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DEGREES IN WHICH THE RECURSIVE SETS ARE UNIFORMLY RECURSIVE

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1. Introduction. One of the most fundamental and characteristic features of recursion theory is the fact that the recursive sets are not uniformly recursive. In this paper we consider the degrees $\mathbf{a}$ such that the recursive sets are uniformly of degree $\leqq \mathbf{a}$ and characterize them by the condition $\mathbf{a}' \geqq \mathbf{0}''$. A number of related results will be proved, and one of these will be combined with a theorem of Yates to show that there is no r.e. degree $\mathbf{a} < \mathbf{0}'$ such that the r.e. sets of degree $\leqq \mathbf{a}$ are uniformly of degree $\leqq \mathbf{a}$. This result and a generalization will be used to study the relationship between Turing and many-one reducibility on the r.e. sets.

2. Terminology. Our notation generally follows that of [**7**]. In particular we use letters such as A, B, W for sets of integers, f, g, h for total (number theoretic) functions, and ψ, φ for partial functions. We write $\lambda nf(n)$ for the function f, μs for the least number s, N for the set of natural numbers, φ_e for the eth partial recursive function, and W_e for the eth r.e. set.

We let $\varphi_e^s(x)$ be $\varphi_e(x)$ if $\varphi_e(x)$ is computed within s steps, and otherwise $\varphi_e^s(x)$ is undefined. We fix a recursive pairing function from $N \times N$ onto N and write $\langle e, i\rangle$ for the code number of the pair (e, i). A *degree* is a Turing degree, although the latter term is sometimes used for emphasis. Boldface symbols such as $\mathbf{a}$, $\mathbf{b}$ are used for degrees and $\mathbf{d}(A)$ denotes the degree of the set A. We write $\mathbf{0}$ for the degree of the recursive sets, $\mathbf{a}'$ for the jump of the degree $\mathbf{a}$, and $\mathbf{a} \cup \mathbf{b}$ for the least upper bound of the degrees $\mathbf{a}$, $\mathbf{b}$. For sets A, B we write $A \leqq_T B$ ($A \leqq_m B$) if A is Turing (many-one) reducible to B, and $A \oplus B$ for $\{2n : n \in A\} \cup \{2n + 1 : n \in B\}$. If ψ is a partial function, $\rho\psi$ denotes the range of ψ, and ψ is called *recursively extendible* if it can be extended to a (total) recursive function. For functions g, h we say that g *majorizes* h if $g(n) \geqq h(n)$ for all $n \in N$ and g *dominates* h if $g(n) \geqq h(n)$ for all but finitely many $n \in N$ (in which case $(\lambda n)[i + g(n)]$ majorizes h for some fixed $i \in N$). We shall frequently use the result of Martin [**6**, Lemmas 1.1 and 1.2] that for any degree $\mathbf{a}$, $\mathbf{a}' \geqq \mathbf{0}''$ if and only if there is a function g of degree $\leqq \mathbf{a}$ which dominates every recursive function.

If f is a binary function, then f_e denotes $(\lambda n)f(e, n)$. If $\mathscr{C}$ is a class of (unary) functions and $\mathbf{a}$ is a degree, $\mathscr{C}$ is called $\mathbf{a}$-*uniform* ($\mathbf{a}$-*subuniform*) if there is a

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binary function f of degree $\leqq \mathbf{a}$ such that

$$\mathscr{C} = \{f_e : e \in N\} \, (\mathscr{C} \subseteq \{f_e : e \in N\}).$$

If $\mathscr{C}$ is a class of sets, the preceding definition is to be interpreted by identifying each element of $\mathscr{C}$ with its characteristic function.

3. Basic results.

THEOREM 1. *If* $\mathbf{a}$ *is any degree, statements* (i)–(iv) *are equivalent.*

(i) $\mathbf{a}' \geqq \mathbf{0}''$.

(ii) *the recursive functions are* $\mathbf{a}$*-uniform.*

(iii) *the recursive functions are* $\mathbf{a}$*-subuniform.*

(iv) *the recursive sets are* $\mathbf{a}$*-uniform.*

If $\mathbf{a}$ *is* r.e., *then* (i)–(iv) *are each equivalent to* (v).

(v) *the recursive sets are* $\mathbf{a}$*-subuniform.*

Proof. (i) $\Rightarrow$ (ii). Assume $\mathbf{a}' \geqq \mathbf{0}''$, and let g be a function of degree $\leqq \mathbf{a}$ which dominates all recursive functions. Define the binary partial function ψ by $\psi(\langle e, i\rangle, n) \simeq \varphi_e^{i+g(n)}(n)$. Let $f(\langle e, i\rangle, n) = \psi(\langle e, i\rangle, n)$ if $\psi(\langle e, i\rangle, m)$ is defined for all $m \leqq n$; otherwise let $f(\langle e, i\rangle, n) = 0$. Then if

$$\psi_{\langle e,i\rangle}(= (\lambda n)\psi(\langle e, i\rangle, n))$$

is total, $f_{\langle e,i\rangle} = \varphi_e$, and otherwise $f_{\langle e,i\rangle}$ is nonzero for only finitely many arguments. Hence $f_{\langle e,i\rangle}$ is recursive in either case. Also if φ_e is total, then g dominates $(\lambda n)(\mu s(\varphi_e^s(n))$ is defined$))$ and so $f_{\langle e,i\rangle} = \varphi_e$ for all sufficiently large i. This proves (ii), and the implication (ii) $\Rightarrow$ (iii) is trivial.

(iii) $\Rightarrow$ (i). Let $f(e, n)$ be a function of degree $\leqq \mathbf{a}$ such that every recursive function is an f_e. Define $g(n) = \max\{f_e(n) : e \leqq n\}$. Then g dominates every f_e and hence every recursive function. Since $\mathbf{d}(g) \leqq \mathbf{a}$, (i) follows. Therefore the equivalence of (i)–(iii) is established. Also the implication (ii) $\Rightarrow$ (iv) is immediate.

To show (iv) $\Rightarrow$ (i) we need a simple lemma which will also be useful elsewhere in the paper. The motivation of this lemma will be explained after the proof of Proposition 3.

LEMMA 2. *There is a recursive function g such that for every e, $\rho\varphi_{g(e)} \subseteq \{0, 1\}$ and*

(a) φ_e *total* $\Rightarrow \varphi_{g(e)}$ *total,*

(b) φ_e *not total* $\Rightarrow \varphi_{g(e)}$ *is not recursively extendible.*

Proof. Let φ_k be a fixed partial recursive function such that $\rho\varphi_k \subseteq \{0, 1\}$ and φ_k is not recursively extendible. For any pair (e, n), let $\psi(e, n)$ be the least number s such that either $\varphi_k^s(n)$ is defined or $\varphi_e^s(0), \varphi_e^s(1), \ldots, \varphi_e^s(n)$ are all defined, and let $\psi(e, n)$ be undefined if no such s exists. By the s-m-n theorem there is a recursive function g such that for all e and n, $\varphi_{g(e)}(n) = \varphi_k(n)$ if $\psi(e, n)$ is defined via the first alternative, $\varphi_{g(e)}(n) = 0$ if $\psi(e, n)$ is defined through the second alternative and $\varphi_{g(e)}(n)$ is undefined otherwise. If φ_e is total, then $\psi(e, n)$ is defined for all n and so $\varphi_{g(e)}$ is total. If φ_e is not total, then $\varphi_{g(e)}(n) \simeq \varphi_k(n)$ for all sufficiently large n and so $\varphi_{g(e)}$ is not recursively extendible, and the Lemma is proved.

(iv) $\Rightarrow$ (i). Assume that f has degree $\leqq \mathbf{a}$ and the f_e's are exactly the recursive characteristic functions. Then for all e,

$$(1) \qquad \varphi_e \text{ total} \Leftrightarrow (\exists i)[f_i \text{ extends } \varphi_{g(e)}] \Leftrightarrow (\exists i)(\forall n)(\forall s)(\forall y)[\varphi_{g(e)}{}^s(n) = y \Rightarrow f_i(n) = y]$$

where g is the function from the Lemma. But if $T = \{e : \varphi_e \text{ total}\}$, the above equivalences show that T is $\Sigma_2{}^0(\mathbf{a})$ (i.e., $\Sigma_2{}^0$ in the degree $\mathbf{a}$). Since T is $\Pi_2{}^0$, it follows that T is $\Delta_2{}^0(\mathbf{a})$ and so of degree $\leqq \mathbf{a}'$ by Post's Hierarchy Theorem. Since $\mathbf{d}(T) = \mathbf{0}''$ [**7**, p. 264], (i) follows.

Since the implication (iv) $\Rightarrow$ (v) is trivial, it remains only to show that (v) $\Rightarrow$ (i) assuming $\mathbf{a}$ to be r.e. Assume that (i) is false and that f is a binary function of degree $\leqq \mathbf{a}$. We must show that there is a recursive function with $\rho r \subseteq \{0, 1\}$ such that $r \neq f_e$ for all e. The construction of r is similar to the diagonal proof that the recursive functions are not uniformly recursive, except that during the construction we must work with an approximation to f rather than with f itself. Since f has degree $\leqq \mathbf{0}'$, it follows from [**10**, Theorem 2] that there is a recursive function $g(e, n, s)$ such that $f(e, n) = \lim_s g(e, n, s)$ for all e, n. In fact, since f has degree $\leqq \mathbf{a}$ and $\mathbf{a}$ is r.e., it follows from the proof of [**10**, Theorem 2] that g may be chosen so that there is a function h of degree $\leqq \mathbf{a}$ such that $g(e, n, s) = f(e, n)$ for all $s \geqq h(e, n)$. Now define $p(n) = \max\{h(e, \langle e, n\rangle) : e \leqq n\}$. Since p has degree $\leqq \mathbf{a}$ and $\mathbf{a}' \ngeq \mathbf{0}''$, there is a recursive function q which p fails to dominate. Finally define $r(\langle e, n\rangle) = 1 \dot{-} g(e, \langle e, n\rangle, q(n))$. Then r is a recursive function and $r(\langle e, n\rangle) \neq f_e(\langle e, n\rangle)$ whenever $n \geqq e$ and $q(n) \geqq p(n)$ (since then $q(n) \geqq h(e, \langle e, n\rangle)$ and so $g(e, \langle e, n\rangle, q(n)) = f(e, \langle e, n\rangle) = f_e(\langle e, n\rangle))$. This completes the proof of Theorem 1.

The next result shows that the implication (v) $\Rightarrow$ (i) of Theorem 1 is not true in general.

PROPOSITION 3. *There is a degree* $\mathbf{a}$ *such that* $\mathbf{a}' = \mathbf{0}'$ *and the recursive sets are* $\mathbf{a}$*-subuniform.*

Proof. Let the predicate $P(f)$ be true of the function f in case

$$\rho f \subseteq \{0, 1\} \ \& \ (\forall e)(\forall n)[\varphi_e(n) \text{ defined} \rightarrow f(\langle e, n\rangle) = \min\{1, \varphi_e(n)\}].$$

Then routine expansion shows that P is a $\Pi_1{}^0$ predicate and clearly $(\exists f)P(f)$ holds. Also P is *recursively bounded* because of the clause $\rho f \subseteq \{0, 1\}$. It now follows from a basis theorem of Soare and the author [**4**, Theorem 2.1] that there is a function f such that $P(f)$ holds and $\mathbf{a}' = \mathbf{0}'$, where $\mathbf{a} = \mathbf{d}(f)$. Clearly the recursive sets are $\mathbf{a}$-subuniform, and so Proposition 3 is proved.

It was the proof of Proposition 3 which led us to the proof of (iv) $\Rightarrow$ (i) in Theorem 1. The relevant observation is that if f is any function satisfying $P(f)$ and φ_k is as in the proof of Lemma 2 (i.e., $\{0, 1\}$-valued and not recursively extendible), then $(\lambda n)f(\langle k, n\rangle)$ is nonrecursive. Thus the mere existence of such a φ_k makes it immediately clear that the construction for Proposition 3 cannot yield a counterexample to (iv) $\Rightarrow$ (i), while a slightly more elaborate use of φ_k suffices to prove (iv) $\Rightarrow$ (i).

The proof of (iv) $\Rightarrow$ (i) yields a useful characterization of degrees satisfying (v).

PROPOSITION 4. *For any degree* $\mathbf{a}$, *assertion* (v) *is equivalent to the disjunction* (i) $\vee$ (vi) *where* (i), (v) *are as in Theorem* 1 *and* (vi) *is the following:*

(vi) *there is a complete extension of first-order Peano arithmetic of degree* $\leqq \mathbf{a}$.

Proof. Clearly (i) $\Rightarrow$ (v) since (i) $\Rightarrow$ (iv) $\Rightarrow$ (v) by Theorem 1. Also (vi) $\Rightarrow$ (v) since if T is any complete extension of Peano arithmetic of degree $\leqq \mathbf{a}$, the family of sets definable in T is $\mathbf{a}$-uniform and includes all recursive sets. It remains to show that (v) $\Rightarrow$ (i) $\vee$ (vi), so assume that f is a function of degree $\leqq \mathbf{a}$ and every recursive characteristic function is an f_e. Let g be the function from Lemma 2, and assume that φ_k in the proof of Lemma 2 was chosen so that $\varphi_k^{-1}(0)$ and $\varphi_k^{-1}(1)$ are effectively inseparable. Now reconsider the equivalence (1) used in the proof of (iv) $\Rightarrow$ (i). It is no longer necessarily valid because we are not assuming that all f_e's are recursive. However, if (1) is valid, then $\mathbf{a}' \geqq \mathbf{0}''$ follows as before. So assume (1) is not valid. Since the left-right implication of (1) still follows from our weaker hypothesis, we see that there must be numbers e, i such that φ_e is not total and f_i extends $\varphi_{g(e)}$. But $\varphi_{g(e)}$ differs only finitely from φ_k, and so $f_i^{-1}(0)$ separates a pair of effectively inseparable sets (i.e., $\varphi_{g(e)}^{-1}(0)$ and $\varphi_{g(e)}^{-1}(1)$). It now follows from [**4**, Proposition 6.1] that there is a complete extension of Peano arithmetic recursive in f_i and thus of degree $\leqq \mathbf{a}$.

Degrees of complete extensions of Peano arithmetic were originally studied by Scott and Tennenbaum [**9**] and more recently by Soare and the author [**4**;**5**]. For instance, in [**4**, Corollary 2.2] it is proved that there is a complete extension of Peano arithmetic whose degree $\mathbf{a}$ satisfies $\mathbf{a}' = \mathbf{0}'$ and in [**5**, Corollary 4.3] it is proved that $\mathbf{0}'$ is the only r.e. degree satisfying (vi). From these results and Proposition 4, we immediately obtain new (but rather indirect) proofs of Proposition 3 and (v) $\Rightarrow$ (i) for r.e. degrees. Similarly, we have the following corollary.

COROLLARY 5. *If the recursive sets are* $\mathbf{a}$-*subuniform, then either* $\mathbf{a}' \geqq \mathbf{0}''$, *or every countable partially ordered set can be embedded in the degrees* $\leqq \mathbf{a}$.

Proof. It is shown in [**4**, Corollary 4.4] that if $\mathbf{a}$ is the degree of any complete extension of Peano arithmetic, then every countable partially ordered set can be embedded in the degrees $\leqq \mathbf{a}$.

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The converse of Corollary 5 is false. To see this let $\mathbf{a}$ be a nonzero r.e. degree with $\mathbf{a}' = \mathbf{0}'$ [**8**, § 6, Corollary 2]. By a theorem of Sacks [**8**, § 5, Theorem 2] every countable partially ordered set can be embedded in the degrees $\leqq \mathbf{a}$, and yet the recursive sets are not $\mathbf{a}$-subuniform by Theorem 1. More generally, we suspect that there is no degree-theoretic characterization whatever of the degrees $\mathbf{a}$ such that the recursive sets are $\mathbf{a}$-subuniform. We remark also that Corollary 5 becomes false if either alternative is dropped from the conclusion. For the first alternative this follows from the theorem of Cooper [**1**, Theorem 1] that if $\mathbf{b} \geqq \mathbf{0}'$ (in particular $\mathbf{b} = \mathbf{0}''$) there is a minimal degree $\mathbf{a}$ with $\mathbf{a}' = \mathbf{b}$; for the second alternative this follows from Proposition 3.

4. Applications.

COROLLARY 6. *If* $\mathbf{a}$ *is an* r.e. *degree and* $\mathbf{a} < \mathbf{0}'$, *then the class of* r.e. *sets of degree* $\leqq \mathbf{a}$ *is not* $\mathbf{a}$*-uniform.*

Proof. Assume the degree $\mathbf{a}$ yields a counterexample. Then the recursive sets are $\mathbf{a}$-subuniform and so $\mathbf{a}' = \mathbf{0}''$ by (v) $\Rightarrow$ (i) of Theorem 1. On the other hand, since the r.e. sets of degree $\leqq \mathbf{a}$ are $\mathbf{a}$-uniform, they are $\mathbf{0}'$-uniform and so $\mathbf{a}'' = \mathbf{0}''$ by a theorem of Yates [**11**, Theorem 9].

Corollary 6 answers a question raised by Yates at the end of [**11**]. S. B. Cooper and the author independently proved it by rather involved direct constructions before this simple argument was found. However, the present methods yield a strong generalization of Corollary 6 which does not seem accessible to direct proof.

COROLLARY 7. *If* $\mathbf{a}$, $\mathbf{b}$ *are* r.e. *degrees*, $\mathbf{b} \leqq \mathbf{a}$, *and* $\mathbf{b} < \mathbf{0}'$, *then the following three statements are equivalent:*

(a) *the* r.e. *sets of degree* $\leqq \mathbf{b}$ *are* $\mathbf{a}$*-subuniform;*

(b) $\mathbf{b}'' = \mathbf{a}' = \mathbf{0}''$;

(c) *there is an* r.e. *sequence of* r.e. *sets which is uniformly of degree* $\leqq \mathbf{a}$ *and consists exactly of the* r.e. *sets of degree* $\leqq \mathbf{b}$.

Proof. The proof that (a) $\Rightarrow$ (b) is the same as for Corollary 6, except that one should note that only subuniformity (not uniformity) is actually used by Yates [**11**, Theorems 8 and 9] to show $\mathbf{b}'' = \mathbf{0}''$. Since (c) $\Rightarrow$ (a) is trivial, it remains only to prove that (b) $\Rightarrow$ (c). From the assumptions $\mathbf{b} \leqq \mathbf{a}$ and $\mathbf{b}'' = \mathbf{a}'$ it follows by relativizing [**6**, Lemma 1.2] to $\mathbf{b}$ that there is a function g^* of degree $\leqq \mathbf{a}$ which dominates every function of degree $\leqq \mathbf{b}$. It then follows from the proof of [**10**, Theorem 2] that there is a recursive function $g(n, s)$ and a function h of degree $\leqq \mathbf{a}$ such that $g(n, s) = g^*(n)$ for all $s \geqq h(n)$. Also, since $\mathbf{b}'' = \mathbf{0}''$ it follows from [**11**, Theorem 9] that there is a uniformly r.e. sequence of r.e. sets $R_0, R_1, \ldots$ consisting exactly of the r.e. sets of degree $\leqq \mathbf{b}$. Let $R_e(s)$ be the finite subset of R_e obtained by stage s in a fixed simultaneous recursive enumeration of this sequence. Define

$$S_{\langle e, i\rangle} = \{n : (\exists s)[n \in R_e(i + g(n, s))]\}$$

The sequence $\{S_{\langle e,i\rangle}\}$ is clearly uniformly r.e. and also is uniformly of degree $\leqq \mathbf{a}$ since the quantifier over s in its definition is implicitly bounded by $h(n)$. Furthermore if i is chosen so large that $(\lambda n)[i + g^*(n)]$ majorizes $(\lambda n)[(\mu s)[n \in R_e(s) \vee n \notin R_e]]$ then $S_{\langle e,i\rangle} = R_e$. However, this is not yet our desired sequence of sets because there is no reason to believe that every $S_{\langle e,i\rangle}$ is of degree $\leqq \mathbf{b}$. To obtain the desired sequence, we employ the trick of [**11**, Theorem 8] and define

$$T_{\langle e,i\rangle} = \{n : n \in S_{\langle e,i\rangle} \,\&\, (\forall m)_{<n}[m \in R_e(n) \Rightarrow m \in S_{\langle e,i\rangle}]\}$$

Then $\{T_{\langle e,i\rangle}\}$ is uniformly r.e. and uniformly of degree $\leqq \mathbf{a}$ because $\{S_{\langle e,i\rangle}\}$ has these properties. Furthermore, if $S_{\langle e,i\rangle} = R_e$ then $T_{\langle e,i\rangle} = R_e$ and otherwise $S_{\langle e,i\rangle}$ is finite. From this and the other properties of $\{S_{\langle e,i\rangle}\}$, it follows that $\{T_{\langle e,i\rangle}\}$ satisfies the requirements of part (c).

Our original interest in the topic of this paper grew out of the following question: is there an r.e. Turing degree other than $\mathbf{0}$ or $\mathbf{0}'$ which contains a maximum r.e. m-degree? (We remark for background that $\mathbf{0}'$ has no maximum among all its m-degrees, but there do exist nonzero Turing degrees having maximum m-degrees (cf. [**3**, Problem 14-14] or [**7**, § 4]). Also by [**7**, §§ 7.6 and 8.4] every truth-table degree has a maximum m-degree and every m-degree has a maximum 1-degree.) The question posed above remains unanswered, but the following result gives some information on it and answers the corresponding question for reductions by primitive recursive functions. (We write $B \leqq_{\mathrm{pr}} A$ if $B = f^{-1}(A)$ for some primitive recursive function f.)

COROLLARY 8. (i) *If* $\mathbf{a}$ *is an* r.e. *Turing degree having a maximum among its r.e. m-degrees and* $\mathbf{a} < \mathbf{0}'$, *then* $\mathbf{a}'' = \mathbf{0}''$.

(ii) *If A is a noncreative* r.e. *set, then there is an* r.e. *set B of the same Turing degree as A such that* $B \nleqq_{\mathrm{pr}} A$.

(iii) *There exists a Turing incomplete* r.e. *set A and a nonzero* r.e. *Turing degree* $\mathbf{b}$ *such that every* r.e. *set of Turing degree* $\leqq \mathbf{b}$ *is* $\leqq_{\mathrm{pr}} A$.

Proof. (i) Let A be an r.e. set in the maximum r.e. m-degree of $\mathbf{a}$. Let $G(\leqq_m A) = \{e : W_e \leqq_m A\}$. As in [**11**], let $G(\leqq \mathbf{a}) = \{e : W_e \leqq_T A\}$. We claim $G(\leqq_m A) = G(\leqq \mathbf{a})$. Clearly $G(\leqq_m A) \subseteq G(\leqq \mathbf{a})$. Suppose $W_e \leqq_T A$. Then $A \oplus W_e$ has degree $\mathbf{a}$ and so $A \oplus W_e \leqq_m A$ by the maximality of A. Hence $W_e \leqq_m A$. Direct expansion shows that $G(\leqq_m A)$ is a Σ_3^0 set. In [**11**, Theorem 9], Yates showed that if $G(\leqq \mathbf{a})$ is Σ_3^0 and $\mathbf{a} < \mathbf{0}'$, then $\mathbf{a}'' = \mathbf{0}''$.

(ii) Assume it is false. Then A cannot have Turing degree $\mathbf{0}'$ since we could then take B to be creative. Since the family of sets $\leqq_{\mathrm{pr}} A$ is $\mathbf{a}$-uniform, it follows from Corollary 6 that there is an r.e. set W such that $W \leqq_T A$ and $W \nleqq_{\mathrm{pr}} A$. Clearly if $B = A \oplus W$, then B satisfies the conclusion of (ii).

(iii) By [**8**, § 6, Corollary 5] there is an r.e. degree $\mathbf{a} < \mathbf{0}'$ such that $\mathbf{a}' = \mathbf{0}''$. By [**8**, § 6, Corollary 2] there is a nonzero r.e. degree $\mathbf{b} \leqq \mathbf{a}$ such that $\mathbf{b}' = \mathbf{0}'$ and so $\mathbf{b}'' = \mathbf{0}'' = \mathbf{a}'$. By Corollary 7, there is a uniformly r.e. sequence of sets $\{T_n\}$ which is uniformly of degree $\leqq \mathbf{a}$ and includes all r.e. sets of degree $\leqq \mathbf{b}$. Let $A = \{2^n 3^j : j \in T_n\}$, so A is r.e. and has degree $\leqq \mathbf{a}$. Then A is incomplete and every r.e. set of degree $\leqq \mathbf{b}$ is $\leqq_{\text{pr}} A$.

5. A variant of the basic result. Suppose f is a function of degree $\leqq \mathbf{a}$ such that the f_e's are exactly the recursive functions. Then there is, of course, a function h such that $f_e = \varphi_{h(e)}$ for all e, but in general there is no reason to suppose that h can be chosen to have degree $\leqq \mathbf{a}$. Let us say that the recursive functions are $\mathbf{a}$-*superuniform* if there is a function h of degree $\leqq \mathbf{a}$ such that $\varphi_{h(0)}, \varphi_{h(1)}, \ldots$ are precisely the recursive functions. (The notion is defined analogously for the recursive sets.)

THEOREM 9. *For any degree* $\mathbf{a}$, *the following three statements are equivalent:*
(vii) *the recursive functions are* $\mathbf{a}$-*superuniform;*
(viii) *the recursive sets are* $\mathbf{a}$-*superuniform;*
(ix) $\mathbf{a} \cup \mathbf{0}' \geqq \mathbf{0}''$.

Proof. The implication (vii) $\Rightarrow$ (viii) is immediate. To prove (viii) $\Rightarrow$ (ix) assume that h is a function of degree $\leqq \mathbf{a}$ such that the $\varphi_{h(e)}$'s are exactly the recursive characteristic functions. Define $f(i, n) = \varphi_{h(i)}(n)$ and let g be the recursive function of Lemma 2. Then the equivalence (1) from the proof of (iv) $\Rightarrow$ (i) in Theorem 1 holds. Since $f_i = \varphi_{h(i)}$ for all i, (1) can be rewritten as

$$\varphi_e \text{ total} \Leftrightarrow (\exists\, i) C(h(i), g(e)) \tag{2}$$

where $C(n, k)$ is the assertion that φ_n and φ_k are *compatible* (i.e. agree on the intersection of their domains). But $C(n, k)$ is easily seen to be Π_1^0 and hence of degree $\leqq \mathbf{0}'$. Thus $C(h(i), g(e))$ has degree $\leqq \mathbf{a} \cup \mathbf{0}'$ and so (2) shows that $T(=\{e : \varphi_e \text{ total}\})$ is $\Sigma_1^0(\mathbf{a} \cup \mathbf{0}')$. But also T is $\Pi_1^0(\mathbf{0}')$ and so $\Pi_1^0(\mathbf{a} \cup \mathbf{0}')$. It follows that $\mathbf{0}'' = \mathbf{d}(T) \leqq \mathbf{a} \cup \mathbf{0}'$.

To prove (ix) $\Rightarrow$ (vii) we need a more useful form of the assumption $\mathbf{0}'' \leqq \mathbf{a} \cup \mathbf{0}'$.

LEMMA 10. *If* $\mathbf{0}'' \leqq \mathbf{a} \cup \mathbf{0}'$, *then there is an* r.e. *set* K *and a binary function* p *of degree* $\leqq \mathbf{a}$ *such that for all* e, φ_e *is total if and only if* $(\exists\, i)[p(e, i) \notin K]$.

Proof. Let K be a creative set and let A be any set of degree $\mathbf{a}$. Since $T \leqq_T A \oplus K$ (where $T = \{e : \varphi_e \text{ total}\}$) it follows from the formalism of relative computation of [**7**, Chapter 9] that we can effectively find for each e an index of an r.e. set S_e such that φ_e is total if and only if

$$(\exists\, \langle u, v, w, z\rangle)[\langle u, v, w, z\rangle \in S_e \,\&\, D_u \subseteq A \,\&\, D_v \subseteq \bar{A} \,\&\, D_w \subseteq K \,\&\, D_z \subseteq \bar{K}].$$

Since $\{z : D_z \cap K \neq \emptyset\} \leqq_m K$ by the m-completeness of K, there is a recursive function q such that for all z, $D_z \subseteq \bar{K}$ if and only if $q(z) \in \bar{K}$. For each e, let

$$T_e = \{q(z) : (\exists\, u)(\exists\, v)(\exists\, w)[\langle u, v, w, z\rangle \in S_e \,\&\, D_u \subseteq A \,\&\, D_v \subseteq \bar{A} \,\&\, D_w \subseteq K]\} \cup \{k\},$$

where k is a fixed element of K. Then $\{T_e\}$ is uniformly r.e. in A and all T_e are nonempty, so there is a function $p(e, i)$ of degree $\leqq \mathbf{a}$ such that T_e is the range of $(\lambda i)[p(e, i)]$ for all e. This is the desired function.

To prove (ix) $\Rightarrow$ (vii), first let $s(e, i, n)$ be the least s such that either $\varphi_e^s(n)$ is defined or $p(e, i) \in K^s$. (Here p, K are from the Lemma and K^s denotes the subset of K obtained after s steps in a recursive enumeration. Note that $s(e, i, n)$ is always defined since otherwise $\varphi_e(n)$ is undefined and $p(e, i) \notin K$, in contradiction to the Lemma.) Since the process of computing $s(e, i, n)$ is effective once $p(e, i)$ is given, there is a function h recursive in p (and therefore of degree $\leqq \mathbf{a}$) such that $\varphi_{h(\langle e, i \rangle)}(n) = \varphi_e(n)$ if $s(e, i, n)$ is defined through the first alternative and $\varphi_{h(\langle e, i \rangle)}(n) = 0$ otherwise. Thus $\varphi_{h(\langle e, i \rangle)}$ is total for all e and i; furthermore, if φ_e is total, then there is an i such that $p(e, i) \notin K$ and clearly $\varphi_e = \varphi_{h(\langle e, i \rangle)}$ for this i.

COROLLARY 11. *The recursive functions are not* $\mathbf{0}'$*-superuniform but there is a degree* $\mathbf{a}$ *incomparable with* $\mathbf{0}'$ *such that the recursive functions are* $\mathbf{a}$*-superuniform.*

Proof. In view of Theorem 9, the first assertion is immediate and the second assertion follows from the theorem of Friedberg [**2**] that there is a degree $\mathbf{a}$ such that $\mathbf{a}' = \mathbf{a} \cup \mathbf{0}' = \mathbf{0}''$.

The first part of Corollary 11 can be easily proved by a direct diagonal argument; however we do not know how to prove the second part without using Theorem 9.

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