+1)X_i)),$$

where $X_i := X \cap D_i$ (we will use both notations). By the residue map we get $\mathrm{Res}(A\Omega_\sigma / f^{p+1}) \in H^{d-3}(X \cap D_i)$. The next statement shows that up to a constant, $(\tilde{\omega}_A^i)_I$ is a Čech cocycle which represents this residue.

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PROPOSITION 6.2. *Let $X \subset \mathbf{P}_\Sigma$ be a d-semiample nondegenerate hypersurface defined by $f \in S_\beta$. Given $\rho_i \subset \sigma \in \Sigma_X(2)$ such that $\rho_i \notin \Sigma_X(1)$, and $A \in S_{(p+1)\beta-\beta_0+\beta_1^\sigma}$, then, under the natural map*

$$\check{\mathrm{H}}^p(\mathcal{U}|_{X\cap D_i}, \Omega^{d-3-p}_{X\cap D_i}) \longrightarrow H^p(X\cap D_i, \Omega^{d-3-p}_{X\cap D_i}) \cong H^{d-3-p,p}(X\cap D_i),$$

the Hodge component $\mathrm{Res}(A\Omega_\sigma/f^{p+1})^{d-3-p,p}$ *is represented by the Čech cocycle*

$$\frac{(-1)^{d-3+(p(p+1)/2)}}{p!}\left\{\frac{AK_{i_p}\cdots K_{i_0}\Omega_\sigma}{f_{i_0}\cdots f_{i_p}}\right\}_I \in C^p(\mathcal{U}|_{X\cap D_i}, \Omega^{d-3-p}_{X\cap D_i}).$$

Proof. The proof of this is similar to the proof of Theorem 3.3 in [M] (see also [CaG]). We only need to show that

$$\mathrm{d}f\wedge\Omega_\sigma \equiv 0 \text{ modulo multiples of } f \text{ and } x_i. \tag{15}$$

Note

$$\mathrm{d}f\wedge\Omega_\sigma = \mathrm{d}f\wedge\frac{x_{l_0}x_{l_{n(\sigma)+1}}K_{l_{n(\sigma)+1}}K_{l_0}\Omega}{\mathrm{mult}(\sigma)\prod_{\rho_k\subset\sigma}x_k} = \frac{x_{l_0}x_{l_{n(\sigma)+1}}K_{l_{n(\sigma)+1}}K_{l_0}(\mathrm{d}f\wedge\Omega)}{\mathrm{mult}(\sigma)\prod_{\rho_k\subset\sigma}x_k} - \\ - \frac{x_{l_0}f_{l_0}x_{l_{n(\sigma)+1}}K_{l_{n(\sigma)+1}}\Omega}{\mathrm{mult}(\sigma)\prod_{\rho_k\subset\sigma}x_k} + \frac{x_{l_0}x_{l_{n(\sigma)+1}}f_{l_{n(\sigma)+1}}K_{l_0}\Omega}{\mathrm{mult}(\sigma)\prod_{\rho_k\subset\sigma}x_k}.$$

The first summand is divisible by f, because $\mathrm{d}f\wedge\Omega\equiv 0$ modulo multiples of f, as in Lemma 4.10, and because f is not divisible by any variable x_k, corresponding to $\rho_k\subset\sigma$, since X is nondegenerate. The sum of the other two terms is a multiple of x_i, because, by the argument after Definition 4.1, $x_{l_j}f_{l_j}$ are divisible by all variables x_k, corresponding to the cones $\rho_k\subset\sigma$ not contained in $\Sigma_X(1)$, and because of an Euler identity similar to (7). Hence, Equation (15) follows.

We also verify that $(\tilde{\omega}^i_A)_I$ is a Čech cocycle. The Čech coboundary of $(\tilde{\omega}^i_A)_I$ is

$$(-1)^{p^2/2}\left\{A\sum_{k=0}^{p+1}\frac{(-1)^k f_{i_k}K_{i_{p+1}}\cdots\widehat{K_{i_k}}\cdots K_{i_0}\Omega_\sigma}{f_{i_0}\cdots f_{i_{p+1}}}\right\}_I.$$

On the other hand,

$$\sum_{k=0}^{p+1}(-1)^k f_{i_k}K_{i_{p+1}}\cdots\widehat{K_{i_k}}\cdots K_{i_0}\Omega_\sigma = K_{i_{p+1}}\cdots K_{i_0}(\mathrm{d}f\wedge\Omega_\sigma) - (-1)^{p+2}\mathrm{d}f\wedge K_{i_{p+1}}\cdots K_{i_0}\Omega_\sigma.$$

Applying Equation (15) and $\mathrm{d}f=0$ on X, we can see that the image of $(\tilde{\omega}^i_A)_I$ under the Čech coboundary map is zero. □

Denote by $\tilde{\omega}^i_A$ the image of the cocycle $(\tilde{\omega}^i_A)_I$ in $H^p(X\cap D_i, \Omega^{d-3-p}_{X\cap D_i})$. In the next step we show a relation between $\tilde{\omega}^i_A$ and ω^i_A.

PROPOSITION 6.3. *Let $X\subset\mathbf{P}_\Sigma$ be a d-semiample nondegenerate hypersurface defined by $f\in S_\beta$. Then $\varphi_{i!}\tilde{\omega}^i_A = \omega^i_A$, where $\varphi_{i!}$ is the Gysin map for $\varphi_i\colon X\cap D_i\hookrightarrow X$.*

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Proof. It suffices to show that $\varphi_{i!}\widetilde{\omega}^i_A$, for $A \in S_{p\beta-\beta_0+\beta_1^\sigma}$, is represented by the Čech cocycle $(\omega^i_A)_I$.

The Gysin map $\varphi_{i!}$ we can compute, using the following commutative diagram

$$\begin{array}{ccccccc} 0 \to & C^p(\mathcal{V}^\sigma, \Omega_X^{d-1-p}) & \to & C^p(\mathcal{V}^\sigma, \Omega_X^{d-1-p}(\log X_i)) & \overset{\text{Res}}{\to} & C^p(\mathcal{V}^\sigma_i, \Omega_{X\cap D_i}^{d-2-p}) & \\ & \uparrow & & \uparrow & & \uparrow & \\ 0 \to & C^{p-1}(\mathcal{V}^\sigma, \Omega_X^{d-1-p}) & \to & C^{p-1}(\mathcal{V}^\sigma, \Omega_X^{d-1-p}(\log X_i)) & \overset{\text{Res}}{\to} & C^{p-1}(\mathcal{V}^\sigma_i, \Omega_{X\cap D_i}^{d-2-p}), & \end{array}$$

where the vertical arrows are the Čech coboundary maps, $\mathcal{V}^\sigma$ denotes the open cover $\mathcal{U}^\sigma|_X$, and the cover $\mathcal{V}^\sigma_i$ is the restriction $\mathcal{V}^\sigma|_{X_i}$, $X_i = X \cap D_i$. By the residue map, the cocycle $(\widetilde{\omega}^i_A)_{\tilde{I}}$ is lifted to the cochain

$$\psi_{\tilde{I}} = (-1)^{(p-1)^2/2}\left\{\frac{AK_{\tilde{i}_{p-1}}\cdots K_{\tilde{i}_0}\Omega_\sigma}{f_{\tilde{i}_0}\cdots f_{\tilde{i}_{p-1}}} \wedge \sum_{k=1}^n \langle m_{\tilde{j}_0}, e_k\rangle \frac{\mathrm{d}x_k}{x_k}\right\}_I$$

in $C^{p-1}(\mathcal{V}^\sigma, \Omega_X^{d-1-p}(\log X_i))$, where $\tilde{I}$ is the index set $\{(\tilde{i}_0, \tilde{j}_0), \ldots, (\tilde{i}_{p-1}, \tilde{j}_{p-1})\}$, corresponding to the cover $\mathcal{V}^\sigma$, and where $m_{\tilde{j}_0} \in M_{\mathbb{R}}$, for $\sigma_{\tilde{j}_0} \supset \rho_i$ generated by e_i and e_s, satisfies $\langle m_{\tilde{j}_0}, e_i\rangle = 1$, $\langle m_{\tilde{j}_0}, e_s\rangle = 0$, and $m_{\tilde{j}_0} = 0$ in all other cases. Appropriately, this can be obtained, using some affine open cover on X, where $X \cap D_i$ is given by $\prod_{k=1}^n x_k^{\langle m, e_k\rangle} = 0$ up to some multiplicity (we omit the details).

The image of $\psi_{\tilde{I}}$ under the Čech coboundary map should represent $\varphi_{i!}\widetilde{\omega}^i_A$. Using the diagram, we can see that changing of $\psi_{\tilde{I}}$ by a cochain in $C^{p-1}(\mathcal{V}^\sigma, \Omega_X^{d-1-p})$ does not affect the image. Notice that $\psi_{\tilde{I}}$ is equivalent to

$$(-1)^{(p-1)^2/2}\left\{\frac{AK_{\tilde{i}_{p-1}}\cdots K_{\tilde{i}_0}}{f_{\tilde{i}_0}\cdots f_{\tilde{i}_{p-1}}}\left(\Omega_\sigma \wedge \sum_{k=1}^n \langle m_{\tilde{j}_0}, e_k\rangle \frac{\mathrm{d}x_k}{x_k}\right)\right\}_{\tilde{I}}$$

modulo some cochain in $C^{p-1}(\mathcal{V}^\sigma, \Omega_X^{d-1-p})$. Assume for a moment that

$$\left(\Omega_\sigma \wedge \sum_{k=1}^n \langle m_{\tilde{j}_0}, e_k\rangle \frac{\mathrm{d}x_k}{x_k}\right) - (-1)^d \frac{\partial^i_{\tilde{j}_0} \lrcorner\, \Omega}{\prod_{\rho_k\subset\sigma} x_k} \tag{16}$$

is well defined on $U_{\sigma_{\tilde{j}_0}}$. Then $\psi_{\tilde{I}}$ is actually equivalent to

$$(-1)^{d+((p-1)^2/2)}\left\{\frac{AK_{\tilde{i}_{p-1}}\cdots K_{\tilde{i}_0}}{f_{\tilde{i}_0}\cdots f_{\tilde{i}_{p-1}}}\left(\frac{\partial^i_{\tilde{j}_0}\lrcorner\,\Omega}{\prod_{\rho_k\subset\sigma} x_k}\right)\right\}_{\tilde{I}}$$

modulo some cochain in $C^{p-1}(\mathcal{V}^\sigma, \Omega_X^{d-1-p})$. The image of this under the Čech coboundary map is clearly $(\omega^i_A)_I$. We are left to show that (16) is well defined on $U_{\sigma_{\tilde{j}_0}}$. The case $\sigma_{\tilde{j}_0} \not\supset \rho_i$ is trivial because $m_{\tilde{j}_0} = 0$ and $\partial^i_{\tilde{j}_0} = 0$. The cases left are $\tilde{j}_0 = k, k+1$ for $i = l_k$ as in (6); we only check the case $\tilde{j}_0 = k$ (then $\langle m_{\tilde{j}_0}, e_{l_{k-1}}\rangle = 0$,

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$\partial^i_{\tilde{j}_0} = x_{l_{k-1}}\partial_{l_{k-1}}/(\mathrm{mult}(\sigma_k))$, the other case is similar. It is enough to verify that multiples of $(\mathrm{d}x_i)/x_i$ cancel each other in the difference (16). We defined $\Omega = \sum_{|I|=d}\det(e_I)\hat{x}_I\mathrm{d}x_I$; note that the multiples of $(\mathrm{d}x_i)/x_i$ in (16) are

$$\left(\sum_{|J|=d-2}\left(\frac{\det(e_{\{l_0,l_{n(\sigma)+1}\}\cup J})}{\mathrm{mult}(\sigma)} - (-1)^d\frac{\det(e_{\{l_{k-1}\}\cup J\cup\{i\}})}{\mathrm{mult}(\sigma_k)}\right)\frac{\hat{x}_J\mathrm{d}x_J}{\prod_{\rho_k\subset\sigma}x_k}\right)\wedge\frac{\mathrm{d}x_i}{x_i},$$

where the sum is over all $(d-2)$-element subsets $J\subset\{1,\dots,n\}$. Interchanging $i=l_k$ with the ordered set $\{l_{k-1}\}\cup J$ in $\det(e_{\{l_{k-1}\}\cup J\cup\{i\}})$ and using the relations of the cone generators (see [D, Section 8.2])

$$\frac{e_{l_k}}{\mathrm{mult}(\sigma_k)} = \frac{\mathrm{mult}(\sigma_{0,k})e_{l_{k-1}}}{\mathrm{mult}(\sigma_k)\mathrm{mult}(\sigma_{0,k-1})} - \frac{e_{l_0}}{\mathrm{mult}(\sigma_{0,k-1})},$$
$$\frac{\mathrm{mult}(\sigma)e_{l_{k-1}}}{\mathrm{mult}(\sigma_{0,k-1})\mathrm{mult}(\sigma_{k-1,n(\sigma)+1})} = \frac{e_{l_{n(\sigma)+1}}}{\mathrm{mult}(\sigma_{k-1,n(\sigma)+1})} + \frac{e_{l_0}}{\mathrm{mult}(\sigma_{0,k-1})},$$

where $\sigma_{s,t}$ denotes the cone generated by e_{l_s} and e_{l_t}, we get that the multiples of $(\mathrm{d}x_i)/x_i$ in (16) cancel each other. The proposition is proved. □

The last proposition shows the relation of ω^i_A to the description of the middle cohomology of X given in Equation (3). But we also need to understand the relation of ω^i_A to the description of the cohomology in Theorem 2.10. For this, we will have to consider some toric subvarieties of codimension 2 in $\mathbf{P}_\Sigma$, and to study the relation of some quotients of the homogeneous coordinate rings of these toric subvarieties and $\mathbf{P}_\Sigma$. This work will culminate in Theorem 6.7, which generalizes Theorem 2.10.

As in [M, Section 5], we consider a two-dimensional cone $\sigma'\in\Sigma$ contained in $\sigma\in\Sigma_X(2)$ and containing ρ_i (in the notation of (6), we have $i=l_k$ and $\sigma'=\sigma_k$ or σ_{k+1}), and let $S(V(\sigma'))=\mathbb{C}[x_{\gamma'}:\sigma'\subset\gamma'\in\Sigma(3)]$ be the coordinate ring of the $(d-2)$-dimensional complete simplicial toric variety $V(\sigma')\subset\mathbf{P}_\Sigma$. From Lemma 1.4 in [M], it follows that $X_{\sigma'}:=X\cap V(\sigma')$ (we will use both notations) has a positive self-intersection number inside $V(\sigma')$, implying $X_{\sigma'}$ is a big and nef hypersurface. We have a natural commutative diagram:

$$\begin{array}{ccc} S_{*\beta} & \cong & H^0(\mathbf{P}_\Sigma, O_{\mathbf{P}_\Sigma}(*X)) \\ {\scriptstyle\varphi^*_{\sigma'}}\big\downarrow & & {\scriptstyle\varphi^*_{\sigma'}}\big\downarrow \\ S(V(\sigma'))_{*\beta^{\sigma'}} & \cong & H^0(V(\sigma'), O_{V(\sigma')}(*X_{\sigma'})), \end{array}$$

where $\beta^{\sigma'}\in A_{d-3}(V(\sigma'))$ is the restriction of β, and the vertical arrows are the restriction maps induced by the inclusion $\varphi_{\sigma'}: V(\sigma\prime)\subset\mathbf{P}_\Sigma$. To describe the vertical arrow on the left one first has to restrict a Cartier divisor $D=\sum_{k=1}^n a_kD_k$ (as in [F1, Section

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5.1], assuming that $a_k = 0$ for $\rho_k \subset \sigma'$) in degree β to $V(\sigma')$:

$$D|_{V(\sigma')} = \sum_{\gamma'} a_{k(\gamma')} \frac{\text{mult}(\sigma')}{\text{mult}(\gamma')} V(\gamma'),$$

where the sum is over all $\gamma' \in \Sigma(3)$ spanned by σ' and a generator $e_{k(\gamma')}$. Then a monomial $\prod_{k=1}^n x_k^{qa_k + \langle m, e_k\rangle}$ in $S_{q\beta}$ with $m \in \sigma'^{\perp}$ is sent by the restriction map $\varphi_{\sigma'}^*$ to $\prod_{\gamma'} x_{\gamma'}^{qa_{\gamma'} + \langle m, e_{\gamma'}\rangle}$, where $a_{\gamma'} = a_{k(\gamma')}\text{mult}(\sigma')/\text{mult}(\gamma')$ and $e_{\gamma'} = e_{k(\gamma')}\text{mult}(\sigma')/\text{mult}(\gamma')$; if $m \notin \sigma'^{\perp}$, the monomial is sent to 0. Hence, we can see that the restriction map $S_{*\beta} \longrightarrow S(V(\sigma'))_{*\beta^{\sigma'}}$ is surjective, and its kernel is the ideal in $S_{*\beta}$ generated by all variables x_k such that $\rho_k \subset \sigma$, by the argument in the proof of Lemma 4.2. Therefore, we have an isomorphism:

$$\varphi_{\sigma'}^*\colon (S/\langle x_k : \rho_k \subset \sigma\rangle)_{*\beta} \cong S(V(\sigma'))_{*\beta^{\sigma'}}. \tag{17}$$

If X is defined by $f \in S_\beta$, then the restriction of f, denoted by $f_{\sigma'}$, determines exactly the hypersurface $X_{\sigma'} \subset V(\sigma')$.

We also have a natural map

$$S_{(p+1)\beta - \beta_0 + \beta_1^\sigma} \longrightarrow H^0(D_i, \Omega_{D_i}^{d-2}((p+1)X_i)), \tag{18}$$

sending A to the rational $(d-2)$-form $(A\Omega_\sigma/f^{p+1})$ considered after Definition 6.1. Let us determine the restriction of this form with respect to the map

$$H^0(D_i, \Omega_{D_i}^{d-2}((p+1)X_i)) \xrightarrow{\varphi_{i,\sigma'}^*} H^0(V(\sigma'), \Omega_{V(\sigma')}^{d-2}((p+1)X_{\sigma'})),$$

induced by the inclusion $\varphi_{i,\sigma'}\colon V(\sigma') \subset D_i$. The form Ω in Definition 2.7 is determined up to ± 1, depending on the choice of the basis for the lattice M. We have fixed one basis $m_1, \ldots, m_d$, but it is always possible to find another basis $m_1^\sigma, \ldots, m_d^\sigma$, for $\sigma \in \Sigma_X(2)$, so that the corresponding Ω is the same as before and $m_1^\sigma, \ldots, m_{d-2}^\sigma$ form a basis for the lattice $M \cap \sigma^\perp$. With the new choice of the basis, the proof of Proposition 9.5 in [BC] shows that

$$\Omega = \prod_{k=1}^n x_k \left(\sum_{k=1}^n \langle m_1^\sigma, e_k\rangle \frac{\mathrm{d}x_k}{x_k}\right) \wedge \cdots \wedge \left(\sum_{k=1}^n \langle m_d^\sigma, e_k\rangle \frac{\mathrm{d}x_k}{x_k}\right).$$

Using this, we compute

$$\begin{aligned}\Omega_\sigma &= \frac{x_{l_0} x_{l_{n(\sigma)+1}} K_{l_{n(\sigma)+1}} K_{l_0} \Omega}{\text{mult}(\sigma) \prod_{\rho_k \subset \sigma} x_k} \\ &= \frac{\prod_{\rho_k \not\subset \sigma} x_k}{\text{mult}(\sigma)} \mathrm{e}^{d-1,d} \left(\sum_{k=1}^n \langle m_1^\sigma, e_k\rangle \frac{\mathrm{d}x_k}{x_k}\right) \wedge \cdots \wedge \left(\sum_{k=1}^n \langle m_{d-2}^\sigma, e_k\rangle \frac{\mathrm{d}x_k}{x_k}\right),\end{aligned}$$

where $\mathrm{e}^{d-1,d}$ denotes $(\langle m_{d-1}^\sigma, e_{l_0}\rangle\langle m_d^\sigma, e_{l_{n(\sigma)+1}}\rangle - \langle m_d^\sigma, e_{l_0}\rangle\langle m_{d-1}^\sigma, e_{l_{n(\sigma)+1}}\rangle)$. By the properties of mult(σ) in [D, Section 8], we can see that $\mathrm{e}^{d-1,d}/\text{mult}(\sigma)$ is ± 1. There were two (reverse to each other) possibilities of labeling the generators of σ when we chose

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the order in (6). *In further calculations we assume* such a choice of ρ_{l_0} and $\rho_{l_{n(\sigma)+1}}$ that $\mathrm{e}^{d-1,d}/\mathrm{mult}(\sigma) = 1$. Set $t_j^\sigma = \prod_{k=1}^n x_k^{\langle m_j^\sigma, e_k\rangle}$, then $t_1^\sigma, \dots, t_{d-2}^\sigma$ are the coordinates on the torus $\mathbf{T}_{\sigma'}$. In terms of the homogeneous coordinates $x_{\gamma'}$ on $V(\sigma')$, the affine coordinates t_j^σ are identified with $\prod_{\gamma'} x_{\gamma'}^{\langle m_j^\sigma, e_{\gamma'}\rangle}$. Hence,

$$\begin{aligned}\varphi_{i,\sigma'}^*\left(\frac{A\Omega_\sigma}{f^{p+1}}\right) &= \varphi_{i,\sigma'}^*\left(\frac{A\prod_{\rho_k\not\subset\sigma} x_k}{f^{p+1}}\frac{\mathrm{d}t_1^\sigma}{t_1^\sigma}\wedge\cdots\wedge\frac{\mathrm{d}t_{d-2}^\sigma}{t_{d-2}^\sigma}\right)\\ &= \frac{\varphi_{\sigma'}^*(A\prod_{\rho_k\not\subset\sigma} x_k)}{f_{\sigma'}^{p+1}}\left(\sum_{\gamma'}\langle m_1^\sigma, e_{\gamma'}\rangle\frac{\mathrm{d}x_{\gamma'}}{x_{\gamma'}}\right)\wedge\cdots\wedge\left(\sum_{\gamma'}\langle m_{d-2}^\sigma, e_{\gamma'}\rangle\frac{\mathrm{d}x_{\gamma'}}{x_{\gamma'}}\right)\\ &= \frac{\varphi_{\sigma'}^*(A\prod_{\rho_k\not\subset\sigma} x_k)}{(\prod_{\gamma'} x_{\gamma'})f_{\sigma'}^{p+1}}\Omega_{V(\sigma')},\end{aligned}$$

where, as in Definition 2.7, $\Omega_{V(\sigma')}$ is the $(d-2)$-form on the toric variety $V(\sigma')$, corresponding to the basis $m_1^\sigma, \dots, m_{d-2}^\sigma$. A monomial in $S_{(p+1)\beta-\beta_0+\beta_1^\sigma}$ with $\beta = [\sum_{k=1}^n a_k D_k]$ corresponds to a lattice point m, satisfying the inequalities $(p+1)a_k + \langle m, e_k\rangle \geqslant 0$, for $\rho_k \subset \sigma$, and $(p+1)a_k + \langle m, e_k\rangle \geqslant 1$, for $\rho_k \not\subset \sigma$. Then, by the earlier explicit description of $\varphi_{\sigma'}^*$, we can see that the restriction $\varphi_{\sigma'}^*(A\prod_{\rho_k\not\subset\sigma} x_k)$ is a polynomial in $S(V(\sigma'))_{(p+1)\beta^{\sigma'}}$, divisible by $\prod_{\gamma'} x_{\gamma'}$. Therefore, we get the following commutative diagram

$$\begin{array}{ccc} S_{(p+1)\beta-\beta_0+\beta_1^\sigma} & \longrightarrow & H^0(D_i, \Omega_{D_i}^{d-2}((p+1)X_i)) \\ \varphi_{\sigma'}^*\downarrow & & \varphi_{i,\sigma'}^*\downarrow \\ S(V(\sigma'))_{(p+1)\beta^{\sigma'}-\beta_0^{\sigma'}} & \longrightarrow & H^0(V(\sigma'), \Omega_{V(\sigma')}^{d-2}((p+1)X_{\sigma'})), \end{array}$$

where $\beta_0^{\sigma'} := \deg(\prod_{\gamma'} x_{\gamma'}) \in A_{d-3}(V(\sigma'))$ is the anticanonical degree, and the horizontal arrows are given by (18) and a similar one sending a polynomial A to the form $(A\Omega_{V(\sigma')}/f_{\sigma'}^{p+1})$. Recall from Section 2 that for the hypersurface $X_{\sigma'} \subset V(\sigma')$ we have the residue map

$$\mathrm{Res}\colon S(V(\sigma'))_{(p+1)\beta^{\sigma'}-\beta_0^{\sigma'}} \to H^p(X, \Omega_{X\cap V(\sigma')}^{d-3-p}),$$

sending a polynomial B to the Hodge component $\mathrm{Res}(\omega_B)^{d-3-p,p}$. As in Section 3, denote $[\omega_B] = (-1)^{p/2}p!\mathrm{Res}(\omega_B)^{d-3-p,p}$. By the naturality of the residue map and Proposition 6.2, we obtain the following result.

PROPOSITION 6.4. *Let $X \subset \mathbf{P}_\Sigma$ be a d-semiample nondegenerate hypersurface defined by $f \in S_\beta$. Given $\rho_i \subset \sigma \in \Sigma_X(2)$ such that $\rho_i \notin \Sigma_X(1)$, and given $\sigma' \in \Sigma(2)$ such that $\rho_i \subset \sigma' \subset \sigma$, then we have a commutative diagram:*

$$\begin{array}{ccc} S_{(p+1)\beta-\beta_0+\beta_1^\sigma} & \xrightarrow{\tilde\omega_-^i} & H^p(X\cap D_i, \Omega_{X\cap D_i}^{d-3-p}) \\ \varphi_{\sigma'}^*\downarrow & & \varphi_{i,\sigma'}^*\downarrow \\ S(V(\sigma'))_{(p+1)\beta^{\sigma'}-\beta_0^{\sigma'}} & \xrightarrow{(-1)^{d-3-p}[\omega_-]} & H^p(X\cap V(\sigma'), \Omega_{X\cap V(\sigma')}^{d-3-p}). \end{array}$$

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From Section 2 we know that the map

$$\mathrm{Res}\colon R_1(f_{\sigma'})_{(p+1)\beta^{\sigma'}-\beta_0^{\sigma'}} \to H^p(X\cap V(\sigma'), \Omega^{d-3-p}_{X\cap V(\sigma')})$$

is well defined. The map $\tilde{\omega}^i_-$ should also be well defined on some quotient of the coordinate ring S. In Definition 2.9 we had the rings $R_0(f) = S/J_0(f)$ and $R_1(f) = S/J_1(f)$. Now introduce the following similar rings.

DEFINITION 6.5. Given $f \in S_\beta$ of d-semiample degree $\beta \in A_{d-1}(\mathbf{P}_\Sigma)$ and $\sigma \in \Sigma_\beta(2)$ (see Remark 1.5), let $J_0^\sigma(f)$ be the ideal in S generated by the ideal $J_0(f)$ and all x_k such that $\rho_k \subset \sigma$, and let $J_1^\sigma(f)$ be the ideal quotient $J_0^\sigma(f) : (\prod_{\rho_k \not\subset \sigma} x_k)$. Then we get the quotient rings $R_0^\sigma(f) = S/J_0^\sigma(f)$ and $R_1^\sigma(f) = S/J_1^\sigma(f)$ graded by the Chow group $A_{d-1}(\mathbf{P}_\Sigma)$.

We have the toric morphism $\pi\colon \mathbf{P}_\Sigma \to \mathbf{P}_{\Sigma_X}$, associated with a d-semiample hypersurface $X \subset \mathbf{P}_\Sigma$. By the previous discussion, for $\sigma' \subset \sigma \in \Sigma_X(2)$, $X_{\sigma'} = X \cap V(\sigma')$ is a big and nef hypersurface, defined by $f_{\sigma'}$, in the toric variety $V(\sigma')$. It follows from Proposition 1.6 that the restriction of π is the toric morphism $\pi_{\sigma'} : V(\sigma') \to V(\sigma)$, associated with the semiample divisor $X_{\sigma'}$. In particular, we have a ring homomorphism $\pi_{\sigma'*} : S(V(\sigma')) \to S(V(\sigma))$ between the coordinate rings of the toric varieties. The image of $f_{\sigma'}$ is a polynomial $f_\sigma \in S(V(\sigma))_{\beta^\sigma}$, which determines the ample hypersurface $Y_\sigma := \pi_{\sigma'}(X_{\sigma'})$ in $V(\sigma)$.

PROPOSITION 6.6. *Let $\beta \in A_{d-1}(\mathbf{P}_\Sigma)$ be d-semiample and let $\beta_0^{\sigma'} = \deg(\prod_{\gamma'} x_{\gamma'}) \in A_{d-3}(V(\sigma'))$, $\beta_0^\sigma = \deg(\prod_\gamma y_\gamma) \in A_{d-3}(V(\sigma))$ be the anticanonical degrees. Then, there are natural isomorphisms:*

(i) $R_0^\sigma(f)_{*\beta} \cong R_0(f_{\sigma'})_{*\beta^{\sigma'}} \cong R_0(f_\sigma)_{*\beta^\sigma}$,
(ii) $R_1^\sigma(f)_{*\beta-\beta_0+\beta_1^\sigma} \cong R_1(f_{\sigma'})_{*\beta^{\sigma'}-\beta_0^{\sigma'}} \cong R_1(f_\sigma)_{*\beta^\sigma-\beta_0^\sigma}$.

Proof. (i) To show the first isomorphism, induced by $\varphi^*_{\sigma'}$, it suffices, because of equation (17), to check that the ideal $J_0(f)$ in S is mapped onto the ideal $J_0(f_{\sigma'})$ in $S(V(\sigma'))$. By Proposition 5.3 in [C2], the ideal $J_0(f)$ is generated by f and $x_{i_1}\partial f/\partial x_{i_1}, \ldots, x_{i_d}\partial f/\partial x_{i_d}$ for linearly independent $e_{i_1}, \ldots, e_{i_d}$. We can assume that $e_{i_1}, \ldots, e_{i_d}$ are generators of some simplicial cone τ, containing σ', and $e_{i_{d-1}}, e_{i_d}$ are generators of σ'. By the explicit description of the restriction map $\varphi^*_{\sigma'}$, f is sent to $f_{\sigma'}$, while $x_{i_{d-1}}\partial f/\partial x_{i_{d-1}}$ and $x_{i_d}\partial f/\partial x_{i_d}$ are sent to 0. To understand the image of the other polynomials, as in [BC], we write $f = \sum_{m\in\Delta\cap M} a_m \prod_{k=1}^n x_k^{b_k+\langle m,e_k\rangle}$, where Δ is the polytope associated with a torus invariant divisor $\sum_{k=1}^n b_k D_k$ (assuming $b_{i_{d-1}} = b_{i_d} = 0$) in degree β. Then

$$x_{i_s}\frac{\partial f}{\partial x_{i_s}} = \sum_{m\in\Delta\cap M} a_m(b_{i_s} + \langle m, e_{i_s}\rangle)\prod_{k=1}^n x_k^{b_k+\langle m,e_k\rangle}.$$

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Applying the restriction map $\varphi^*_{\sigma'}$ to this, we get, for $s \neq d-1, d$,

$$\sum_{m\in\Delta\cap M\cap\sigma^\perp} a_m(b_{i_s} + \langle m, e_{i_s}\rangle)\prod_{\gamma'} x_{\gamma'}^{b_{\gamma'}+\langle m,e_{\gamma'}\rangle} = \frac{\mathrm{mult}(\gamma'_s)}{\mathrm{mult}(\sigma')} x_{\gamma'_s}\frac{\partial f}{\partial x_{\gamma'_s}},$$

where the cone γ'_s is spanned by σ' and the generator e_{i_s}, and where $b_{\gamma'} = b_{k(\gamma')}\mathrm{mult}(\sigma')/\mathrm{mult}(\gamma')$, $e_{\gamma'} = e_{k(\gamma')}\mathrm{mult}(\sigma')/\mathrm{mult}(\gamma')$ correspond to the cone γ' spanned by σ' and a generator $e_{k(\gamma')}$. Therefore, we get the first isomorphism $\varphi^*_{\sigma'} : R_0^\sigma(f)_{*\beta} \cong R_0(f_{\sigma'})_{*\beta^{\sigma'}}$.

For the second isomorphism, induced by $\pi_{\sigma'*} : S(V(\sigma'))_{*\beta^{\sigma'}} \cong S(V(\sigma))_{*\beta^\sigma}$ (see Section 1), it is enough to show that $J_0(f_{\sigma'})$ is mapped onto the ideal $J_0(f_\sigma)$ in $S(V(\sigma))$. This can be easily achieved by the argument in the previous paragraph.

(ii) By the construction of the maps $\varphi^*_{\sigma'}$ and $\pi_{\sigma'*}$, we get the commutative diagram:

$$\begin{array}{ccccc} R_0^\sigma(f)_{*\beta} & \cong & R_0(f_{\sigma'})_{*\beta^{\sigma'}} & \cong & R_0(f_\sigma)_{*\beta^\sigma} \\ \uparrow \prod_{\rho_k\not\subset\sigma} x_k & & \uparrow \prod_{\gamma'} x_{\gamma'} & & \uparrow \prod_\gamma y_\gamma \\ R_1^\sigma(f)_{*\beta-\beta_0+\beta_1^\sigma} & \longrightarrow & R_1(f_{\sigma'})_{*\beta^{\sigma'}-\beta_0^{\sigma'}} & \longrightarrow & R_1(f_\sigma)_{*\beta^\sigma-\beta_0^\sigma}, \end{array}$$

where the vertical arrows are injections, induced by the multiplication. To show that the bottom arrows are isomorphisms it suffices to check that the images of the spaces from the bottom into the spaces on the top correspond to each other under the isomorphisms of part (i). Note that these images are the ideals generated by $\prod_{\rho_k\not\subset\sigma} x_k$, $\prod_{\gamma'} x_{\gamma'}$ and $\prod_\gamma y_\gamma$, respectively. By the explicit description of the maps $\varphi^*_{\sigma'}$ and $\pi_{\sigma'*}$, one can see that these are mapped onto each other. □

Finally, we can put all of the above together and generalize equation (3), describing the middle cohomology of a big and nef nondegenerate hypersurface.

THEOREM 6.7. *Let $X \subset \mathbf{P}_\Sigma$ be a d-semiample nondegenerate hypersurface defined by $f \in S_\beta$, $d = \dim \mathbf{P}_\Sigma$. Then there is a natural isomorphism, for $p = d-1-q$:*

$$H^{p,q}(X) \cong R_1(f)_{(q+1)\beta-\beta_0} \bigoplus \left(\bigoplus_{\sigma\in\Sigma_X(2)} (R_1^\sigma(f)_{q\beta-\beta_0+\beta_1^\sigma})^{n(\sigma)} \right) \bigoplus H^{p,q}_{\mathrm{toric}}(X) \bigoplus C,$$

where $C = \sum_{\tau\in\Sigma(2)} \varphi_{\tau!} H^{p-2,q-2}_{\mathrm{res}}(X\cap V(\tau))$ (the Gysin maps $\varphi_{\tau!}$ are induced by the inclusions $\varphi_\tau : X \cap V(\tau) \subset X$), and the graded pieces of $R_1(f)$ and $R_1^\sigma(f)$ are embedded by the maps $[\omega\_]$ and $\omega^i\_$ for all $\rho_i \notin \Sigma_X$ contained in some $\sigma \in \Sigma_X(2)$ ($n(\sigma)$ is the number of such cones). Moreover, $R_1^\sigma(f)_{q\beta-\beta_0+\beta_1^\sigma} = 0$ for $q = 0, d-1$, and the cup product of any two elements from the distinct summands of the above decomposition vanishes.

Proof. Theorem 4.4 in [M] combined with the diagram (14) gives an isomorphism:

$$H^{d-1-q,q}(X) \cong R_1(f)_{(q+1)\beta-\beta_0} \bigoplus H^{d-1-q,q}_{\mathrm{toric}}(X) \bigoplus \sum_{i=1}^n \varphi_{i!} H^{d-2-q,q-1}_{\mathrm{res}}(X\cap D_i),$$

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where $\varphi_{i!}$ are the Gysin maps induced by the inclusions. Applying (14) to the hypersurface $X \cap D_i$ in D_i, we get an exact sequence

$$\bigoplus_{\rho_i \subset \tau \in \Sigma(2)} H_{\text{res}}^{d-5}(X \cap V(\tau)) \to PH^{d-3}(X \cap D_i) \to \mathrm{Gr}_{d-3}^{W} PH^{d-3}(X \cap \mathbf{T}_{\rho_i}). \tag{19}$$

The space $\mathrm{Gr}_{d-3}^{W} PH^{d-3}(X \cap \mathbf{T}_{\rho_i})$ vanishes, by Equation (11), unless $X \cap D_i$ is a $(d-2)$-semiample hypersurface in D_i. By Proposition 1.6, the latter happens only when $\rho_i \notin \Sigma_X$ lies in some $\sigma \in \Sigma_X(2)$. In this case, there is $\sigma' \in \Sigma(2)$ such that $\rho_i \subset \sigma' \subset \sigma$, and, by equation (12), we have isomorphisms

$$\mathrm{Gr}_{d-3}^{W} H^{d-3}(X \cap \mathbf{T}_{\rho_i}) \cong \mathrm{Gr}_{d-3}^{W} H^{d-3}(\pi(X) \cap \mathbf{T}_{\sigma}) \cong \mathrm{Gr}_{d-3}^{W} H^{d-3}(X \cap \mathbf{T}_{\sigma'})$$

induced by the morphism $\pi \colon \mathbf{P}_{\Sigma} \to \mathbf{P}_{\Sigma_X}$. The hypersurface $X \cap V(\sigma')$ in $V(\sigma')$ is $(d-2)$-semiample (big and nef). So, we can apply Theorem 4.4 in [M] to deduce that the composition

$$R_1(f_{\sigma'})_{q\beta^{\sigma'} - \beta_0^{\sigma'}} \xrightarrow{\text{Res}} H_{\text{res}}^{d-2-q,q-1}(X \cap V(\sigma')) \to H^{d-2-q,q-1}\big(PH^{d-3}(X \cap \mathbf{T}_{\sigma'})\big)$$

is an isomorphism. Using Propositions 6.4 and 6.6, we get that another composition

$$R_1^{\sigma}(f)_{q\beta - \beta_0 + \beta_1^{\sigma}} \xrightarrow{\tilde{\omega}^i} \to H_{\text{res}}^{d-2-q,q-1}(X \cap D_i) \to H^{d-2-q,q-1}\big(PH^{d-3}(X \cap \mathbf{T}_{\rho_i})\big)$$

is also an isomorphism. Hence, by Equation (19),

$$H_{\text{res}}^{d-2-q,q-1}(X \cap D_i) \cong R_1^{\sigma}(f)_{q\beta - \beta_0 + \beta_1^{\sigma}} \bigoplus \sum_{\rho_i \subset \tau \in \Sigma(2)} \varphi_{\tau!}^{i} H_{\text{res}}^{d-3-q,q-2}(X \cap V(\tau))$$

for $\rho_i \notin \Sigma_X$ contained in some $\sigma \in \Sigma_X(2)$, and

$$H_{\text{res}}^{d-2-q,q-1}(X \cap D_i) \cong \sum_{\rho_i \subset \tau \in \Sigma(2)} \varphi_{\tau!}^{i} H_{\text{res}}^{d-3-q,q-2}(X \cap V(\tau))$$

for all other ρ_i (here, $\varphi_{\tau}^{i} : X \cap V(\tau) \subset X \cap D_i$ is the inclusion). From (14) we have an exact sequence

$$\bigoplus_{\tau \in \Sigma(2)} H_{\text{res}}^{d-5}(X \cap V(\tau)) \to \bigoplus_{i=1}^{n} H_{\text{res}}^{d-3}(X \cap D_i) \to H_{\text{res}}^{d-1}(X)$$

which shows that the kernel of the right arrow is included into the parts complementary to $R_1^{\sigma}(f)_{q\beta - \beta_0 + \beta_1^{\sigma}}$ in $H_{\text{res}}^{d-3}(X \cap D_i)$. The direct sum decomposition of the middle cohomology follows.

The fact $R_1^{\sigma}(f)_{-\beta_0 + \beta_1^{\sigma}} = 0$ is obvious, while $R_1^{\sigma}(f)_{(d-1)\beta - \beta_0 + \beta_1^{\sigma}} = 0$ is implied by the isomorphism of Proposition 6.6 and by a dimension argument using the proof of Theorem 11.5 in [BC] and Theorems 2.11, 4.8(v) with Corollary 3.14 in [B1]. From Section 5 we know that $H_{\text{res}}^{*}(X) \cup H_{\text{toric}}^{*}(X) \subset H_{\text{res}}^{*}(X)$. But since $H_{\text{res}}^{2d-2}(X) = 0$, the toric part $H_{\text{toric}}^{d-1}(X)$ is orthogonal to all other summands in the middle cohomology.

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Theorem 4.4 in [M] shows that $R_1(f)_{(q+1)\beta-\beta_0}$ is orthogonal to all other summands as well. The proof of Lemma 6.9 below shows that $\omega^i_A \cup \omega^j_B = 0$ if $\rho_i, \rho_j \notin \Sigma_X$ lie in two distinct two-dimensional cones of $\Sigma_X(2)$. Finally, the projection formula gives:

$$\omega^i_A \cup \varphi_{\tau!} H^{d-5}_{\mathrm{res}}(X \cap V(\tau)) = \varphi_{\tau!}(\varphi^*_\tau \omega^i_A \cup H^{d-5}_{\mathrm{res}}(X \cap V(\tau)).$$

But it can be seen directly that the restriction φ^*_τ of the Čech cocycle $(\omega^i_A)_I$ is a coboundary. The theorem is proved. □

Remark 6.8. The direct summand C in the above theorem vanishes when $q = 0, 1, d-1, d-2$. Therefore, we have a complete description of the middle cohomology in the corresponding Hodge degrees.

Lemma 4.13 tells us that the cup product $\gamma^i_A \cup \omega^j_B$ vanishes in certain cases. Now we show that this is true in more cases.

LEMMA 6.9. *Let $X \subset \mathbf{P}_\Sigma$ be a d-semiample nondegenerate hypersurface defined by $f \in S_\beta$. Then the cup product $\gamma^i_A \cup \omega^j_B = 0$, for $A \in S_{(p-1)\beta+\beta^\sigma_1}$ and $B \in S_{(q-1)\beta+\beta^\sigma_1}$, if $\rho_i, \rho_j \notin \Sigma_X(1)$ lie in two distinct two-dimensional cones of $\Sigma_X(2)$.*

Proof. We use the description of the middle cohomology in Equation (3) and the Poincaré nondegenerate pairing to show that $\gamma^i_A \cup \omega^j_B = 0$ for ρ_i and ρ_j lying in two distinct two-dimensional cones σ^1 and σ^2 of Σ_X. Because of this, it is enough to check that the cup product of $\gamma^i_A \cup \omega^j_B$ with all elements in (3) vanishes.

Take $[\omega_C] \in H^{d-1}(X)$, corresponding to $C \in S_{(d-p-1)\beta-\beta_0}$, in the Hodge component complementary to the one of $\gamma^i_A \cup \omega^j_B$. Then

$$\gamma^i_A \cup \omega^j_B \cup [\omega_C] = \pm\omega^i_{AC} \cup \omega^j_B = \pm\varphi_{i!}\tilde{\omega}^i_{AC} \cup \varphi_{j!}\tilde{\omega}^j_B = \pm\varphi_{j!}((\varphi^*_j \varphi_{i!}\tilde{\omega}^i_{AC}) \cup \tilde{\omega}^j_B),$$

where we use Theorem 4.11, Proposition 6.3 and the projection formula for Gysin homomorphisms. By Lemma 5.4 in [M], there is a commutative diagram:

$$\begin{array}{ccc} H^{d-3}(X \cap D_i) & \xrightarrow{\varphi_{i!}} & H^{d-1}(X) \\ \downarrow{\scriptstyle \varphi^*_{ij}} & & \downarrow{\scriptstyle \varphi^*_j} \\ H^{d-3}(X \cap D_i \cap D_j) & \xrightarrow{\alpha\cdot\varphi_{ji!}} & H^{d-1}(X \cap D_j), \end{array}$$

where φ_{ij}: $X \cap D_i \cap D_j \hookrightarrow X \cap D_i$ is the inclusion map and α is some constant. On the other hand, $\varphi^*_{ij}\tilde{\omega}^i_{AC}$ vanishes, because the cocycle representing $\tilde{\omega}^i_{AC}$ has a multiple of $\mathrm{d}x_j$ or x_j in each term of the form Ω. Therefore, $\gamma^i_A \cup \omega^j_B \cup [\omega_C] = 0$.

The rest of the elements, which span the middle cohomology, have the form $\varphi_{l!}(a)$ for some $a \in H^{d-3}(X \cap D_l)$. The projection formula gives $\gamma^i_A \cup \omega^j_B \cup \varphi_{l!}(a) = \varphi_{l!}(\varphi^*_l(\gamma^i_A \cup \omega^j_B) \cup a)$. Hence, it suffices to show that $\varphi^*_l(\gamma^i_A \cup \omega^j_B) = 0$. In further calculations, for simplicity, we assume that $A \in S_{\beta^{\sigma^1}_1}$ and $B \in S_{\beta-\beta_0+\beta^{\sigma^2}_1}$. We will need to use a refinement $\tilde{\mathcal{U}}$ of the cover $\mathcal{U}$, by the open sets $\tilde{U}_k = \{x \in \mathbf{P}_\Sigma : x_k f_k(x) \neq 0\}$ for

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$k = 1, \dots, n$. Since X is nondegenerate, these sets cover the toric variety P_Σ. In this case, the cup product $\gamma_A^i \cup \omega_B^j$ is represented by the Čech cocycle

$$\left\{ \frac{(-1)^d AB}{(\prod_{\rho_k \subset \sigma^1} x_k)(\prod_{\rho_k \subset \sigma^2} x_k)} \left(\frac{u^i_{i_1,j_1}}{f_{i_1}} - \frac{u^i_{i_0,j_0}}{f_{i_0}} \right) \lrcorner \left(\frac{K_{i_2}(\partial^j_{k_2} \lrcorner\, \Omega)}{f_{i_2}} - \frac{K_{i_1}(\partial^j_{k_1} \lrcorner\, \Omega)}{f_{i_1}} \right) \right\}_I,$$

where the index set $I = \{(i_0, j_0, k_0), (i_1, j_1, k_1), (i_2, j_2, k_2)\}$ corresponds to the refinement of $\tilde{\mathcal{U}}$, $\mathcal{U}^{\sigma^1}$ and $\mathcal{U}^{\sigma^2}$, and where $u^i_{i_t,j_t}$ denotes $\langle \partial_{i_t} \wedge \partial^i_{j_t}, \mathrm{d}f \rangle$. Note that

$$\frac{\langle \partial_{i_0} \wedge \partial^i_{j_0}, \mathrm{d}f \rangle}{f_{i_0}} \lrcorner \frac{K_{i_1}(\partial^j_{k_1} \lrcorner\, \Omega)}{f_{i_1}} = \frac{K_{i_1}(\partial^j_{k_1} \lrcorner\, \partial^i_{j_0} \lrcorner\, \Omega)}{f_{i_1}} + \frac{\langle \partial^i_{j_0}, \mathrm{d}f \rangle K_{i_1} K_{i_0}(\partial^j_{k_1} \lrcorner\, \Omega)}{f_{i_0} f_{i_1}}.$$

For ρ_l, not lying in the cones σ^1 and σ^2, the restriction φ_l^* of the above cocycle vanishes: if i is among $\{i_0, i_1, i_2\}$, then $\tilde{U}_i \cap D_i$ is empty; if $i \notin \{i_0, i_1, i_2\}$, each term of the cocycle is multiple of x_l or $\mathrm{d}x_l$ coming from Ω. We are left to consider $\rho_l \subset \sigma^1 \cup \sigma^2$. For $\rho_l \subset \sigma^1$, we will show that the restriction φ_l^* of the cocycle is a Čech coboundary; the other case is similar. Compute

$$\left(\frac{u^i_{i_1,j_1}}{f_{i_1}} - \frac{u^i_{i_0,j_0}}{f_{i_0}} \right) \lrcorner \left(\frac{K_{i_2}(\partial^j_{k_2} \lrcorner\, \Omega)}{f_{i_2}} - \frac{K_{i_1}(\partial^j_{k_1} \lrcorner\, \Omega)}{f_{i_1}} \right)$$
$$= \sum_{\tilde{I} = I \setminus \{(i_s, j_s, k_s)\}} (-1)^s \left(\frac{K_{\tilde{i}_1}(\partial^j_{\tilde{k}_1} \lrcorner\, (\partial^i_{\tilde{j}_0} - \partial^i_{\tilde{j}_1}) \lrcorner\, \Omega)}{f_{\tilde{i}_1}} + \frac{\langle \partial^i_{\tilde{j}_0}, \mathrm{d}f \rangle K_{\tilde{i}_1} K_{\tilde{i}_0}(\partial^j_{\tilde{k}_0} \lrcorner\, \Omega)}{f_{\tilde{i}_0} f_{\tilde{i}_1}} \right).$$

Using this, we can see that the restriction $\varphi_l^*(\gamma_A^i \cup \omega_B^j)$ is represented by a Čech coboundary because of the following observations. The polynomial $\langle \partial^i_{\tilde{j}_0}, \mathrm{d}f \rangle$ is divisible by x_l. If $\rho_l \not\subset \sigma^1_{j_t}$, then the restricted open set $U_{\sigma^1_{j_t}} \cap D_l$ is empty. If $\sigma^1_{\tilde{j}_0}$ and $\sigma^1_{\tilde{j}_1}$ contain ρ_l, then $K_{\tilde{i}_1}(\partial^i_{\tilde{j}_0} - \partial^i_{\tilde{j}_1}) \lrcorner\, \Omega$ is either 0 or divisible by x_l because of Equation (7). Thus, the restriction $\varphi_l^*(\gamma_A^i \cup \omega_B^j) = 0$, and the result follows. □

At this point, let us summarize our calculations of the cup products $H^*(X, \wedge^* \mathcal{T}_X)$ with the middle cohomology $H^*(X, \Omega_X^{d-1-*})$ for d-semiample nondegenerate hypersurfaces. We have the elements in $H^*(X, \wedge^* \mathcal{T}_X)$ represented by $\gamma_\_$, $\gamma^i_\_$ (with ρ_i lying in some $\sigma \in \Sigma_X(2)$ such that $\rho_i \notin \Sigma_X$), and the corresponding elements in $H^*(X, \Omega_X^{d-1-*})$ represented by $\omega_\_$, $\omega^i_\_$. Theorem 3.4 provides $\gamma_A \cup \omega_B = \omega_{AB}$, while Theorems 4.11 and 4.12 have $\gamma_A \cup \omega_B^i = \omega^i_{AB}$ and $\gamma_A^i \cup \omega_B = \omega^i_{AB}$. Lemmas 4.13 and 6.9 tell us that the cup product $\gamma_A^i \cup \omega_B^j = 0$, for $i \neq j$, unless ρ_i and ρ_j span a 2-dimensional cone of Σ. Thus, for the constructed elements in $H^*(X, \wedge^* \mathcal{T}_X)$ and $H^*(X, \Omega_X^{d-1-*})$, we are missing only the cup products $\gamma_A^i \cup \omega_B^j$ when ρ_i and ρ_j (i may be equal to j) span a cone of Σ contained in some 2-dimensional cone of Σ_X.

First, we consider the nontrivial cup products $\gamma_A^i \cup \omega_B^j$ lying in $H^{d-1}(X, O_X)$, which is isomorphic to $R_1(f)_{d\beta - \beta_0}$, by Theorem 6.7. We note here that the inclusion

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$$\mu : R_1(f)_{d\beta-\beta_0} \xrightarrow{\prod_{k=1}^{n} x_k} R_0(f)_{d\beta} \tag{20}$$

induced by the multiplication is an isomorphism because the dimensions of the spaces is the same number (of the interior integral points of a polytope Δ corresponding to β) by the isomorphism $R_1(f)_{d\beta-\beta_0} = H^{0,d-1}(H^{d-1}(X \cap \mathrm{T}))$ of [M, Theorem 4.4], by [DK, Section 5.8], and by [BC, Theorem 11.5] with [B2, Corollary 3.14]. The cup product should be represented by a polynomial in the above spaces.

PROPOSITION 6.10. *Let $X \subset \mathrm{P}_\Sigma$ be a d-semiample nondegenerate hypersurface defined by $f \in S_\beta$, and denote*

$$G^\sigma(f) := \frac{x_s f_s x_t f_t \prod_{\rho_k \not\subset \sigma} x_k}{\mathrm{mult}(\sigma) \prod_{\rho_k \subset \sigma} x_k} \in S_{2\beta+\beta_0-2\beta_1^\sigma}$$

for $\sigma \in \Sigma_X(2)$ spanned by ρ_s and ρ_t. Given $A \in S_{(p-1)\beta+\beta_1^\sigma}$, $B \in S_{(d-1-p)\beta-\beta_0+\beta_1^\sigma}$, then

(i) *for $\rho_i = \rho_{l_k} \notin \Sigma_X$, as in* (6), *contained in $\sigma \in \Sigma_X(2)$:*

$$\gamma_A^i \cup \omega_B^i = \frac{\mathrm{mult}(\sigma_k + \sigma_{k+1})[\omega_{\mu^{-1}(ABG^\sigma(f))}]}{\mathrm{mult}(\sigma_k)\mathrm{mult}(\sigma_{k+1})} \ \text{in } H^{d-1}(X, O_X),$$

(ii) *for $\rho_i, \rho_j \notin \Sigma_X$ which span a two-dimensional cone $\sigma' \in \Sigma$ contained in $\sigma \in \Sigma_X(2)$:*

$$\gamma_A^i \cup \omega_B^j = -\frac{[\omega_{\mu^{-1}(ABG^\sigma(f))}]}{\mathrm{mult}(\sigma')} \ \text{in } H^{d-1}(X, O_X).$$

Proof. To simplify the proof we assume that $p = 1$.

(i) After a simple modification it follows that the cup product $\gamma_A^i \cup \omega_B^i$ is represented by the cocycle

$$\left\{ \frac{(-1)^{d+((d-3)^2/2)}AB}{(\prod_{\rho_k \subset \sigma} x_k)^2} \sum_{\tilde{I}=I\setminus\{(i_k,j_k)\}} (-1)^k \frac{\langle \partial_{\tilde{i}_0} \wedge \partial^i_{\tilde{j}_0}, \mathrm{d}f \rangle}{f_{\tilde{i}_0}} \lrcorner \frac{K_{\tilde{i}_{d-2}} \cdots K_{\tilde{i}_1}(\partial^i_{\tilde{j}_1} \lrcorner \Omega)}{f_{\tilde{i}_1} \cdots f_{\tilde{i}_{d-2}}} \right\}_I,$$

where $I = \{(i_0, j_0), \dots, (i_{d-1}, j_{d-1})\}$ is the index set corresponding to the open sets $\tilde{U}_{i_k}$ (defined in Lemma 6.9) and $U_{\sigma_{j_k}}$. Note that

$$\begin{aligned} &\frac{AB}{(\prod_{\rho_k \subset \sigma} x_k)^2} \frac{\langle \partial_{\tilde{i}_0} \wedge \partial^i_{\tilde{j}_0}, \mathrm{d}f \rangle}{f_{\tilde{i}_0}} \lrcorner \frac{K_{\tilde{i}_{d-2}} \cdots K_{\tilde{i}_1}(\partial^i_{\tilde{j}_1} \lrcorner \Omega)}{f_{\tilde{i}_1} \cdots f_{\tilde{i}_{d-2}}} \\ &\quad = \frac{(-1)^{d-1}AB}{(\prod_{\rho_k \subset \sigma} x_k)^2} \left(\langle \partial^i_{\tilde{j}_0}, \mathrm{d}f \rangle \frac{K_{\tilde{i}_{d-2}} \cdots K_{\tilde{i}_0}(\partial^i_{\tilde{j}_1} \lrcorner \Omega)}{f_{\tilde{i}_0} \cdots f_{\tilde{i}_{d-2}}} + \frac{K_{\tilde{i}_{d-2}} \cdots K_{\tilde{i}_1}(\partial^i_{\tilde{j}_1} \lrcorner \partial^i_{\tilde{j}_0} \lrcorner \Omega)}{f_{\tilde{i}_1} \cdots f_{\tilde{i}_{d-2}}} \right). \end{aligned}$$

The first summand is well defined on the corresponding open set: if $i \in \{\tilde{i}_0, \dots, \tilde{i}_{d-2}\}$, then $x_i \neq 0$ on the open set; otherwise, $\langle \partial^i_{\tilde{j}_0}, \mathrm{d}f \rangle K_{\tilde{i}_{d-2}} \cdots K_{\tilde{i}_0}(\partial^i_{\tilde{j}_1} \lrcorner \Omega)$ is a multiple of

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$(x_i)^2$. Therefore, the corresponding sum in the above cocycle forms a Čech coboundary, and $\gamma_A^i \cup \omega_B^i$ is represented by

$$\left\{\frac{(-1)^{1+((d-3)^2/2)}AB}{(\prod_{\rho_k\subset\sigma} x_k)^2} \sum_{\tilde{I}=I\setminus\{(i_k,j_k)\}} (-1)^k \frac{K_{\tilde{i}_{d-2}}\cdots K_{\tilde{i}_1}(\partial^i_{\tilde{j}_1} \lrcorner\, \partial^i_{\tilde{j}_0} \lrcorner\, \Omega)}{f_{\tilde{i}_1}\cdots f_{\tilde{i}_{d-2}}}\right\}_I.$$

By Proposition 5.3 in [C2], the polynomials $x_{r_0}f_{r_0},\ldots,x_{r_{d-1}}f_{r_{d-1}}$ do not vanish simultaneously on X if $e_{r_0},\ldots,e_{r_{d-1}}$ are linearly independent. We can always find such generators so that $e_{r_0}=e_{l_0}$ and $e_{r_1}=e_{l_{n(\sigma)+1}}$ as in (6). Since the open sets $\{x\in \mathrm{P}_\Sigma : x_{r_k}f_{r_k}\neq 0\}$ cover the toric variety, we can assume that the first index in I takes only the ordered values $r_0,\ldots,r_{d-1}$. In this case, it is not difficult to check that the above cocycle is different by a coboundary from

$$\left\{\frac{(-1)^{1+((d-3)^2/2)}AB}{(\prod_{\rho_k\subset\sigma} x_k)^2} \sum_{\tilde{I}=I\setminus\{(i_k,j_k)\}} (-1)^k \alpha_{\tilde{I}} \frac{K_{\tilde{i}_{d-2}}\cdots K_{\tilde{i}_1}(\partial_{l_{k+1}} \lrcorner\, \partial_{l_{k-1}} \lrcorner\, \Omega)}{f_{\tilde{i}_1}\cdots f_{\tilde{i}_{d-2}}\mathrm{mult}(\sigma_k)\mathrm{mult}(\sigma_{k+1})}\right\}_I,$$

where

$$\alpha_{\tilde{I}} = \begin{cases} -1 & \text{if } \tilde{i}_0=\tilde{i}_1=r_2, \tilde{j}_0\leqslant k<k+1\leqslant \tilde{j}_1,\\ -1 & \text{if } \tilde{i}_0=r_1, \tilde{i}_1=r_2, \tilde{j}_1\geqslant k+1,\\ 1 & \text{if } \tilde{i}_0=r_0, \tilde{i}_1=r_2, \tilde{j}_1\leqslant k,\\ 0 & \text{in all other cases.}\end{cases}$$

Using the Euler identities in the proof of Proposition 6.3, the last cocycle converts to

$$\frac{\mathrm{mult}(\sigma_k+\sigma_{k+1})}{\mathrm{mult}(\sigma_k)\mathrm{mult}(\sigma_{k+1})}\left\{\frac{(-1)^{(d-3)^2/2}ABx_{l_0}f_{l_0}x_{l_{n(\sigma)+1}}f_{l_{n(\sigma)+1}}}{\mathrm{mult}(\sigma)(\prod_{\rho_k\subset\sigma}x_k)^2}\frac{K_{i_{d-1}}\cdots K_{i_0}\Omega}{f_{i_0}\cdots f_{i_{d-1}}}\right\}_I.$$

This cocycle represents

$$\frac{\mathrm{mult}(\sigma_k+\sigma_{k+1})[\omega_{\mu^{-1}(ABG^\sigma(f))}]}{\mathrm{mult}(\sigma_k)\mathrm{mult}(\sigma_{k+1})}$$

in $H^{d-1}(X,O_X)$.

(ii) Similar to the proof of the previous part and Lemma 6.9, the cup product $\gamma_A^i\cup\omega_B^j$ is represented by the following cocycle:

$$\left\{\frac{(-1)^{1+((d-3)^2/2)}AB}{(\prod_{\rho_k\subset\sigma} x_k)^2} \sum_{\tilde{I}=I\setminus\{(i_k,j_k)\}} (-1)^k \frac{K_{\tilde{i}_{d-2}}\cdots K_{\tilde{i}_1}(\partial^j_{\tilde{j}_1} \lrcorner\, (\partial^i_{\tilde{j}_0}-\partial^i_{\tilde{j}_1})\lrcorner\, \Omega)}{f_{\tilde{i}_1}\cdots f_{\tilde{i}_{d-2}}}\right\}_I.$$

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The one-dimensional cones ρ_i and ρ_j span one of the 2-dimensional cones $\sigma_k \subset \sigma$ as in (6). The cocycle differs by a coboundary from

$$\left\{ \frac{(-1)^{1+((d-3)^2/2)}AB}{(\prod_{\rho_k\subset\sigma} x_k)^2} \sum_{\tilde{I}=I\setminus\{(i_k,j_k)\}} (-1)^k \alpha_{\tilde{I}} \frac{-K_{\tilde{i}_{d-2}}\cdots K_{\tilde{i}_1}(\partial_{l_k}\lrcorner\, \partial_{l_{k-1}}\lrcorner\, \Omega)}{f_{\tilde{i}_1}\cdots f_{\tilde{i}_{d-2}}\mathrm{mult}(\sigma_k)^2} \right\}_I,$$

where $\alpha_{\tilde{I}}$ is the same as in part (i). The Euler identities show that this represents

$$-\frac{[\omega_{\mu^{-1}(ABG^{\sigma}(f))}]}{\mathrm{mult}(\sigma_k)} \in H^{d-1}(X, O_X). \qquad \square$$

The restriction maps φ_l^*, induced by the inclusions $\varphi_l : X\cap D_l \hookrightarrow X$, give some information about the nontrivial cup products $\gamma_A^i \cup \omega_B^i$ in $H^{d-1}(X)$. We will use this in Section 7 to calculate nontrivial triple products on the chiral ring of anticanonical hypersurfaces.

PROPOSITION 6.11. *Let $X \subset \mathrm{P}_\Sigma$ be a d-semiample nondegenerate hypersurface defined by $f \in S_\beta$, and let, as in (6), $\rho_i = \rho_{l_k} \notin \Sigma_X$ be in some $\sigma \in \Sigma_X(2)$. Then, for $A \in S_{p\beta+\beta_1^\sigma}$ and $B \in S_{q\beta-\beta_0+\beta_1^\sigma}$,*

(i) $\varphi_{l_{k\pm1}}^*(\gamma_A^i \cup \omega_B^i) = \pm\varphi_{l_{k\pm1}}^*(\omega^i_{ABH^\sigma_{i,\pm1}(f)})$, *where $H^\sigma_{i,\pm1}(f)$ is a polynomial in $S_{\beta-\beta_1^\sigma}$ equal to $\sqrt{-1}x_{l_{k\pm1}}f_{l_{k\pm1}}/(\mathrm{mult}(\sigma_{k,k\pm1})\prod_{\rho_k\subset\sigma}x_k)$ at $x_{l_k}=0$ and $x_{l_{k\pm1}}=0$, where $\sigma_{s,t}$ denotes the cone spanned by ρ_{l_s} and ρ_{l_t}.*

(ii) $\varphi_i^*(\gamma_A^i \cup \omega_B^i) = \varphi_i^*\left(\omega^i_{ABH_i^\sigma(f)}\right)$, *where $H_i^\sigma(f)$ is a polynomial in $S_{\beta-\beta_1^\sigma}$ equal to*

$$\frac{\sqrt{-1}x_{l_{k+1}}f_{l_{k+1}}}{\mathrm{mult}(\sigma_{k,k+1})\prod_{\rho_k\subset\sigma}x_k} - \frac{\sqrt{-1}x_{l_{k-1}}f_{l_{k-1}}}{\mathrm{mult}(\sigma_{k-1,k})\prod_{\rho_k\subset\sigma}x_k}$$

with $x_i = x_{l_{k-1}} = x_{l_{k+1}} = 0$.

Proof. For simplicity, we assume that $A \in S_{\beta_1^\sigma}$ and $B \in S_{\beta-\beta_0+\beta_1^\sigma}$. The cup product $\gamma_A^i \cup \omega_B^i$ is represented by the Čech cocycle:

$$\left\{ \frac{(-1)^d AB}{(\prod_{\rho_k\subset\sigma}x_k)^2}\left(\frac{u^i_{i_1,j_1}}{f_{i_1}} - \frac{u^i_{i_0,j_0}}{f_{i_0}}\right)\lrcorner\left(\frac{K_{i_2}(\partial^j_{k_2}\lrcorner\,\Omega)}{f_{i_2}} - \frac{K_{i_1}(\partial^j_{k_1}\lrcorner\,\Omega)}{f_{i_1}}\right)\right\}_I,$$

where $u^i_{i_k,j_k}$ denotes $\langle\partial_{i_k}\wedge\partial^i_{j_k}, \mathrm{d}f\rangle$, and $I = \{(i_0,j_0),(i_1,j_1),(i_2,j_2)\}$ is the index set. Compute

$$\begin{aligned}&\frac{\langle\partial_{i_0}\wedge\partial^i_{j_0}, \mathrm{d}f\rangle}{f_{i_0}}\lrcorner\,\frac{K_{i_1}(\partial^i_{j_1}\lrcorner\,\Omega)}{f_{i_1}}\\ &\quad= \frac{K_{i_1}(\partial^i_{j_1}\lrcorner\,\partial^i_{j_0}\lrcorner\,\Omega)}{f_{i_1}} + \frac{\langle\partial^i_{j_0}, \mathrm{d}f\rangle K_{i_1}K_{i_0}(\partial^i_{j_1}\lrcorner\,\Omega)}{f_{i_0}f_{i_1}}\end{aligned}$$

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$$= \frac{K_{i_0}(\partial^i_{j_1} \lrcorner\, \partial^i_{j_0} \lrcorner\, \Omega)}{f_{i_0}} + \frac{\langle \partial^i_{j_1}, \mathrm{d}f \rangle K_{i_1} K_{i_0}(\partial^i_{j_0} \lrcorner\, \Omega)}{f_{i_0} f_{i_1}} + \frac{K_{i_1} K_{i_0}(\partial^i_{j_1} \lrcorner\, \partial^i_{j_0}) \lrcorner\, (\mathrm{d}f \wedge \Omega)}{f_{i_0} f_{i_1}}$$
$$\equiv \frac{K_{i_0}(\partial^i_{j_1} \lrcorner\, \partial^i_{j_0} \lrcorner\, \Omega)}{f_{i_0}} + \frac{\langle \partial^i_{j_1}, \mathrm{d}f \rangle K_{i_1} K_{i_0}(\partial^i_{j_0} \lrcorner\, \Omega)}{f_{i_0} f_{i_1}},$$

where, as in Lemma 4.10, we used $\mathrm{d}f \wedge \Omega \equiv 0$ modulo multiples of f. Hence, the cup product $\gamma^i_A \cup \omega^i_B$ is represented by the Čech cocycle

$$\left\{ \frac{(-1)^d AB}{(\prod_{\rho_k \subset \sigma} x_k)^2} \sum_{\tilde{I} = I \setminus \{(i_s, j_s)\}} (-1)^s \left(\frac{K_{\tilde{i}_0}(\partial^i_{\tilde{j}_1} \lrcorner\, \partial^i_{\tilde{j}_0} \lrcorner\, \Omega)}{f_{\tilde{i}_0}} + \frac{\langle \partial^i_{\tilde{j}_1}, \mathrm{d}f \rangle K_{\tilde{i}_1} K_{\tilde{i}_0}(\partial^i_{\tilde{j}_0} \lrcorner\, \Omega)}{f_{\tilde{i}_0} f_{\tilde{i}_1}} \right) \right\}_I.$$

For part (i), consider the restriction $\varphi^*_{l_{k\pm1}}$ of this cocycle. Note that the open set $U_{\sigma_j} \cap D_{l_{k\pm1}}$ is empty, if σ_j does not contain $\rho_{l_{k\pm1}}$, and that $\partial^i_j = 0$, if the corresponding cone σ_j does not contain ρ_i. Using this and $\partial^i_j \wedge \partial^i_j = 0$, we get that the restriction $\varphi^*_{l_{k\pm1}}$ of $\gamma^i_A \cup \omega^i_B$ is represented by

$$\pm\sqrt{-1} \left\{ \frac{(-1)^{d+(1/2)} AB}{(\prod_{\rho_k \subset \sigma} x_k)^2} \sum_{\tilde{I} = I \setminus \{(i_s, j_s)\}} (-1)^s \frac{x_{l_{k\pm1}} f_{l_{k\pm1}}}{\mathrm{mult}(\sigma_{k,k\pm1})} \frac{K_{\tilde{i}_1} K_{\tilde{i}_0}(\partial^i_{\tilde{j}_0} \lrcorner\, \Omega)}{f_{\tilde{i}_0} f_{\tilde{i}_1}} \right\}_I,$$

where the index set I corresponds to the restricted open cover $\mathcal{U}^\sigma|_{X \cap D_{l_{k\pm1}}}$, and where $\sigma_{k,k\pm1}$ is the cone generated by e_{l_k} and $e_{l_{k\pm1}}$. Notice that this cocycle is similar to the restriction $\varphi^*_{l_{k\pm1}}(\omega^i_C)$ for some polynomial C. The problem here is that $x_{l_{k\pm1}} f_{l_{k\pm1}}$ is not necessarily divisible by $\prod_{\rho_k \subset \sigma} x_k$. So some work is required to get the correct polynomial. Let X be linearly equivalent to a torus invariant divisor $D = \sum_{k=1}^n b_k D_k$ with the associated polytope $\Delta = \Delta_D$ given by the conditions $b_l + \langle m, e_l \rangle \geqslant 0$. Then we can write $f = \sum_{m \in \Delta \cap M} a_m x^{D(m)}$, where $x^{D(m)}$ denotes $\prod_{l=1}^n x_l^{b_l + \langle m, e_l \rangle}$. Note that

$$x_{l_{k\pm1}} f_{l_{k\pm1}} = \sum_{m \in \Delta \cap M} a_m (b_{l_{k\pm1}} + \langle m, e_{l_{k\pm1}} \rangle) x^{D(m)}.$$

If $b_{l_{k\pm1}} + \langle m, e_{l_{k\pm1}} \rangle = 0$, then the corresponding monomial $x^{D(m)}$ is not present in $x_{l_{k\pm1}} f_{l_{k\pm1}}$. On the other hand, if $b_{l_{k\pm1}} + \langle m, e_{l_{k\pm1}} \rangle > 1$, then the multiple of the corresponding monomial $x^{D(m)}$ in (22) vanishes, since $\partial^i_{\tilde{j}_0} = \mp x_{l_{k\pm1}} \partial_{l_{k\pm1}}$ or 0. By the argument in the proof of Lemma 4.2, $b_{l_{k\pm1}} + \langle m, e_{l_{k\pm1}} \rangle = 1$ implies that $b_l + \langle m, e_l \rangle > 0$ for all $\rho_l \subset \sigma$ such that $\rho_l \notin \Sigma_X(1)$. If $b_i + \langle m, e_i \rangle > 1$, the multiple of the monomial $x^{D(m)}$ in (22) forms a coboundary. Therefore, only the monomials $x^{D(m)}$ in $x_{l_{k\pm1}} f_{l_{k\pm1}}$, satisfying $b_{l_{k\pm1}} + \langle m, e_{l_{k\pm1}} \rangle = 1$ and $b_i + \langle m, e_i \rangle = 1$, have a nontrivial contribution in the Čech cocycle (22). For all such monomials, it follows from the relations of the cone generators in the proof of Proposition 6.3 that $b_s + \langle m, e_s \rangle > 0$ with $s = l_0, l_{n(\sigma)+1}$. Hence, the monomials are divisible by $\prod_{\rho_k \subset \sigma} x_k$.

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Thus, $\varphi^*_{l_{k\pm1}}(\gamma^i_A \cup \omega^i_B) = \pm\varphi^*_{l_{k\pm1}}\left(\omega^i_{ABH^\sigma_{i,\pm1}(f)}\right)$, where $H^\sigma_{i,\pm1}(f)$ is the polynomial

$$\sum_m \frac{\sqrt{-1}\sum_m a_m x^{D(m)}}{\mathrm{mult}(\sigma_{k,k\pm1})\prod_{\rho_k\subset\sigma} x_k}$$

with the sum over all $m \in \Delta \cap M$, satisfying the equalities $b_{l_{k\pm1}} + \langle m, e_{l_{k\pm1}}\rangle = 1$ and $b_i + \langle m, e_i\rangle = 1$. This is the same as $\sqrt{-1}x_{l_{k\pm1}}f_{l_{k\pm1}}/(\mathrm{mult}(\sigma_{k,k\pm1})\prod_{\rho_k\subset\sigma} x_k)$ evaluated at $x_i = 0$ and $x_{l_{k\pm1}} = 0$.

In part (ii), we will need to use the refinement $\tilde{\mathcal{U}}$ of the cover $\mathcal{U}$ defined in Lemma 6.9. From (21) we get that the cup product $\gamma^i_A \cup \omega^i_B$ is represented by the Čech cocycle

$$\left\{\frac{(-1)^d AB}{(\prod_{\rho_k\subset\sigma} x_k)^2}\sum_{\tilde{I}=I\setminus\{(i_s,j_s)\}}(-1)^s\left(\frac{K_{\tilde{i}_1}(\partial^i_{\tilde{j}_1}\lrcorner\,\partial^i_{\tilde{j}_0}\lrcorner\,\Omega)}{f_{\tilde{i}_1}} + \frac{\langle\partial^i_{\tilde{j}_0}, \mathrm{d}f\rangle K_{\tilde{i}_1}K_{\tilde{i}_0}(\partial^i_{\tilde{j}_1}\lrcorner\,\Omega)}{f_{\tilde{i}_0}f_{\tilde{i}_1}}\right)\right\}_I,$$

where the index set $I = \{(i_0, j_0), (i_1, j_1), (i_2, j_2)\}$ corresponds to the refinement of $\tilde{\mathcal{U}}$ and $\mathcal{U}^\sigma|_X$. Notice

$$\begin{aligned}
&\frac{K_{i_1}(\partial^i_{j_1}\lrcorner\,\partial^i_{j_0}\lrcorner\,\Omega)}{f_{i_1}} - \frac{K_{i_2}(\partial^i_{j_1}\lrcorner\,\partial^i_{j_0}\lrcorner\,\Omega)}{f_{i_2}}\\
&\quad= \frac{\langle\partial^i_{j_0}, \mathrm{d}f\rangle K_{i_2}K_{i_1}(\partial^i_{j_1}\lrcorner\,\Omega)}{f_{i_1}f_{i_2}} - \frac{\langle\partial^i_{j_1}, \mathrm{d}f\rangle K_{i_2}K_{i_1}(\partial^i_{j_0}\lrcorner\,\Omega)}{f_{i_1}f_{i_2}} - \frac{K_{i_2}K_{i_1}(\partial^i_{j_1}\lrcorner\,\partial^i_{j_0})\lrcorner\,(\mathrm{d}f\wedge\Omega)}{f_{i_1}f_{i_2}}\\
&\quad\equiv \frac{\langle\partial^i_{j_0}, \mathrm{d}f\rangle K_{i_2}K_{i_1}(\partial^i_{j_1}\lrcorner\,\Omega)}{f_{i_1}f_{i_2}} - \frac{\langle\partial^i_{j_1}, \mathrm{d}f\rangle K_{i_2}K_{i_1}(\partial^i_{j_0}\lrcorner\,\Omega)}{f_{i_1}f_{i_2}},
\end{aligned}$$

since $\mathrm{d}f\wedge\Omega \equiv 0$ modulo multiples of f. Using this, we compute

$$\begin{aligned}
&\frac{AB}{(\prod_{\rho_k\subset\sigma} x_k)^2}\sum_{\tilde{I}=I\setminus\{(i_s,j_s)\}}(-1)^s\left(\frac{K_{\tilde{i}_1}(\partial^i_{\tilde{j}_1}\lrcorner\,\partial^i_{\tilde{j}_0}\lrcorner\,\Omega)}{f_{\tilde{i}_1}} + \frac{\langle\partial^i_{\tilde{j}_0}, \mathrm{d}f\rangle K_{\tilde{i}_1}K_{\tilde{i}_0}(\partial^i_{\tilde{j}_1}\lrcorner\,\Omega)}{f_{\tilde{i}_0}f_{\tilde{i}_1}}\right)\\
&\quad= \sum_{\tilde{I}=I\setminus\{(i_s,j_s)\}}(-1)^s\frac{AB\langle\partial^i_{\tilde{j}_0}, \mathrm{d}f\rangle K_{\tilde{i}_1}K_{\tilde{i}_0}((\partial^i_{\tilde{j}_1} - \partial^i_{\tilde{j}_0})\lrcorner\,\Omega)}{(\prod_{\rho_k\subset\sigma} x_k)^2 f_{\tilde{i}_0}f_{\tilde{i}_1}} +\\
&\qquad+ \frac{ABK_{i_2}((\partial^i_{j_0}\wedge\partial^i_{j_1} - \partial^i_{j_0}\wedge\partial^i_{j_2} + \partial^i_{j_1}\wedge\partial^i_{j_2})\lrcorner\,\Omega)}{(\prod_{\rho_k\subset\sigma}} x_k)^2 f_{i_2} +\\
&\qquad+ \frac{AB\langle\partial^i_{j_1}, \mathrm{d}f\rangle}{(\prod_{\rho_k\subset\sigma} x_k)^2}\left(\frac{K_{i_2}K_{i_0}(\partial^i_{j_0}\lrcorner\,\Omega)}{f_{i_0}f_{i_2}} - \frac{K_{i_1}K_{i_0}(\partial^i_{j_0}\lrcorner\,\Omega)}{f_{i_0}f_{i_1}} - \frac{K_{i_2}K_{i_1}(\partial^i_{j_0}\lrcorner\,\Omega)}{f_{i_1}f_{i_2}}\right) +\\
&\qquad+ \frac{AB(\langle\partial^i_{j_0}, \mathrm{d}f\rangle + \langle\partial^i_{j_1}, \mathrm{d}f\rangle)}{(\prod_{\rho_k\subset\sigma} x_k)^2}\left(\sum_{\tilde{I}=I\setminus\{(i_s,j_s)\}}(-1)^s\frac{K_{\tilde{i}_1}K_{\tilde{i}_0}(\partial^i_{\tilde{j}_0}\lrcorner\,\Omega)}{f_{\tilde{i}_0}f_{\tilde{i}_1}}\right).
\end{aligned}$$

It is not difficult to see that the first summand produces a coboundary. The open set $U_{\sigma_j}\cap D_i$ is empty unless σ_j contains ρ_i. Therefore, applying the restriction φ^*_i, we can

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assume that the second component of the index (i_0, j_0) takes only values l_{k-1} or l_{k+1}. In this case, $\partial^i_{j_0} \wedge \partial^i_{j_1} - \partial^i_{j_0} \wedge \partial^i_{j_2} + \partial^i_{j_1} \wedge \partial^i_{j_2}$ (and the corresponding summand in the cocycle) vanishes. The third summand also ends up contributing zero:

$$\begin{aligned}
&\frac{AB\langle \partial^i_{j_1}, \mathrm{d}f\rangle}{(\prod_{\rho_k\subset\sigma} x_k)^2}\left(\frac{K_{i_2}K_{i_0}(\partial^i_{j_0}\lrcorner\,\Omega)}{f_{i_0}f_{i_2}} - \frac{K_{i_1}K_{i_0}(\partial^i_{j_0}\lrcorner\,\Omega)}{f_{i_0}f_{i_1}} - \frac{K_{i_2}K_{i_1}(\partial^i_{j_0}\lrcorner\,\Omega)}{f_{i_1}f_{i_2}}\right)\\
&\quad = \frac{AB\langle \partial^i_{j_1}, \mathrm{d}f\rangle}{(\prod_{\rho_k\subset\sigma} x_k)^2}\left(\frac{\langle \partial^i_{j_0}, \mathrm{d}f\rangle K_{i_2}K_{i_1}K_{i_0}\Omega}{f_{i_0}f_{i_1}f_{i_2}} + \frac{K_{i_2}K_{i_1}K_{i_0}\partial^i_{j_0}\lrcorner\,(\mathrm{d}f\wedge\Omega)}{f_{i_0}f_{i_1}f_{i_2}}\right)\\
&\quad \equiv \frac{AB\langle \partial^i_{j_0}, \mathrm{d}f\rangle\langle \partial^i_{j_1}, \mathrm{d}f\rangle}{(\prod_{\rho_k\subset\sigma} x_k)^2}\frac{K_{i_2}K_{i_1}K_{i_0}\Omega}{f_{i_0}f_{i_1}f_{i_2}}.
\end{aligned}$$

If i is among $\{i_0, i_1, i_2\}$, then this restricts to an empty set since $\tilde{U}_i \cap D_i$ is empty. In the opposite case, this gives 0 under the restriction since x_i or $\mathrm{d}x_i$ is present in each term of Ω. Thus, the restriction $\varphi_i^*(\gamma^i_A \cup \omega^i_B)$ is represented by the cocycle

$$\left\{\frac{(-1)^d AB(\langle \partial^i_{j_0}, \mathrm{d}f\rangle + \langle \partial^i_{j_1}, \mathrm{d}f\rangle)}{(\prod_{\rho_k\subset\sigma} x_k)^2}\sum_{\tilde{I}=I\setminus\{(i_s,j_s)\}}(-1)^s\frac{K_{\tilde{i}_1}K_{\tilde{i}_0}(\partial^i_{\tilde{j}_0}\lrcorner\,\Omega)}{f_{\tilde{i}_0}f_{\tilde{i}_1}}\right\}_I,$$

where the index set I now corresponds to the open cover $\mathcal{U}^\sigma|_{X\cap D_i}$. However, the last calculation shows that if j_0 coincides with j_1, then the expression in the above cocycle vanishes on the given open set. Hence, this cocycle is the same as

$$\left\{\frac{(-1)^d AB}{(\prod_{\rho_k\subset\sigma} x_k)^2}\left(\frac{x_{l_{k-1}}f_{l_{k-1}}}{\mathrm{mult}(\sigma_k)} - \frac{x_{l_{k+1}}f_{l_{k+1}}}{\mathrm{mult}(\sigma_{k+1})}\right)\sum_{\tilde{I}=I\setminus\{(i_s,j_s)\}}(-1)^s\frac{K_{\tilde{i}_1}K_{\tilde{i}_0}(\partial^i_{\tilde{j}_0}\lrcorner\,\Omega)}{f_{\tilde{i}_0}f_{\tilde{i}_1}}\right\}_I.$$

By the arguments similar to part (i), one can show that this coincides with the restriction $\varphi_i^*\big(\omega^i_{ABH^\sigma_i(f)}\big)$, where $H^\sigma_i(f)$ is equal to

$$\frac{\sqrt{-1}x_{l_{k+1}}f_{l_{k+1}}}{\mathrm{mult}(\sigma_{k,k+1})\prod_{\rho_k\subset\sigma}x_k} - \frac{\sqrt{-1}x_{l_{k-1}}f_{l_{k-1}}}{\mathrm{mult}(\sigma_{k-1,k})\prod_{\rho_k\subset\sigma}x_k}$$

at $x_{l_k} = x_{l_{k-1}} = x_{l_{k+1}} = 0$. □

Remark 6.12. In the anticanonical case $\beta = \beta_0$ the polynomials $H^\sigma_{i,\pm1}(f)$, $H^\sigma_i(f)$ of the above proposition can be written in a simpler form. Let $D = \sum_{k=1}^n D_k$ be the anticanonical divisor with the associated polytope $\Delta := \Delta_D$. For $f = \sum_{m\in\Delta\cap M} a_m x^{D(m)}$ and $\sigma \in \Sigma_X(2)$, denote

$$H^\sigma(f) := \sqrt{-1}\sum_{m\in\sigma^\perp\cap\Delta\cap M} a_m\frac{x^{D(m)}}{\prod_{\rho_k\subset\sigma}x_k}.$$

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Then $H^{\sigma}_{i,\pm 1}(f) = H^{\sigma}(f)/\mathrm{mult}(\sigma_{k,k\pm 1})$ and

$$H^{\sigma}_{i}(f) = \left(\frac{1}{\mathrm{mult}(\sigma_{k+1})} - \frac{1}{\mathrm{mult}(\sigma_{k})}\right) H^{\sigma}(f).$$

7. The Chiral Ring for Anticanonical Hypersurfaces

Here, we will apply the results of the previous sections to explicitly describe a subring of the chiral ring $H^*(X, \wedge^* \mathcal{T}_X)$, coming from the graded pieces of $R_1(f)$ and $R_1^{\sigma}(f)$, for semiample anticanonical nondegenerate hypersurfaces. By Proposition 2.6, such hypersurfaces are Calabi–Yau. The description of the chiral ring is complete for Calabi–Yau threefolds.

Let P_{Σ} be a complete simplicial toric variety, and let $X \subset \mathrm{P}_{\Sigma}$ be a big and nef nondegenerate hypersurface defined by $f \in S_{\beta}$. From Theorem 6.7, we know the following part of the middle cohomology of X:

$$[\omega_{\_}] \bigoplus \left(\bigoplus_i \omega_{\_}^i \right) : R_1(f)_{(^*+1)\beta - \beta_0} \bigoplus \left(\bigoplus_{\sigma \in \Sigma_X(2)} \left(R_1^{\sigma}(f)_{^*\beta - \beta_0 + \beta_1^{\sigma}} \right)^{n(\sigma)} \right) \hookrightarrow H^{d-1-^*,^*}(X).$$

Now suppose that β is the anticanonical degree β_0. In this case, the isomorphism (5) and Theorems 3.4, 4.11 give us:

THEOREM 7.1. *Let $X \subset \mathrm{P}_{\Sigma}$ be a semiample anticanonical nondegenerate hypersurface defined by $f \in S_{\beta}$. Then there is a natural inclusion*

$$\gamma_{\_} \bigoplus \left(\bigoplus_i \gamma_{\_}^i \right) : R_1(f)_{^*\beta} \bigoplus \left(\bigoplus_{\sigma \in \Sigma_X(2)} \left(R_1^{\sigma}(f)_{(^*-1)\beta + \beta_1^{\sigma}} \right)^{n(\sigma)} \right) \hookrightarrow H^*(X, \wedge^* \mathcal{T}_X),$$

where the sum $\bigoplus_i \gamma_{\_}^i$ is over $\rho_i \subset \sigma \in \Sigma_X(2)$ such that $\rho_i \notin \Sigma_X$ and $n(\sigma)$ is the number of such cones. Also, $R_1^{\sigma}(f)_{(q-1)\beta + \beta_1^{\sigma}} = 0$ for $q = 0, d-1$.

Remark 7.2. The map given by $\gamma_{\_} \bigoplus (\bigoplus_i \gamma_{\_}^i)$ is an isomorphism onto $H^q(X, \wedge^q \mathcal{T}_X)$ if $q = 0, 1, d-2, d-1$ and $d \neq 1, 3$. In particular, for semiample anticanonical nondegenerate hypersurfaces of dimension 3, we get a complete description of the chiral ring.

We claim that the part of $H^*(X, \wedge^* \mathcal{T}_X)$ given in the above theorem is a subring. Let us describe the product structure on this part. First, note that the ring $\bigoplus_p H^p(X, \wedge^p \mathcal{T}_X)$ is commutative. Theorems 3.3, 4.7, Lemmas 4.8, 6.9 and Equation (5) give us all information about the ring structure except for the products $\gamma_A^i \cup \gamma_B^j$ when ρ_i and ρ_j span a cone of Σ contained in some two-dimensional cone $\sigma \in \Sigma_X$. For such ρ_i and ρ_j, we first show that $\gamma_A^i \cup \omega_B^j$ is in the part of the middle cohomology represented by $[\omega_{\_}] \bigoplus (\bigoplus_k \omega_{\_}^k)$. It is easy to see that $\gamma_A^i \in H^*(X, \wedge^* \mathcal{T}_X)$ can be 'lifted' to γ in $H^*(\mathrm{P}_{\Sigma}, \wedge^* \mathcal{T}_{\mathrm{P}_{\Sigma}})$ with respect to the maps of the following lemma.

LEMMA 7.3. *Let $i: L \to K$ be a morphism of orbifolds, and let $a \in H^p(K, \wedge^q \mathcal{T}_K)$ be*

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such that $\widetilde{i^*}a = \eta^*\tilde{a}$ *for some* $\tilde{a}$ *under the maps*

$$H^p(K, \wedge^q \mathcal{T}_K) \xrightarrow{\tilde{i}^*} H^p(L, i^* \wedge^q \mathcal{T}_K) \xleftarrow{\eta^*} H^p(L, \wedge^q \mathcal{T}_L).$$

Then $i^*(a \cup b) = \tilde{a} \cup i^*b$ *for* $b \in H^r(K, \Omega_K^s)$.

Proof. First, note that we have natural maps between the sheaves:

$$i^*\Omega_K^* \to \Omega_L^* \quad \text{and} \quad \wedge^* \mathcal{T}_L \to i^* \wedge^* \mathcal{T}_K.$$

The corresponding maps in cohomology of L with coefficients in the sheaves are denoted by η_* and η^*, respectively. Then the restriction map $i^* : H^*(K, \Omega_K^*) \to H^*(L, \Omega_L^*)$ decomposes as

$$H^*(K, \Omega_K^*) \xrightarrow{\widetilde{i^*}} H^*(L, i^*\Omega_K^*) \xrightarrow{\eta_*} H^*(L, \Omega_L^*).$$

Therefore,

$$i^*(a \cup b) = \eta_*\tilde{i}^*(a \cup b) = \eta_*(\tilde{i}^*a \cup \tilde{i}^*b) = \eta_*(\eta^*\tilde{a} \cup \tilde{i}^*b) = \tilde{a} \cup \eta_*\tilde{i}^*b = \tilde{a} \cup i^*b,$$

where we use the projection formula. □

Using the above lemma, for $h \in H^{d-1}(\mathrm{P}_\Sigma)$, we have

$$\gamma_A^i \cup \omega_B^j \cup i^*h = \pm\gamma_A^i \cup i^*h \cup \omega_B^j = \pm i^*(\gamma \cup h) \cup \omega_B^j = 0$$

since the toric part is orthogonal to the residue part in the middle cohomology. Similarly, $\widetilde{\varphi_\tau^*\gamma_A^i} = \eta^*\widetilde{\gamma_A^i}$ for a corresponding $\widetilde{\gamma_A^i} \in H^*(X \cap V(\tau), \wedge^*\mathcal{T}_{X\cap V(\tau)})$, $\tau \in \Sigma(2)$. Therefore, for $h' \in H^{d-5}(X \cap V(\tau))$,

$$\begin{aligned}\gamma_A^i \cup \omega_B^j \cup \varphi_{\tau!}h' &= \varphi_{\tau!}(\varphi_\tau^*(\gamma_A^i \cup \omega_B^j) \cup h' = \varphi_{\tau!}(\widetilde{\gamma_A^i} \cup \varphi_\tau^*\omega_B^j \cup h') \\ &= \pm\varphi_{\tau!}(\widetilde{\gamma_A^i} \cup h') \cup \omega_B^j = 0\end{aligned}$$

where we use the projection formula for Gysin homomorphisms and Theorem 6.7. Hence, by the same Theorem and because of the nondegenerate pairing on the middle cohomology, the cup product $\gamma_A^i \cup \omega_B^j$ lies in the space given by $[\omega_-] \bigoplus (\bigoplus_k \omega_-^k)$. By the isomorphism (5), the cup product $\gamma_A^i \cup \gamma_B^j$ is in the part of the chiral ring described in Theorem 7.1. Thus, this part is a subring of $H^*(X, \wedge^*\mathcal{T}_X)$.

Since X is Calabi–Yau, we have natural isomorphisms

$$H^{d-1}(X, \wedge^{d-1}\mathcal{T}_X) \cong H^{d-1}(X, \mathcal{O}_X) \cong H^0(X, \Omega_X^{d-1}) \cong \mathbb{C}.$$

The cup product on the middle cohomology induces a nondegenerate pairing on the chiral ring and its subring represented by $\gamma_- \bigoplus (\bigoplus_k \gamma_-^k)$. Therefore, one can recover the product structure of the subring, knowing the triple products on this subring. Because of Lemmas 4.8 and 6.9 it suffices to consider the product of three elements $\gamma_A^i \cup \gamma_B^j \cup \gamma_C^l \in H^{d-1}(X, \wedge^{d-1}\mathcal{T}_X)$ in the cases $i = j = l$ and $i = j$ with l such that ρ_i, ρ_l

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span a 2-dimensional cone of Σ contained in some $\sigma \in \Sigma_X(2)$. For this, compute

$$
\begin{aligned}
&(\gamma^i_A \cup \gamma^i_B \cup \gamma^i_C \cup [\omega_1]) \cup [\omega_1]\\
&\quad = \varepsilon\gamma^i_A \cup \omega^i_B \cup \omega^i_C = \varepsilon\gamma^i_A \cup \omega^i_B \cup \varphi_i!\tilde{\omega}^i_C\\
&\quad = \varepsilon\varphi_i!(\varphi^*_i(\gamma^i_A \cup \omega^i_B) \cup \tilde{\omega}^i_C) = \varepsilon\varphi_i!((\varphi^*_i\omega^i_{ABH^\sigma_i(f)}) \cup \tilde{\omega}^i_C)\\
&\quad = \varepsilon\omega^i_{ABH^\sigma_i(f)} \cup \varphi_i!\tilde{\omega}^i_C = \varepsilon\omega^i_{ABH^\sigma_i(f)} \cup \omega^i_C\\
&\quad = (\gamma^i_{ABH^\sigma_i(f)} \cup \omega^i_C) \cup [\omega_1]\\
&\quad = \frac{\mathrm{mult}(\sigma_k + \sigma_{k+1})[\omega_{\mu^{-1}}(ABCH^\sigma_i(f)G^\sigma(f))] \cup [\omega_1]}{\mathrm{mult}(\sigma_k)\mathrm{mult}(\sigma_{k+1})},
\end{aligned}
$$

where we used Propositions 6.3, 6.11, 6.10 and the projection formula for Gysin homomorphisms, and where ε is a sign depending on the degree of C. Similarly, in the other case (as in (6), $\rho_i = \rho_{l_k}$):

$$
\begin{aligned}
&(\gamma^i_A \cup \gamma^i_B \cup \gamma^{l_{k\pm1}}_C \cup [\omega_1]) \cup [\omega_1]\\
&\quad = \varepsilon\gamma^i_A \cup \omega^i_B \cup \omega^{l_{k\pm1}}_C = \varepsilon\gamma^i_A \cup \omega^i_B \cup \varphi_{l_{k\pm1}!}\tilde{\omega}^{l_{k\pm1}}_C\\
&\quad = \varepsilon\varphi_{l_{k\pm1}!}(\varphi^*_{l_{k\pm1}}(\gamma^i_A \cup \omega^i_B) \cup \tilde{\omega}^{l_{k\pm1}}_C) = \pm\varepsilon\varphi_{l_{k\pm1}!}((\varphi^*_{l_{k\pm1}}\omega^i_{ABH^\sigma_{i,\pm1}(f)}) \cup \tilde{\omega}^{l_{k\pm1}}_C)\\
&\quad = \pm\varepsilon\omega^i_{ABH^\sigma_{i,\pm1}(f)} \cup \omega^{l_{k\pm1}}_C = \pm(\gamma^i_{ABH^\sigma_{i,\pm1}(f)} \cup \omega^{l_{k\pm1}}_C) \cup [\omega_1]\\
&\quad = \mp\frac{[\omega_{\mu^{-1}(ABCH^\sigma_{i,\pm1}(f)G^\sigma(f))}] \cup [\omega_1]}{\mathrm{mult}(\sigma_{k,k\pm1})}.
\end{aligned}
$$

Since there is an isomorphism $\cup[\omega_1] : H^{d-1}(X, O_X) \cong H^{d-1}(X, \Omega^{d-1}_X)$, from the above calculation we get an explicit product structure on the chiral ring.

THEOREM 7.4. *Let $X \subset \mathrm{P}_\Sigma$ be a semiample anticanonical nondegenerate hypersurface defined by $f \in S_\beta$. Then, under the identifications of Theorem* 7.1, *we have*

(i) $\gamma_A \cup \gamma_B = \gamma_{AB}$,
(ii) $\gamma_A \cup \gamma^i_B = \gamma^i_{AB}$,
(iii) $\gamma^i_A \cup \gamma^j_B = 0$, *$i \neq j$, unless ρ_i and ρ_j span a cone of Σ contained in a two-dimensional cone of Σ_X,*
(iv) *for $\rho_i = \rho_{l_k} \notin \Sigma_X$, as in* (6), *contained in $\sigma \in \Sigma_X(2)$ and $A, B \in R^\sigma_1(f)_{(*-1)\beta+\beta^\sigma_1}$ such that $AB \in R^\sigma_1(f)_{(d-3)\beta+2\beta^\sigma_1}$,*

$$\gamma^i_A \cup \gamma^i_B = \frac{\mathrm{mult}(\sigma_k + \sigma_{k+1})\gamma_{\mu^{-1}(ABG^\sigma(f))}}{\mathrm{mult}(\sigma_k)\mathrm{mult}(\sigma_{k+1})} \quad \text{in} \quad H^{d-1}(X, \wedge^{d-1}\mathcal{T}_X),$$

where the map μ and $G^\sigma(f) \in S_{3\beta-2\beta^\sigma_1}$ are defined in (20) *and Proposition* 6.10,
(v) *for $\rho_i, \rho_j \notin \Sigma_X$ which span a* 2*-dimensional cone $\sigma' \in \Sigma$ contained in $\sigma \in \Sigma_X(2)$ and A, B as in* (iv),

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$$\gamma^i_A \cup \gamma^j_B = -\frac{\gamma_{\mu^{-1}(ABG^\sigma(f))}}{\mathrm{mult}(\sigma')} \quad in \quad H^{d-1}(X, \wedge^{d-1}\mathcal{T}_X),$$

(vi) *for* $\rho_i = \rho_{l_k}$ *as in* (6) *and* $A, B, C \in R^\sigma_1(f)_{(*-1)\beta+\beta^\sigma_1}$ *such that* $ABC \in R^\sigma_1(f)_{(d-4)\beta+3\beta^\sigma_1}$,

$$\gamma^i_A \cup \gamma^i_B \cup \gamma^i_C = \frac{(\mathrm{mult}(\sigma_k) - \mathrm{mult}(\sigma_{k+1}))\mathrm{mult}(\sigma_k + \sigma_{k+1})\gamma_{\mu^{-1}(ABCH^\sigma(f)G^\sigma(f))}}{(\mathrm{mult}(\sigma_k)\mathrm{mult}(\sigma_{k+1}))^2},$$

where $H^\sigma(f) \in S_{\beta-\beta^\sigma_1}$ *is defined in Remark* 6.12*,*

(vii) *for* $\rho_i = \rho_{l_k}$ *as in* (6) *and* A, B, C *as in* (vi)*,*

$$\gamma^i_A \cup \gamma^i_B \cup \gamma^{l_{k\pm1}}_C = \mp\frac{\gamma_{\mu^{-1}(ABCH^\sigma(f)G^\sigma(f))}}{\mathrm{mult}(\sigma_{k,k\pm1})^2},$$

where $\sigma_{k,k\pm1}$ *denotes the cone spanned by* ρ_i *and* $\rho_{l_{k\pm1}}$.

Remark 7.5. If the multiplicities of the two-dimensional cones of the fan Σ, lying inside a cone of $\Sigma_X(2)$, are equal to 1, then $\gamma^i_A \cup \gamma^i_B \cup \gamma^i_C = 0$ in part (vi) of the above theorem. In particular, this holds for the minimal Calabi–Yau hypersurfaces in [B2]. □

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