\[
< \sum_{s=0}^{r-1} \frac{2^{r-s}}{(r-s)!} \binom{r-1}{s} \frac{1}{r(r-1) \ldots (r-s+1)n-s} \frac{n-r}{n-r+1} \frac{n-2r+s+1}{n-r+1}
\]

\[
< \sum_{s=0}^{r-1} \frac{2^{r-s}}{(r-s)!} \binom{r-1}{s} = \frac{2}{r!} \sum_{s=0}^{r-1} 2^{r-1-s} \binom{r-1}{s} = \frac{2}{r!} (2+1)^{r-1} = \frac{2.3^{r-1}}{r!} < \frac{3^r}{r!}.
\]

Also, \( \sum_{r=0}^{\infty} \frac{3^r}{r!} \) converges, so by Tannery's theorem [2, pp. 135-136], perhaps nowadays referred to simply as a 'dominated convergence' theorem, and far from a trivial result, it follows that

\[
\frac{p_n}{n!} \to \sum_{r=0}^{\infty} u_r = e^{-2} \text{ as } n \to \infty.
\]

References

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90.55 The ‘derangements’ relation visualised
Each of the \( n \) objects \( p_1, p_2, \ldots, p_n \) is paired with a corresponding location \( P_1, P_2, \ldots, P_n \). A set of pairings in which all \( n \) objects are mislocated is known as a ‘derangement’. A succinct argument [1] shows that the number of possible derangements of \( n \) objects, \( D_n \), is given by the recurrence relation:

\[
D_n = (n - 1) \left( D_{n-1} + D_{n-2} \right).
\]

The object of this note is to interpret the relation visually.

We can represent an arrangement on an array (Figure 1). There must be exactly one entry in each row and column. We restrict the possibilities to derangements by excluding cells on the main diagonal (Figure 2).
(Note in passing that $D_n$ answers the following question in recreational mathematics: ‘In how many ways can you arrange $n$ non-taking rooks on an $n \times n$ chessboard without using any square on the main diagonal?’)

Imagine we have found all $D_{n-1}$ and all $D_{n-2}$ solutions and wish to use these to generate all $D_n$ solutions.

Consider first the $D_{n-1}$ solutions. In Figure 3 we show a $D_{n-1}$ solution inset in a potential $D_n$ solution. For the latter we choose one of the $n - 1$ cells available in the first row. To produce a solution we must displace the $D_{n-1}$ entry which occurs in this column along its row into the first column.

![Figure 3]

We can do this for any of the $n - 1$ available cells in the first row and for each of the $D_{n-1}$ solutions, producing $(n - 1)D_{n-1}$ solutions in all.

Which $D_n$ solutions are not obtained by this process? Precisely those with entries in row 1, column $k$ and row $k$, column 1 (for some $k \neq 1$). There are $n - 1$ such pairings. A specimen combination is shown in Figure 4.

![Figure 4]
We note that we can complete a solution by combining each $D_{n-2}$ solution with such a pairing, giving $(n-1)D_{n-2}$ solutions overall. This gives the grand total:

$$D_n = (n - 1)D_{n-1} + (n - 1)D_{n-2} = (n - 1)(D_{n-1} + D_{n-2}),$$

as required.

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Reference


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90.56 A lack of memory

As part of some work I was doing recently (involving age-dependent branching processes), I needed to know the answer to the following question:

‘If $n$ individuals each have lifetimes that are independent and identically distributed as $X$, where $X$ follows a negative exponential distribution with parameter $\lambda$, then what can we say about the distribution of the lifetime of the longest lived individual?’

The following result emerged without too much difficulty:

**Theorem**

Let $X$ be a continuous random variable following a negative exponential distribution with parameter $\lambda$. Next, let $M_n = \max\{X_1, X_2, \ldots, X_n\}$, where $X_1, X_2, \ldots, X_n$ are a set of independent random variables, each identically distributed as $X$. Then $M_n$ is distributed as

$$X_1 + \frac{1}{2}X_2 + \ldots + \frac{1}{n}X_n = \sum_{k=1}^{n} \frac{1}{k}X_k.$$

**Proof:**

$X$ has probability density function $f_X(x) = \lambda e^{-\lambda x}$ and cumulative distribution function $F_X(x) = 1 - e^{-\lambda x}$, $x \geq 0$. Now let $f_{M_n}(m)$ and $F_{M_n}(m)$ denote the probability density function and cumulative distribution function of $M_n$ respectively. We have

$$P(M_n \leq m) = P(X_1 \leq m \text{ and } X_2 \leq m \text{ and... and } X_n \leq m)$$